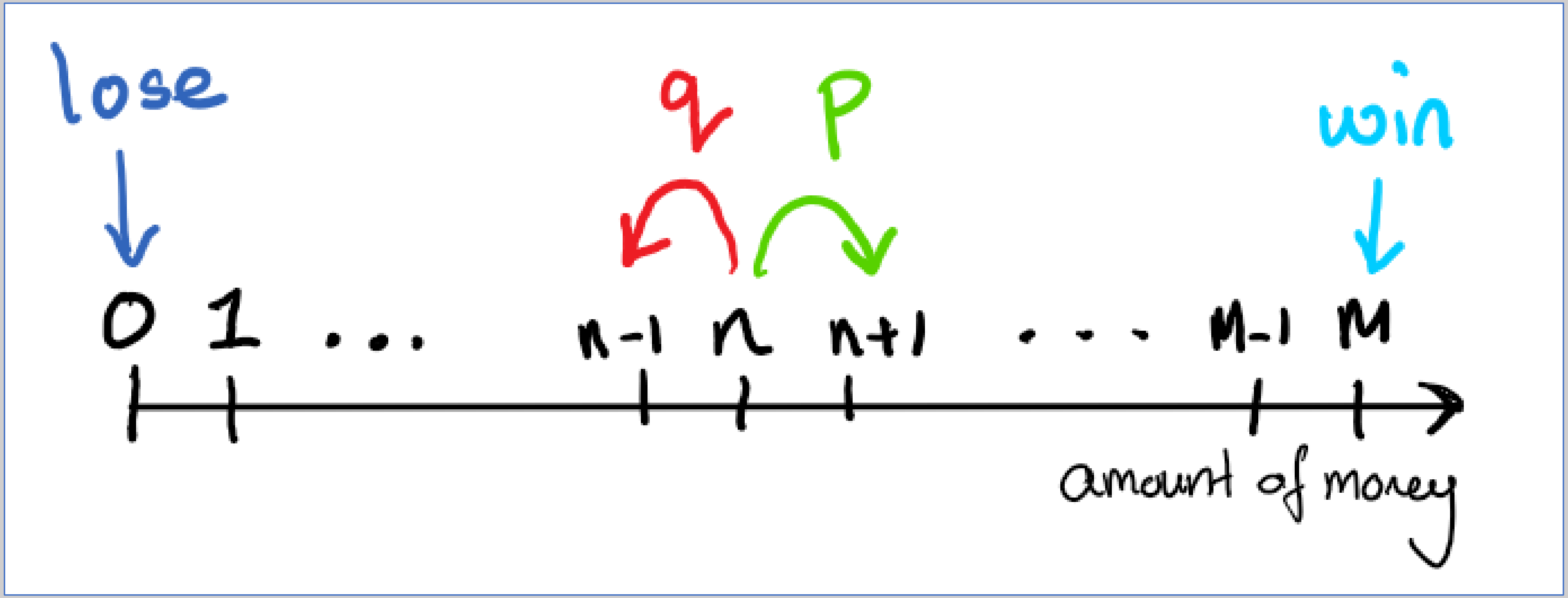
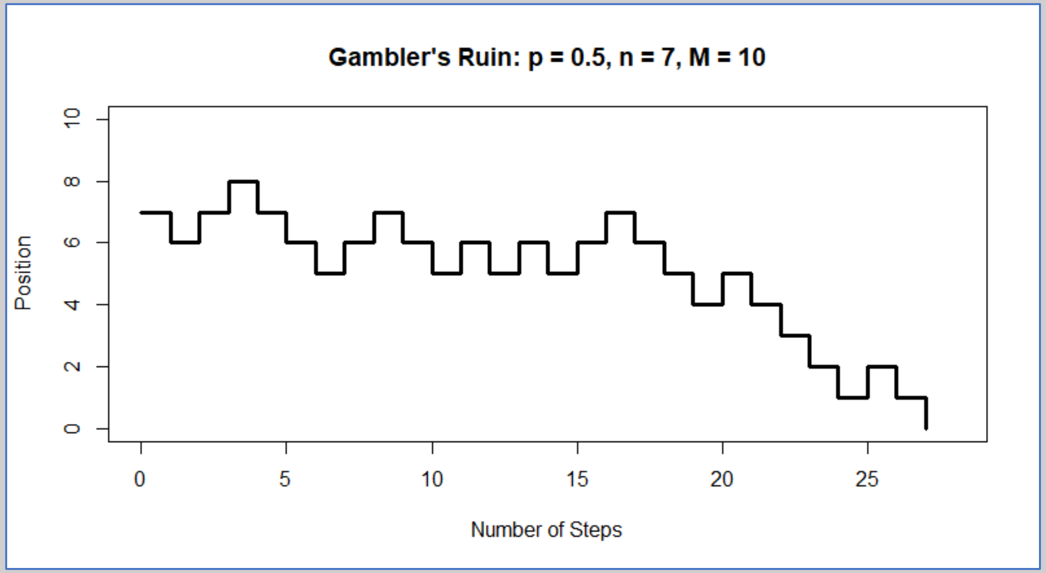
First Year Probability

