Proof of Thm:

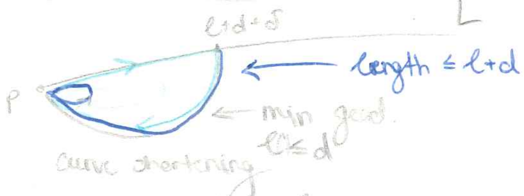
$\Omega_p M^n$ = p.w. diff curves on M^n beginning and ending at p

$H^*(\Omega_p M^n, \mathbb{Q})$ $\quad u, u^2, u^3, \dots$

⇝ concatenation of loops: produces Pontryagin products

In general not possible to control length of curves in sweep outs

Proof of observation.

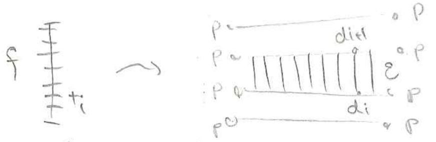


← length $\le \ell + d$ ⇒ decreased length of curve by δ

⇒ repeat

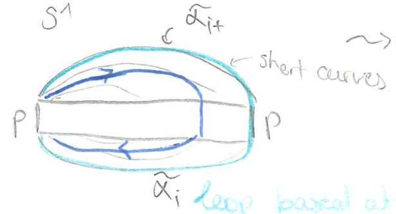
Consider $f: S^1 \to \Omega_p M^n$

construct $\tilde{f}$



$d(f(t_i), f(t_{i+1})) < \varepsilon$ for some small ε

$f(t_i) = d_i$



⇝ replace by short loops

⇒ path homotopy btw. $\tilde{\alpha}_i$ and $\tilde{\alpha}_{i+1}$ over short curves

We have $f, \hat{f}$ ⇝ construct f_t s.th. $f_0 = f$, $f_1 = \hat{f}$

⇝ generalize to higher dim

problems appl. of Farrgas thm

"critical points of distance fcts & appl. to geometry" - Cheeger

Projection complexes and applications to mapping class groups

Mladen Bestvina

Asymptotic dimension (gromov)

Motivation: (covering dim)

X cpt m.s.: $\dim(X) \le n \iff \forall \varepsilon > 0$ $\exists$ open cover $\mathcal{U}$ of X with with $\underbrace{\text{multiplicity} \le n+1}_{\text{intersection of } n+2 \text{ sets is } \emptyset}$, $\text{mesh}(\mathcal{U}) < \varepsilon$

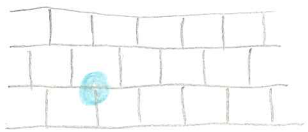
$\text{mesh}(\mathcal{U}) = \sup_{U \in \mathcal{U}} \dim U$

Def: X m.s.: asdim $X \le n \iff \forall R > 0$ $\exists$ (open) cover $\mathcal{U}$ s.t.

1) every metric R-ball intersects $\le n+1$ elts of $\mathcal{U}$

2) $\text{mesh}(\mathcal{U}) < \infty$ (uniformly bdd)

Ex: $\mathbb{R}^2$: asdim $\mathbb{R}^2 \le 2$



⇒ asdim $\mathbb{R}^n \le n$

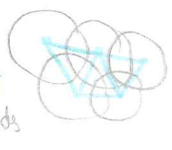
Fact: asdim $\mathbb{R}^n = n$

Suppose $\mathcal{U}$ is an open cover of $\mathbb{R}^n$ by unif bdd sets

Need to show multiplicity is $\ge n+1$

Let N = nerve of the cover

$\exists$ map $f: \mathbb{R}^n \to N$ induced $\underbrace{\text{by a part. of unity}}_{\text{gives barycentric coords of } f}$ subordinate to $\mathcal{U}$.



