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Regularity of minimizers of variational integrals with wide range of anisotropy

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Abstract

In this paper we characterize the minimizers of integral functionals of the form

$$\mathfrak{F}(v,\Omega)=\int_{\Omega}F(Dv(x))\,\mathrm{d}x,$$

where the integrands F are autonomous and suitably convex but not subjected to any growth conditions from above and defined for mappings $v\colon\Omega\subset\mathbb{R}^n\to\mathbb{R}^N$ in the Sobolev class $\mathrm{W}^{1,1}$, satisfying a Dirichlet boundary condition. We establish, under a natural regularity assumption on the boundary datum, that the minimizers of $\mathfrak{F}$ are precisely the energy solutions to the Euler–Lagrange system for $\mathfrak{F}$. More precisely it is shown that u is minimizing precisely when $F'(Du)$ is integrable, row–wise solenoidal and $F'(Du)\cdot Du$ is integrable. As an application, we also deduce a higher differentiability result for the minimizers in a special case.

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1 Introduction and Statement of Results

We consider convex variational integrals of the form

$$\mathfrak{F}(v,\Omega)=\int_{\Omega}F(Dv(x))\,\mathrm{d}x,\tag{1.1}$$

that are defined for $\mathrm{W}^{1,1}$ Sobolev maps $v\colon\Omega\subset\mathbb{R}^n\to\mathbb{R}^N$. The integrand $F\colon\mathbb{R}^{N\times n}\to\mathbb{R}$ is continuous and convex, and Ω is an open subset of $\mathbb{R}^n$ and we are mainly interested in the multi–dimensional vectorial case $n,\,N\geq 2$.

The integral in (1.1) is understood in the usual sense of Lebesgue integration. In fact, as is well–known, for autonomous convex integrands F the pointwise definition (1.1) coincides with Lebesgue–Serrin type definitions of $\mathfrak{F}(v, \Omega)$ also for non–smooth maps v, and there is no instance of the Lavrentiev phenomenon to complicate matters.
The precise meanings of minimizer and energy–extremal are important for us here and we start by displaying their respective definitions. Let $g \in \mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)$ be the boundary datum (further conditions on g will follow in due course).

Definition 1. *A mapping $u \in \mathrm{W}^{1,1}_g(\Omega, \mathbb{R}^N)$ is a minimizer if $F(Du) \in \mathrm{L}^1(\Omega)$ and*

$$\int_\Omega F(Du) \le \int_\Omega F(Dv)$$

for any $v \in \mathrm{W}^{1,1}_g(\Omega, \mathbb{R}^N)$.

Definition 2. *A mapping $u \in \mathrm{W}^{1,1}_g(\Omega, \mathbb{R}^N)$ is an extremal if $F'(Du) \in \mathrm{L}^1(\Omega)$ is row–wise solenoidal, i.e.*

$$\int_\Omega \langle F'(Du), D\varphi \rangle = 0\,,$$

for any $\varphi \in \mathrm{C}^\infty_c(\Omega, \mathbb{R}^N)$. It is an energy–extremal if in addition $F'(Du) \cdot Du \in \mathrm{L}^1(\Omega)$.

In case the integrand F is a C^1 function satisfying the so-called (p, q)-growth conditions

$$|\xi|^p \le F(x, \xi) \le C(1 + |\xi|^q) \qquad 1 < p < q,$$

then a standard argument shows that minimality of a $\mathrm{W}^{1,p}$ map u implies its extremality too provided the growth exponents are related by

$$p \le q < p + 1. \tag{1.2}$$

In fact, under the assumption (1.2) the field $F'(Du)$ is integrable, actually $p/(q-1)$–integrable, and is row–wise solenoidal.
In [4] we were able to remove the growth assumption from above. In fact, we showed that if F is a convex integrand, the minimality implies the extremality of a $\mathrm{W}^{1,p}_g$ map, just imposing a p–convexity condition

$$F(\xi) - c|\xi|^p \qquad \text{is convex}$$

where $c > 0$ and $p > 1$, and a suitable regularity for the boundary datum.