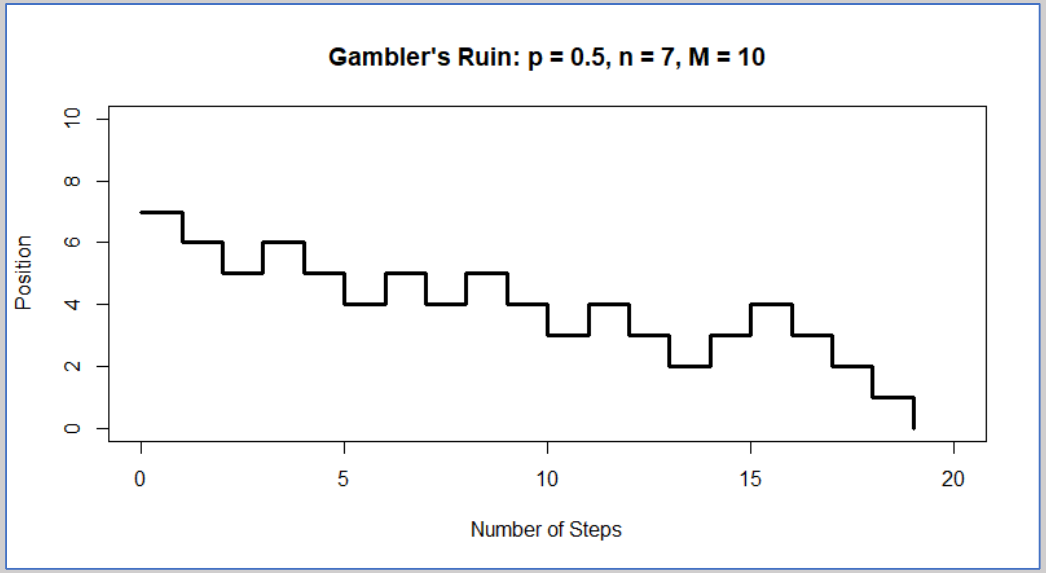
Gambler’s Ruin and Random Walks

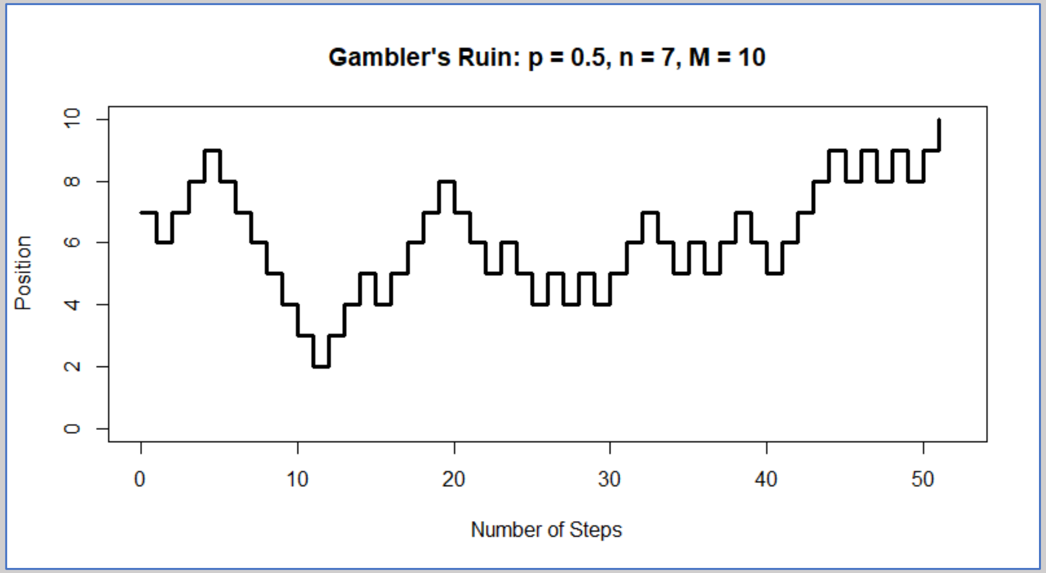
Gambler's Ruin: set up

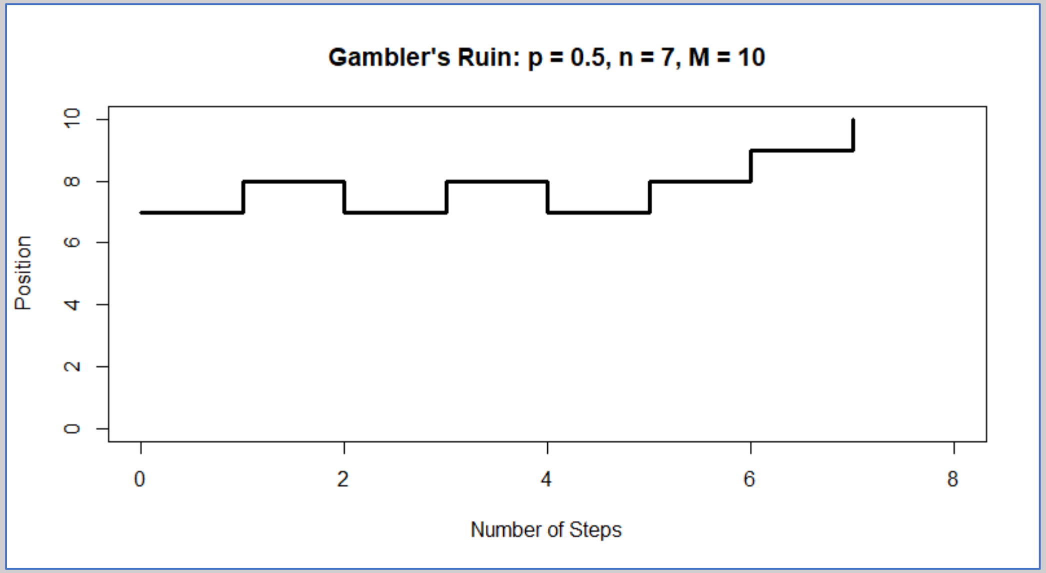


QUESTION: how likely are we to run out of money before we win?