$\exists$ map $g: N \to \mathbb{R}^n$, $g(\overset{\substack{\text{vertex of } N\\ \uparrow}}{U}) \in \mathcal{U}$ extend linear on simplices

unif. bdd ⇝ $gf \overset{\text{bdd}}{\simeq} id$

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "f"] & N \arrow[r, "g"] & \mathbb{R}^n
\end{tikzcd}$$

$$\begin{tikzcd}
H^{\ell f}(\mathbb{R}^n) \arrow[r] \arrow[rr, bend right, "id"'] \arrow[d, phantom, "\not\parallel"] & H_n^{\ell f}(N) \arrow[r] & N_n^{\ell f}(\mathbb{R}^n) \\
0 & &
\end{tikzcd}$$

loc finite

$\Rightarrow H_n^{\ell f}(N) \neq 0$

$\Rightarrow \dim N \ge n$

If $X \overset{q.i.}{\sim} Y \Rightarrow$ asdim X = asdim Y

↳ generated by biLip homeo + cobounded inclusions

$\wr$

can replace by: bijection that sends unif bdd collections on one side to the same on the other

(coarsly equal)

Ex: Finitely generated groups have a well-defined asdim

Monotonicity: $H < \mathcal{G} \Rightarrow \operatorname{asdim} H \le \operatorname{asdim} \mathcal{G}$

Ex:
- $\operatorname{asdim}(\ell^2) = \infty$ $\quad \ell^2 > \mathbb{R}^n \ \forall n$
- Thompson's groups has asdim $= \infty$ $(> \mathbb{Z}^n \ \forall n)$
- Bdd metric spaces have asdim $= 0$.
- More generally, spaces with "archipelago structure" have asdim $= 0$:
  $\exists$ sequence of partitions of X (pairw. disj.) $P_1, P_2, \dots$ s.th.
  - mesh $P_n < \infty$
  - dist. btw. distinct islands in P_n is $\geq a_n$ for $a_n \to \infty$

Ex: $\mathbb{Q}$ with the standard norm (as $\subset \mathbb{R}$) asdim $\mathbb{Q} = 1$

$\rightsquigarrow$ p-adic norm $\|p^n \frac{a}{b}\|_p = p^{-n}$, $p \nmid a, b$.

ultra norm: $\|\alpha + \beta\| \le \max\{\|\alpha\|, \|\beta\|\}$ $\Rightarrow \operatorname{asdim} \mathbb{Q}_p = 0$

$\rightsquigarrow$ Any ultra-metric space has an archipelago structure:

R-balls are same $\qquad$ R-balls are disjoint, $\geq R$ apart

P_n = coll. of n-balls in X

Hurewicz Thm

X, Y cpt. m.s., $f: X \to Y$ map s.th. $\dim f^{-1}(y) \le n \ \forall y \in Y$

$\Rightarrow \dim X \le \dim Y + n$

Bell-Dranishnikov Hurewicz Thm

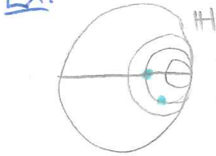
X, Y m.s., $f: X \to Y$ Lipschitz s.th. $\forall R$ the collection $\{f^{-1}(B_R) \mid B_R$ R-ball in $Y\}$ has asdim $\le n$ uniformly. Then

$\operatorname{asdim} X \le \operatorname{asdim} Y + n$

Ex: $1 \to A \to B \xrightarrow{\text{Lip}} C \to 1$ short exact seq.

$\Rightarrow \operatorname{asdim} B \le \operatorname{asdim} A + \operatorname{asdim} C$

Ex:



$\mathbb{H}^2$ Lipschitz map $\mathbb{H}^2 \to \mathbb{R}$

$\rightsquigarrow$ Horocycles give projection

$\rightsquigarrow$ preimage of image is segment btw. horocycles

Ex (Lie groups): $\mathcal{G} = KAN$, $X = {}_K\backslash^{\mathcal{G}} = \underset{\substack{\uparrow \\ (\text{"horocycles"})}}{AN} \to A$ $\rightsquigarrow \operatorname{asdim} \underset{\substack{\uparrow \\ \text{symmetric space}}}{X} \le \dim X$

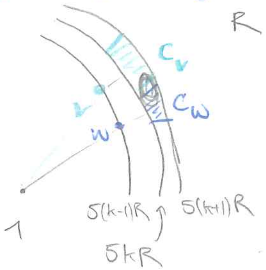
Ex: $\operatorname{asdim}(\text{tree } T) \le 1$. $h: T \to [0, \infty)$ dist. from a vertex

$\nearrow$ bd on diam will increase but sets will have uniform archipelago structure

