By Gromov's compactness thm (after passing to a subseq.) $\exists$ isometric emb. $\phi_i : Y_i \to Y$ into some fixed cpt m.s. Y s.t.

$$\phi_i(Y_i) \xrightarrow{H} \underset{cpt}{K} \subset Y$$

In particular $Y_i \xrightarrow{gH} K$.

Can further assume

$$\phi_{i*}V_i' \xrightarrow{\text{weakly}} V \in I_{k+1}(Y) \quad \text{(Thm 4)}$$

Since $\operatorname{asrk}(X) \le k$, $V = 0$ (since $V \in I_{k+1}$ with image in $\overset{cpt}{K}$)

$\rightsquigarrow$ support of $V = Y_i \to K$

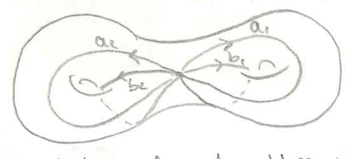
$\hookrightarrow \dim < k+1$ since $\operatorname{asrk}(X) \le k$

$\Rightarrow \phi_{i*}T_i' \Rightarrow \partial V = 0$

Step 3: go back to X and contradict $\operatorname{FillVol}_X(T_i') \geq \frac{\varepsilon}{2} r_i^{k+1}$ $\square$

Metric geometry and analysis on boundaries of Gromov hyperbolic spaces, and applications - Bruce Kleiner

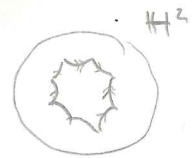
Example: Surface group



$\pi_1(S) = \langle a_1, b_1, a_2, b_2 \mid [a_1,b_1][a_2,b_2] = e \rangle$

$\rightsquigarrow$ solvability of word problem is directly related to isoperimetric inequality on the surface (dehn function)

$\rightsquigarrow$ hyperbolic metric



$$g_{\mathbb{H}^2} = \frac{g_{\mathbb{R}^2}}{(1-|y|^2)^2}$$

$\operatorname{Isom}(\mathbb{H}^2) = SO(2,1)$

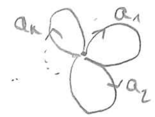
Groups as metric spaces:

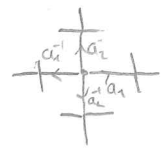
G - finitely gen. group

$\Sigma = \Sigma^{-1}$ - generating set

$\Gamma = \Gamma(G, \Sigma)$ - Cayley graph

$\hookrightarrow$ depends on choice of Σ but any two Γ are q.i.

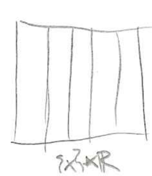




Quasi-isometries:

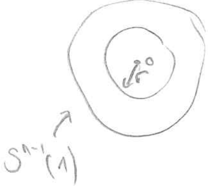
- $X \overset{qi}{\sim} Y$ iff they contain biLipschitz equivalent nets

$\rightsquigarrow$ q.i. = biLip as long as you only consider points that are well-separated



$\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$

$\delta : \mathbb{R}^{n-1} \to \mathbb{R}$



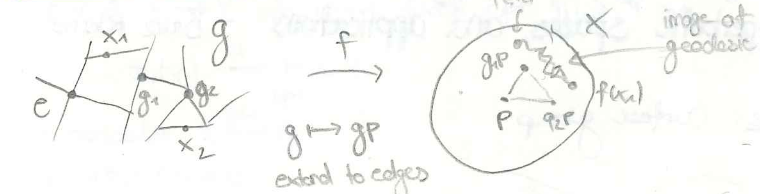
$\mathbb{H}^n = B(1)$

$\varphi : S^{n-1} \to S^{n-1}$ homeo

$\hat{\varphi} : \mathbb{H}^n \to \mathbb{H}^n$

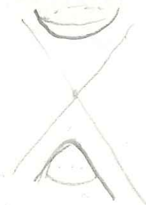
$\rightsquigarrow$ we have canonical q.i. class of m.s. associated to G