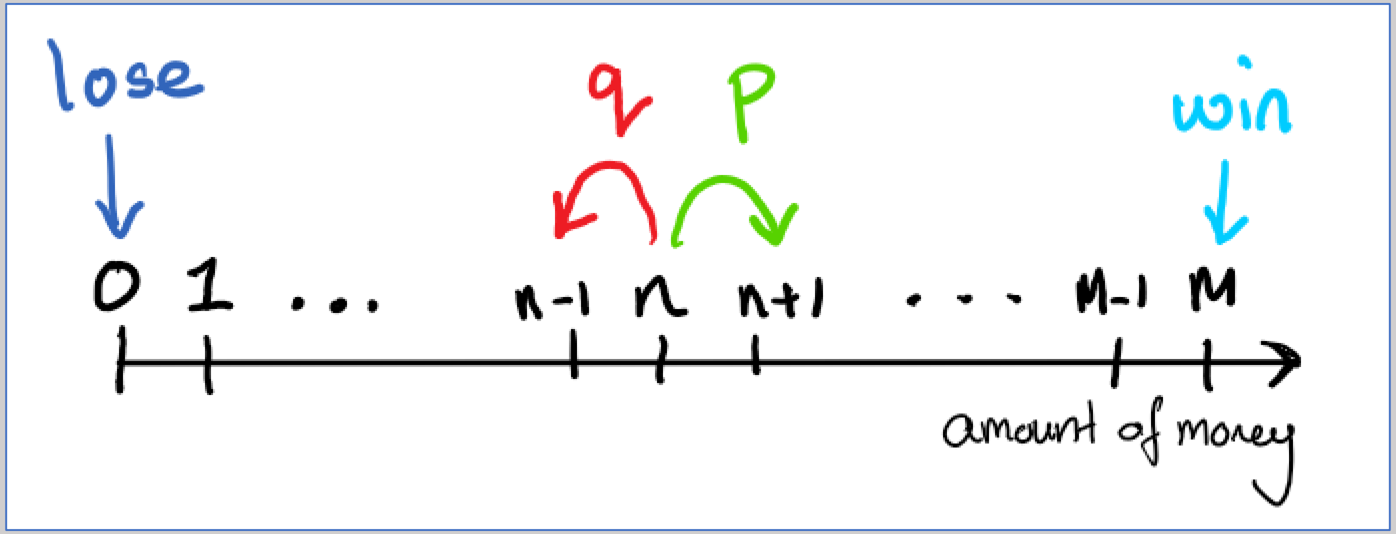
Formulating equations

Define: $u_n = \mathbb{P}(\text{reach £0 before £M} \mid \text{started with £n})$

We have the boundary conditions: $u_0 = \mathbb{P}(\text{reach £0 before £M} \mid \text{started with £0}) = 1$

$u_M = \mathbb{P}(\text{reach £0 before £M} \mid \text{started with £M}) = 0$

We condition on the first step: $u_n = qu_{n-1} + pu_{n+1}$ (for all n such that 1 ≤ n ≤ M-1)