$\rightsquigarrow \{h^{-1}[a,b] \mid b - a = R\}$

Gromov's thm

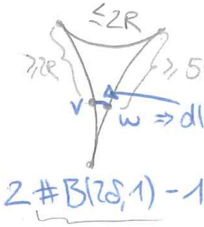
$\mathcal{G}$ word hyperbolic group $\Rightarrow \operatorname{asdim} \mathcal{G} < \infty$



R large integer

For $v \in S(5(k-1)R, 1)$ define

$C_v = \{g \in \mathcal{G} \mid \|g\| \in [5kR, 5(k+1)R]$ and $\exists$ geodesic $[1, g] \ni v\}$

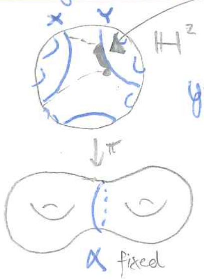


$\Rightarrow d(v,w) \le 2\delta$ $\swarrow$ both are too far away from third side

$\Rightarrow \operatorname{asdim} \mathcal{G} \le 2 \underbrace{\# B(2\delta, 1)}_{= 1 \text{ in free group: } \delta = 0} - 1$

Projection Complexes

Motivating example



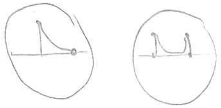
$\{\pi_Y(X) =$ nearest point proj of X to Y

$\mathcal{Y} = \{X, Y, Z, \dots\}$ coll. of lines in $\mathbb{H}^2$ that cover α

$\Rightarrow \exists\, \theta \ge 0$:

(P1) $\operatorname{diam} \pi_Y(X) \le \theta \quad \forall X \ne Y$

($\Rightarrow$ proj. distance: $d_Y(X, Z) := \operatorname{diam}(\pi_Y(X) \cup \pi_Y(Z))$)



ex $\rightarrow$

(P2) $d_Y(X,Z) > \theta$ then $d_X(Y,Z) \le \theta$ Behrstock inequality

(P3) $\forall X, Z$ the $\{Y \mid d_Y(X,Z) > \theta\}$ is finite

Projection data

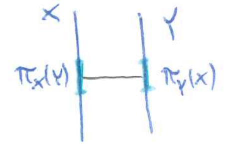
- Collection $\mathcal{Y} = \{X, Y, Z, \ldots\}$ of m.s.
- Collection of Projections $\pi_Y(X) \subseteq Y$ nonempty $\forall X \neq Y$
- Assume $\exists \theta \geq 0$ s.t. (P1)-(P3) hold

⇝ Goal: Build an ambient space $\mathbb{Y}$

Basic strategy for building $\mathbb{Y}$:

quasi-tree of metric spaces

Start with $\coprod_{X \in \mathcal{Y}} X$ ⇝ decide when to connect two of them X, Y



crushing every $X \in \mathcal{Y}$ to a point results in a graph (of — edges)

Main Theorem: This graph is QI to a tree

quasi-tree

called projection complex

Plan: Prove theorem under the stronger hypothesis

(P2 ++) If $d_Y(X, Z) > \theta$ then $\pi_X(Y) = \pi_X(Z)$

⇝ Technical part shows that one can perturb by a bdd amount projection $\pi_Y(X)$ satisfying (P1)-(P3) s.th. (P1), (P2++), (P3) hold

Construction

Choose $K \geq 2\theta$

"path between X and Z"

Define: $\mathcal{Y}_K(X, Z) = \{Y \in \mathcal{Y} \setminus \{X, Z\} \mid d_Y(X, Z) > K\} \cup \{X, Z\}$ finite by (P3)

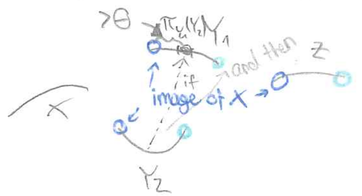
Claim: $\mathcal{Y}_K(X, Z)$ is naturally ordered.

$$\pi_{Y_j}(Y_i) = \begin{cases} \pi_{Y_j}(X), & i < j \\ \pi_{Y_j}(Z) & i > j \end{cases}$$

every space to left projects to this set

If $k = 2$, nothing to prove

$k = 3$



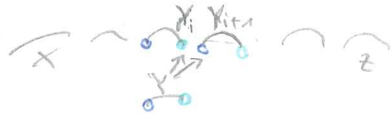
since dist btw ○ & ○ is large.
we have ← say this
$d_{Y_1}(X, Y_2) > \theta$ or $d_{Y_1}(Z_1, Y_2)$