Milnor-Švarc Lemma



Negative curvature

- Quadric model: Quadratic form: $Q(x) = x_1^2 + \dots + x_n^2 - x_{n+1}^2$



Hyp. space: $\{Q = -1\}$

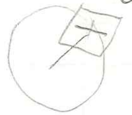
$\rightsquigarrow O(n,1)$

$x_1^2 + \dots + x_{n+1}^2$

Ex: $n \geq 3$ $\mathrm{Isom}(\mathbb{H}^n) \simeq \mathrm{Mob}(S^{n-1}) \simeq \mathrm{Conf}(S^{n-1})$

complex hyperbolic space

$\mathcal{B}(1) \subseteq \mathbb{C}^n$



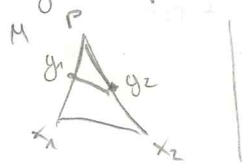
nilpotent group

homogeneous: $I = N \rtimes_\varphi \mathbb{R}$ $\varphi: \mathbb{R} \to \mathrm{Aut}(N)$

e.g. $dx_n^2 + (e^{x_n})^2 g_{\mathbb{R}^{n-1}}$

any metric on N scaled suff. down gives neg. curvature

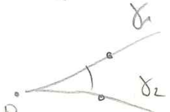
Triangle comparison:



Comparison angle $\tilde{\angle}_p(x_1, x_2)$

$\rightsquigarrow$ monotonicity $\angle_p^n(y_1, y_2) \geq \angle_p^n(x_1, x_2)$

$\rightsquigarrow$ def of bd at inf



(need mon.

$\angle_p^\infty(\gamma_1, \gamma_2) \rightsquigarrow$ metric

$\partial\mathbb{H}^n$

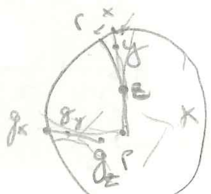
inf paths $\perp$ to tangent.

Ex: $\angle_p^\infty(\gamma_1, \gamma_2) \sim e^{-R(\gamma_1, \gamma_2)}$

$R(\gamma_1, \gamma_2) \simeq (\gamma_1 | \gamma_2)$

Gromov boundary: properties of visual metrics

- ∂X is approx. self-similar



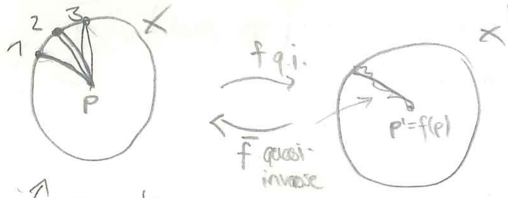
use approx by trees

g - isometry

ball $\hat{=}$ branch off after certain point

$\Rightarrow$ exercise!

- ∂X is Ahlfors Q-regular
- QI $\Rightarrow$ QS at boundary



"basepoint preserving")

$\rightsquigarrow$ else change q.i. with keeping bd at inf

approx by trees

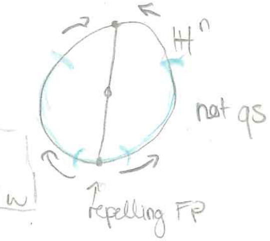
$\rightsquigarrow$ Gromov product

$\rightsquigarrow$ in bd.: add. descripency becomes mult

Morse lemma

related by linear bds since q.i.

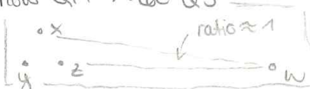
- for hyperbolic spaces: quasi-Möbius



show QM $\Rightarrow$ loc QS

Exercises:

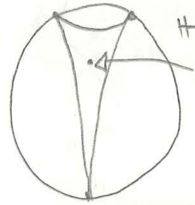
- QS $\Rightarrow$ QM $\Rightarrow$ QC (quant.)