Solving: symmetric case

Suppose we have "fair" bets so:

$$p = q = 1/2$$

Our equation becomes:

$$u_n = (u_{n-1} + u_{n+1})/2$$

(for $1 \leq n \leq M-1$)

We have boundary conditions:

$$u_0 = 1 \;\&\; u_M = 0$$

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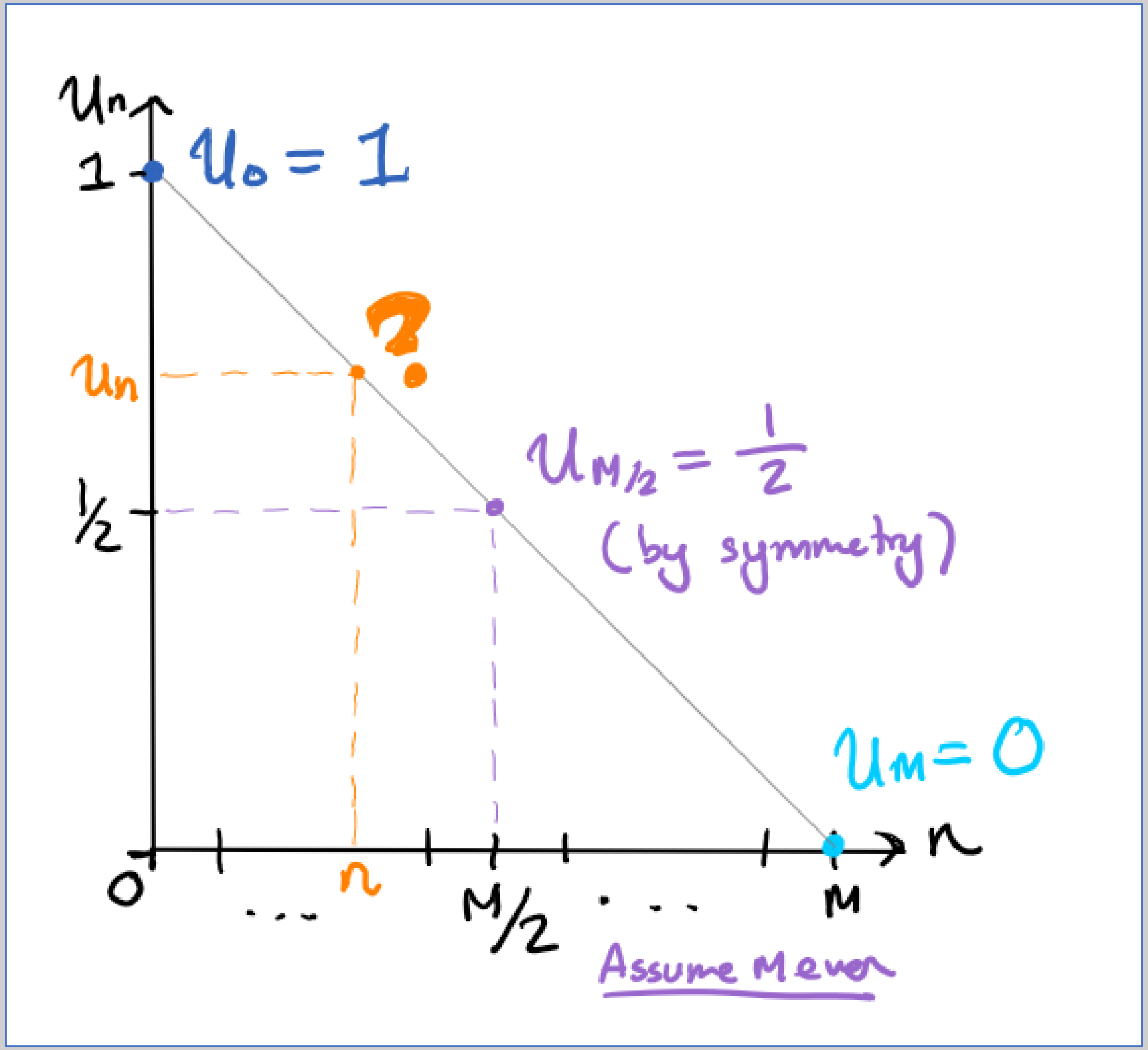
We suspect that the solution may be:

$$u_n = 1 - \frac{n}{M}$$

We can check that this satisfies our equation and the boundary conditions.



Solving the equations (harder!)

We want to solve (for all n such that 1 ≤ n ≤ M-1): $u_n = qu_{n-1} + pu_{n+1}$ (1)

We look for solutions of the form $u_n = \lambda^n$ for some λ that we need to find.

This gives the equation: $\lambda^n = q\lambda^{n-1} + p\lambda^{n+1}$ (2)

Dividing through by λ^{n-1} (assuming $\lambda \neq 0$) gives: $p\lambda^2 - \lambda + q = 0$ (3)

$$p\lambda^2 - (p+q)\lambda + q = 0 \quad (4)$$

$$(\lambda - 1)(p\lambda - q) = 0 \quad (5)$$

So $\lambda = 1$ or $\lambda = \frac{q}{p}$. We assume (for now that) $\frac{q}{p} \neq 1$.

This gives the general solution: $u_n = A \times 1^n + B \times \left(\frac{q}{p}\right)^n = A + B\left(\frac{q}{p}\right)^n$ (6)

A and B are constants that we need to find.

We use the boundary conditions $u_0 = 1$ & $u_M = 0$ to give: $u_n = \frac{\left(\frac{q}{p}\right)^n - \left(\frac{q}{p}\right)^M}{1 - \left(\frac{q}{p}\right)^M}$ (7)

CHALLENGE: use (6) and the boundary conditions to find (7).

Random walks

Other random walks include:

- A frog jumping left or right on a lie of lily pads at random
- A molecule moving randomly in 1D, 2D or 3D space
- The population of a country

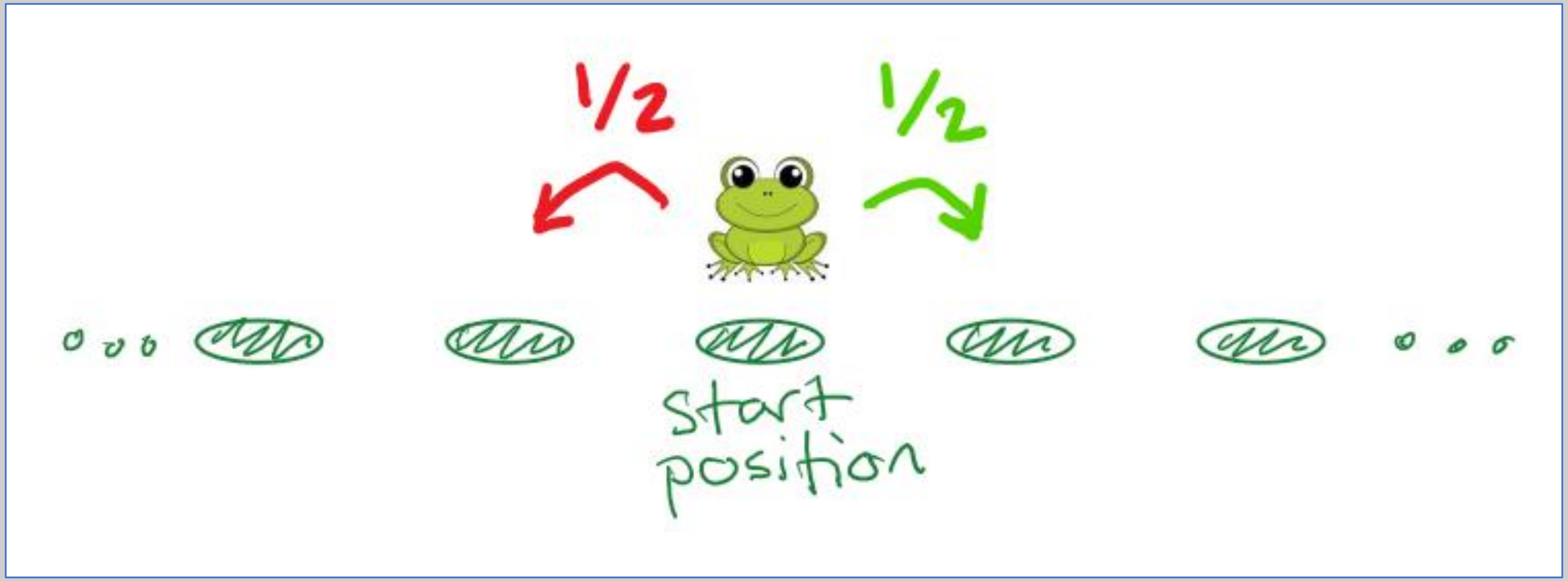
Our total amount of money at time t in the Gambler’s ruin is an example of a Markov chain.