(P2++) for $X, Y_1, Y_2 \Rightarrow \pi_{Y_2}(X) = \pi_{Y_2}(Y_1)$

(P2++) for $Y_1, Y_2, Z \Rightarrow \pi_{Y_1}(Y_2) = \pi_{Y_1}(Z)$

⇒ order: $X \ Y_1 \ Y_2 \ Z$

Continue by induction:



By the same argument applied to Y & Y_i

$\pi_{Y_i}(Y)$ is one of $\pi_{Y_i}(X)$ or $\pi_{Y_i}(Z)$

$\pi_Y(Y_i)$ is one of $\pi_Y(X)$ or $\pi_Y(Z)$

⇝ look for a place where the proj. of Y switches from right to left

(P2++) $X, Y_i, Y \Rightarrow \pi_Y(Y_i) = \pi_Y(X)$

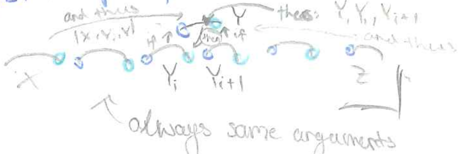
⇝ insert Y between P_i and P_{i+1}.

Def: $P_K(\mathcal{Y})$ graph. Vertices are elts of $\mathcal{Y}$.
connect X to Z by an edge ⟺ $d_Y(X, Z) \leq K \ \forall Y$

Claim: $P_K(\mathcal{Y})$ is connected, and $\mathcal{Y}_K(X, Z)$ is a path from X to Z ← called standard path

Suppose $\exists Y$, $d_Y(Y_i, Y_{i+1}) > K$

⇒ $\mathcal{Y}_K(X, Z)$ is the "standard path" from X to Z



always same arguments

Manning's Bottleneck Criterion

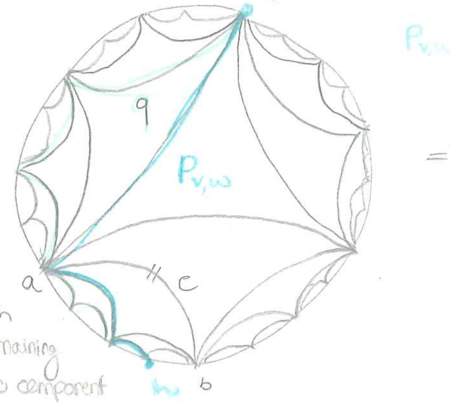
Suppose X is a connected graph, $\Delta \geq 0$ (bottleneck constant).
Assume that for any two vertices v, w $\exists$ path $P_{v,w}$ from v to w s.t. for any path q from v to w, $N_\Delta(q) \supseteq P_{v,w}$. Then X is a quasi-tree

Exercise: Show that the Farey graph is a quasi-tree
with graph metric

$P_{v,w}$ geodesic between v and w

⇒ for any other path q from v to w we have

$P_{v,w} \subseteq N_1(q)$



$P_{v,w} \subseteq N_\Delta(q)$

⇒ by Manning's bottleneck criterion the Farey graph is a quasi-tree

* note that when we delete an edge with its endpoints, the remaining graph consists of two component

Consider a vertex $a \in P_{v,w}$ and let q be any other path from v to w. Choose an edge e that separates v from w. Because of *, the path q has to go through one of its endpoints ⇒ $\text{dist}(a, q) \leq 1$

$\mathcal{Y}=\{X,Y,W\}$ - collection of metric spaces (graphs)

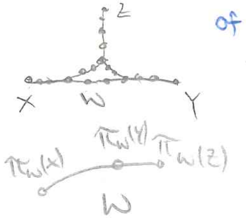
$\theta\geq 0$, $\{\underset{\text{nonempty}}{\pi_Y(X)}\subseteq Y,\ X\neq Y\}$ ⇝ $d_Y(X,Z)=\operatorname{diam}(\pi_Y(X)\cup\pi_Y(Z))$