- comp/inv. of QS/QM are QS/QM
- QS sends balls to quasiballs
- QM = QS on cpt spaces

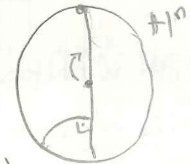
Proof QI $\Rightarrow$ QM (approx by trees)

Mostow rigidity
Statement (ii) => (iii) by passing to boundary



$\mathbb{H}^n$
center of ideal triangle
=> compact ⇝ cocompactness in (iii)

Compactness: Conv. of η-QM to const. map: ⟶



$\mathbb{H}^n$

Sketch of proof Mostow rigidity:

Step 1: $\mathcal{G}$-equiv (via left translation ⇝ from proof)
$\forall g \in \mathcal{G}: f(g_1 \cdot x) = g_2 \cdot f(x)$
↑ (first action)

Step 3: $D(\partial f)(x)$ not conf.

$T_x S^{n-1}$ $\xrightarrow{D(\partial f(x))}$ $T_{f(x)} S^{n-1}$

S^{n-1}
max stretch subspace has dimension d
(measure theory)
a.e. p is density point

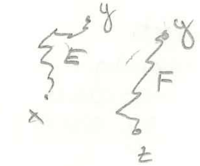
N ← not smooth at northpole *
⇝ fix N ⇝ not poss. since $\mathcal{G}$-equiv
} stereographic proj.
subbundle

⇝ subbundle structure is invariant under group action
?⇝ subbundle would have to be const.

* if pt at inf fixed ⇝ contradiction cocomp., & disc.

Sullivan, Tukia, Gromov
h via intersection of ellipses ⇝ convex set ⇝ John's ellipse

Loewner property
· relative distance: scale invariant
◦ Ferrand cross-ratio



$\boxed{QC \Rightarrow QM}$

QC map quasi-preserves Energy
⟹ Capacity
⟹ Ferrand-cross ratio
Loewner-property ⟹ cross ratio
⟹ is QM

Relation Modulus - Capacity ⇝ exercise

Ex:
· S^{2n-1} with standard Carnot / sub Riem metric
◦ Fuchsian buildings

$\mathbb{C}^n$
S^{2n-1}

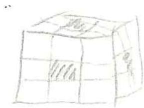
$\mathbb{H}^2$
⇝ isometrics
edge ⇝ geod ⇝ reflection ⟹ discrete subgp & tiling of plane

X 2-complex ⇝ "gluing together isometric copies of polygon"

⇝ simply connected hyperbolic building
⇝ bd at inf 1-dim

◦ Inverse limits of graphs

X_0
X_1
} limit
homeo to menger sponge
1-dim Loewner



Properties:
2) have no cut-points ⇝ LLC cond. (bdd turning?)

Sobolev map on metric measure spaces
⇝ only need norm of gradient
Lipschitz fct:

$\cdot_p \ \overset{\cdot}{x} \quad \rightsquigarrow \quad \frac{|u(x) - u(p)|}{d(x,p)}$ "norm of gradient" $\quad \int \cancel{|Du|}^n (Lu)^n \, d\mu$

Differentiability

- Thm Cheeger ⤳ like Rademacher. think of diff as section of tangent bundle

- (Z, μ) Loewner

$\{Z_i\}$ countable disjoint collection of measurable subsets

$|Z \setminus \cup Z_i| = 0$

for every i we want $\phi_i : Z \to \mathbb{R}^{n_i}$ — generally defin n_i

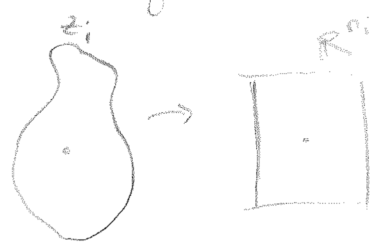
⤳ analogous: any smooth fcts can be expressed by coord fcts

For any u Lipschitz, $\forall i$

uniqueness $\exists ! \, a_{1,i}, \dots, a_{n_i,i} : Z_i \to \mathbb{R}$ L^∞

(⤳ want to say, diff of Lip. fcts span as diff of coord fcts)

For a.e. $x \in Z_i$



$\sum a_{j,i}(x) \phi_{j,i} - u$ constant to first order at x

$\Rightarrow$ def of diff. structure on metric measure space