Random walks: the frog

Suppose a frog jumps left or right (with equal probability) each minute on a infinite line of lily pads. The frog (or perhaps its descendants) continue forever.

QUESTION: will the frog ever return to where it started?

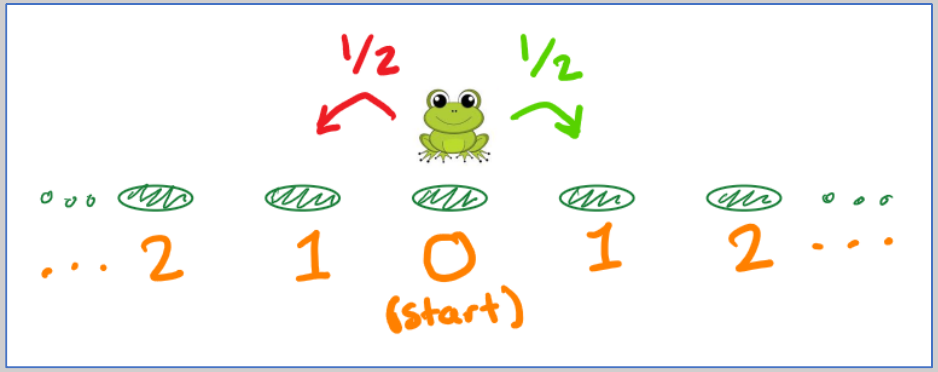
[Hint: try and equate this to the Gambler’s Ruin.]



Random walks: the frog again

QUESTION: will the frog ever return to where it started?

Trick: number the lily pads as shown:



Now wherever the frog jumps first it lands on a lily pad labelled 1.

We want to find $\mathbb{P}(reaches\ 0|starts\ at\ 1)$. This looks a bit like u_1… but what is M?

We don't mind how far the frog goes before it returns so want M very large.

Recall $u_n = 1 - \frac{n}{M}$ so $u_1 = 1 - \frac{1}{M}$

We take M arbitrarily large ie: $M \to \infty$ which gives $u_1 = 1 - \frac{1}{M} \to 1 - 0 = 1$

So $\mathbb{P}(reaches\ 0|starts\ at\ 1) = 1$ meaning the frog will always return eventually.

CHALLENGE: does it matter that there are two lily pads labelled 1? Why?

Linear recurrence equations (harder!)

The equation $u_n = qu_{n-1} + pu_{n+1}$ is called a “linear recurrence equation”.
Here is another (quite cool) application for solving linear recurrence equations.

The Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, . . .
Defined by the linear recurrence equation: $f_{n+2} = f_{n+1} + f_n$ (for n≥0)
With initial conditions: $f_0 = 1 = f_1$

By seeking solutions of the form $f_n = \lambda^n$ in the same way as we did for u_n we get:

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

(amazingly, this does give integers!)

CHALLENGE: find out about the golden ratio and try and link it to this expression.
EXTRA CHALLENGE: can you derive this expression for f_n?

Any Questions?