(P1) diam $\pi_Y(X)\le\theta$

(P2) $d_Y(X,Z)>\theta\Rightarrow d_X(Y,Z)\le\theta$

(P3) $\forall X,Z$ $\{Y\mid d_Y(X,Z)>\theta\}$ finite ⇝ (P2++) $d_Y(X,Z)>\theta\Rightarrow\pi_X(Y)=\pi_X(Z)$

Ⓐ **Fact/Exercise**: $\forall X,Y,Z$: $\mathcal{Y}_K(X,Y)\cup\mathcal{Y}_K(Y,Z)$ contains all but at most two of the vertices on $\mathcal{Y}_K(X,Z)$, and if two, they are adjacent



Hint: given $W\in\mathcal{Y}_K(X,Z)$ either $\underset{①}{d_W(X,Y)>\theta}$ or $\underset{②}{d_W(Y,Z)>\theta}$

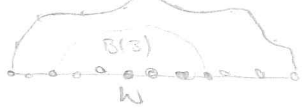
if ①: show that $\forall W'<W$ $W'\in\mathcal{Y}_K(X,Y)$ (apply P2++ to W,W',Y)

Theorem: $P_K(\mathcal{Y})$ is a quasi-tree. ← in particular hyperbolic

Proof: Use Manning's bottleneck criterion with $\Delta=3$ and standard path for $P_{X,Y}$

If P,Q are adjacent & $d(P,Q)=1$, $d(P,W)>0$, $d(Q,W)>0$

$\Rightarrow\pi_W(P)=\pi_W(Q)$



Fact/Exercise: If n = length(std. path from X to Z) subpath of std path is std path

Then $\lfloor\frac{n}{2}\rfloor\underset{-1?}{+1}\le d(X,Z)\le n$

$\Rightarrow$ std. paths are quasi-geodesics.

Hint: Induction on $d(X,Z)$ ⇝ triangles are almost tripods

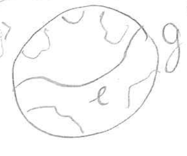
Some examples where (P1)-(P3) hold

① G hyp group, non-elementary, $g\in G$ inf. order $E(g)=\langle g\rangle$

∃ quasi-axis ℓ for g, $g(\ell)=\ell$, $g|_\ell$ translation

$\mathcal{Y}$ = set of translates of ℓ by G

want that translates of ℓ are not parallel



Assume no two distinct elts of $\mathcal{Y}$ are parallel

$\Rightarrow$ (P1)-(P3) hold, just like in the surface case

⇝ $G\curvearrowright P_K(\mathcal{Y})$

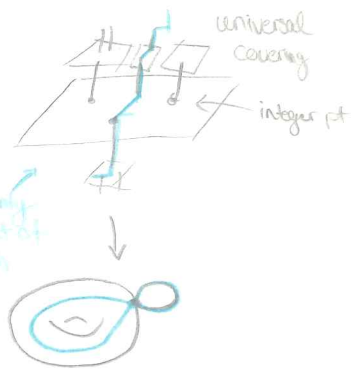
⇝ Every non-elt hyp group has a non-elt action on a quasi-tree

② $\mathbb{Z}^2*\mathbb{Z}$

$\mathcal{Y}$ = collection of points, $\theta=0$

⇝ check (P1)-(P3) exercise

$K=0$ ⇝ collapse planes



speeds only bdd amount of time in each flat

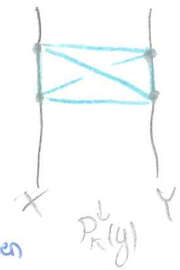
Morse geod. →
Strongly contr. geod.
Rk 1 geod

Quasi-trees of metric spaces (graphs for simplicity)

given $\mathcal{Y}$, $\pi_Y(Z)$, θ, $P_K(\mathcal{Y})$

$\mathcal{C}_K(\mathcal{Y})$ = start with disj. union $\bigsqcup_{X\in\mathcal{Y}}X$

⇝ if $d_{P_K(\mathcal{Y})}(X,Y)=1$ then attach edges from every vertex in $\pi_Y(X)$ to every vertex in $\pi_X(Y)$



Properties

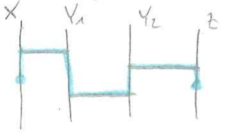
① If we assign length $2K$ to each attached edge, then

1) each $X\in\mathcal{C}_K(\mathcal{Y})$ is convex and isometrically embedded

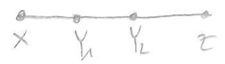
2) Nearest point projection of $X\subset\mathcal{C}_K(\mathcal{Y})$ to $Y\subset\mathcal{C}_K(\mathcal{Y})$ is within uniform distance of $\pi_Y(X)$

② If all $Y\in\mathcal{Y}$ have asdim $\le n$ uniformly then asdim $\mathcal{C}_K(\mathcal{Y})$ is $\le n+1$ (Hurewicz)

③ If all $Y\in\mathcal{Y}$ are quasi-trees with uniform Δ, then $\mathcal{C}_K(\mathcal{Y})$ is a quasi-tree



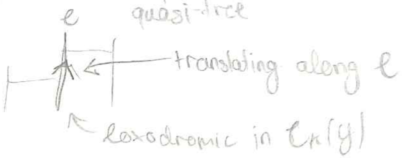
use these paths to argue that $\mathcal{C}_K(\mathcal{Y})$ is a quasi-tree



④ If all $Y\in\mathcal{Y}$ are $\underset{\text{uniform}}{\delta}$-hyperbolic then $\mathcal{C}_K(\mathcal{Y})$ is hyperbolic

Ex:

$G\curvearrowright\underset{\text{quasi-tree}}{\mathcal{C}_K(\mathcal{Y})}$



Exercises

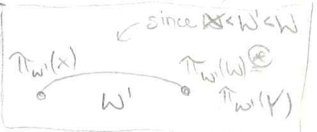
Ⓐ ① $d_W(X,Y) > \theta$

$\forall W' < W$ Show: $d_{W'}(X,Y) > K$ ($\Rightarrow W' \in \mathcal{G}_K(X,Y) \Rightarrow$ there's no more vertex in Δ)

$\hookrightarrow d_W(W',Y) > \theta \Rightarrow \boxed{\pi_{W'}(Y) = \pi_{W'}(W)}$ ⊛

$X < W' < W \Rightarrow \pi_W(W') = \pi_W(X)$

$\overset{①}{\Rightarrow} d_W(W',Y) > \theta$



Mapping class group

S - conn. closed orient. surface - finite set

$\mathrm{Mod}(S) = \mathrm{Homeo}_+(S)/\text{isotopy}$

Ex: $\mathrm{Mod}(S^2) = 1$

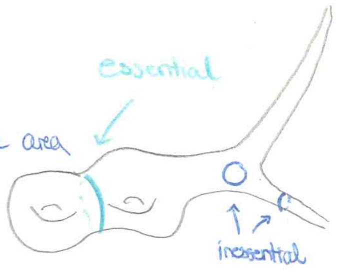
$\mathrm{Mod}(T^2) = SL_2\mathbb{Z}$ $\quad \mathbb{R}^2 \circlearrowleft \downarrow T^2 = \mathbb{R}^2/\mathbb{Z}^2$

Curve complex and graph

When $\chi(S) < 0$, S has a complete hyp metric of finite area

α simple closed curve in S is **essential** if it does not bound D^2 or $D^2 \setminus pt$.

$\Leftrightarrow$ homotopic (isotopic) to a geod. scc

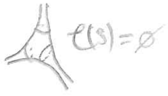


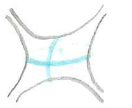
Curve complex: $\mathcal{C}(S)$

Vertices = isotopy classes of essential scc's

Simplices = coll. of ess scc's disjoint up to isotopy ($\Leftrightarrow$ disj. geod. scc's)

Sporadic surfaces



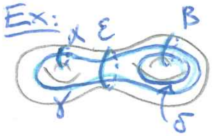


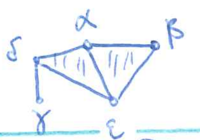
$\rightarrow \mathcal{C}(S)$ is Farey graph

$\rightarrow \mathcal{A}(S)$ filled in Farey graph

modify & replace "disjoint" by "intersect minimally"

Ex:





$\mathcal{C}(S)$ always connected & has infinite diameter

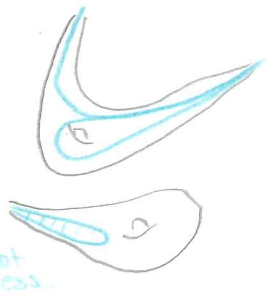
Thm (Masur-Minsky): $\mathcal{C}(S)$ is δ-hyperbolic

The Arc complex

Assume S has at least one puncture

An **arc** in S is either embedded, or has same endpoints and is otherwise embedded.

⤳ **essential**: endpts are distinct, or is same doesn't bound a disc



Arc complex: $\mathcal{A}(S)$

Vertices = isotopy classes of ess. arcs ($\Leftrightarrow$ geodesic arcs)

Simplices = disjointness up to isotopy

⤳ contractible

Thm: (Masur-Schleimer) $\mathcal{A}(S)$ is δ-hyperbolic

Hensel-Przytycki-Webb $\mathcal{A}$, $\mathcal{C}$ } new proofs
P. - Sisto

Guessing Geodesics Lemma

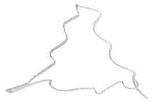
X graph

Assume: $\exists D \geq 0$, $\forall$ two vertices $x, y \in X$ we have a subgraph $P(x,y)$ s.t.

(1) $P(x,y)$ is connected and contain x, y

(2) If $d(x,y) = 1$ then $\mathrm{diam}\, P(x,y) \leq D$.

(3) $\forall x,y,z$ $N_D(P(x,y) \cup P(y,z)) \supseteq P(x,z)$



Then X is hyperbolic (δ-dep only on D ⤳ same δ for any surface!)

Proof that $\mathcal{A}(S)$ is hyperbolic

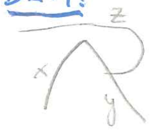
$X = \mathcal{A}(S)^{(1)}$. Define $P(x,y)$, where x,y are arcs ⤳ orient them (wlog geod.)

A **unicorn arc** is the union of an initial segment of x and a terminal segment of y

$P(x,y)$ = set of all union arcs w.r.t. all orientations and orders on $\{x,y\}$

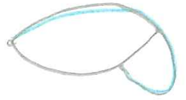
connectivity enlarge first until arc is just first

D = 1:



← this is a unicorn on $P(z,y)$ disjoint from given unicorn

⤳ for $\mathcal{C}(S)$: bicorn (issues with punctures)

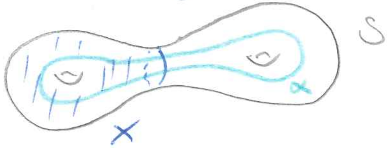


Subsurface projections

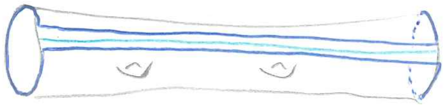
S as before

$X \subset S$ subsurface (conn, π_1-injective, not pair of pants)

There is a partially defined coarse map $\mathcal{C}(S) \to \mathcal{C}(X)$



- If $\alpha \subset X$: projection is $\{\alpha\}$
- If $\alpha \cap X$ is a collection of arcs, close them up along ∂X to get the projection



More generally, if $X, Y \subset S$ are subsurfaces, write $X \pitchfork Y$ if

$$X \cap \partial Y \neq \emptyset \quad \text{and} \quad \partial X \cap Y \neq \emptyset.$$

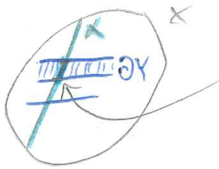
Define $\pi_X(Y) := \pi_X(\partial Y) \subset \mathcal{C}(X)$ unif. bdd subset

Thm:

(1) Let $X, Y \subset S$ subsurfaces, $X \pitchfork Y$, α scc.
If $d_X(\alpha, Y) > 50 \Rightarrow d_Y(\alpha, X) < 50$ (Behrstock ineq)

(2) If α, β are two scc's in S then
$\{Z \mid d_Z(\alpha, \beta) > 50\}$ is finite (exercise)

Proof of (1) intersection number bounds distance $\Rightarrow$ ∂Y & α intersect a lot



arc of intersection b/w α & Y

arc of α disj from ∂X

$\Rightarrow d_Y(\alpha, X)$ small