The aim of this paper is to extend the result of [4] (i.e. minimality implies the extremality) to the case of integrands that satisfy a slightly weaker convexity assumption and to show that minimizers are in fact energy–extremals. In turn this is easily seen to be a characterization of minimality when, as here, the integral functionals are convex. In order to state our result, we shall briefly introduce and discuss our hypotheses.
Let $\phi\colon \mathbb{R}^{N\times n} \to [0,+\infty)$ be a C^1 convex and radial function, that is,

$$\phi(\xi) = \theta(|\xi|),$$

for a convex function $\theta\colon [0,+\infty) \to [0,+\infty)$. Suppose that ϕ is strictly monotone, in the sense that

$$\langle \phi'(\xi) - \phi'(\eta), \xi - \eta\rangle \geq |\xi - \eta| \tag{H0}$$

holds for all $\xi, \eta \in \mathbb{R}^{N\times n}$.
Our main assumption on the integrand is that

$$\xi \mapsto F(\xi) - \theta(|\xi|) \qquad \text{is convex.} \tag{H1}$$

Assumption (H1) is a uniform strong convexity condition for the function F. In fact, since F is a C^1 function, the assumption (H1) is easily seen to be equivalent to the following standard strong monotonicity condition

$$\langle F'(\xi) - F'(\eta), \xi - \eta\rangle \geq \langle \phi'(\xi) - \phi'(\eta), \xi - \eta\rangle \tag{H2}$$

for all $\xi, \eta \in \mathbb{R}^{N\times n}$.
Note that the assumptions (H0) and (H1) are satisfied by a wide class of functionals, from those with almost linear growth to the ones with exponential growth.
The assumption (H1) clearly entails the following growth condition from below

$$\theta(|\xi|)/c - c \leq F(\xi) \tag{1.3}$$

for all $\xi \in \mathbb{R}^{N\times n}$, and a suitable positive constant c. We remark that, in what follows, the requirement that minimizers u must satisfy $F(Du) \in \mathrm{L}^1(\Omega)$ will be crucial.
It is a routine matter to check that $\mathfrak{F}(v,\Omega) = \int_\Omega F(Dv)$, under the assumptions (H0) and (H1), is a lower semicontinuous and proper functional on $\mathrm{W}^{1,1}(\Omega,\mathbb{R}^N)$. Hence, for a given $g \in \mathrm{W}^{1,1}(\Omega,\mathbb{R}^N)$ with $\mathfrak{F}(g,\Omega) < \infty$, the existence and uniqueness of a minimizer u in the Dirichlet class $\mathrm{W}^{1,1}_g(\Omega,\mathbb{R}^N)$ is evident.
The main result of this paper is the following:

Theorem 1. *Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a C^1 function satisfying (H0)–(H1), and let $g \in \mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)$ with $F(2Dg) \in \mathrm{L}^1(\Omega)$. Then the unique minimizer u in $\mathrm{W}^{1,1}_g(\Omega, \mathbb{R}^N)$ is characterized by*

$$F^*(F'(Du)) \in \mathrm{L}^1(\Omega),\ F'(Du)\cdot Du \in \mathrm{L}^1(\Omega) \text{ and}$$

$$F'(Du) \ \text{ is row–wise solenoidal},$$

where F^* denotes the polar of F.

As in our previous paper [4], the main idea in the proof of this theorem is to approximate the integrand F by more regular ones and make substantial use of the dual problems for the corresponding regularized problems. Namely we shall approximate F by strictly convex, uniformly elliptic and Lipschitz continuous functions F_k, whose minimizers u_k strongly converge to the minimizer u in $\mathrm{W}^{1,1}$. To every minimizer u_k of such more regular problems, according to the duality theory of Ekeland and Temam, we can associate a row-wise solenoidal matrix field denoted by σ_k. For the pair (Du_k, σ_k) we shall establish suitable estimates, that are preserved in passing to the limit (compare with [4] and also [5], [6] for a related approach in the case of degenerate convex problems of standard growth).
The study of the regularity properties of minimizers of integral functionals at (1.1) often comes through their extremality. In particular we recall that the case of integrands F with (p, q) growth conditions has been widely investigated in the literature. From very early on it has been clear that little regularity can be expected if the coercitivity and growth exponents, denoted p and q, respectively, are too far apart (see [14, 10]). On the other hand, many regularity results are available if the ratio q/p is bounded above by a suitable constant depending on the dimension n, and converging to 1 when n tends to infinity (incl. [1, 2, 3, 8, 9, 16, 18]). Note that here we do not impose any growth condition from above and we only suppose a very weak condition from below. Nevertheless, having at our disposal Theorem 1, we are able to establish the following higher differentiability result, in the scale of Besov spaces, for functionals depending on the modulus of the gradient.

Theorem 2. *Let $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a C^1 function and assume (H0)–(H1). Suppose moreover that there exists a Young function Φ such that*

$$F(\xi) = \Phi(|\xi|), \tag{1.4}$$

for every $\xi \in \mathbb{R}^{N\times n}$. For $g \in \mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)$ with $F(2Dg) \in \mathrm{L}^1(\Omega)$, let $u \in \mathrm{W}^{1,1}_g(\Omega, \mathbb{R}^N)$ denote the unique minimizer. We then have that

$$Du \in B^{\alpha,2}_{\infty} \qquad \text{locally},$$

for every $\alpha \in (0, 1/2)$.

2 Preliminaries

In this paper we follow the usual convention and denote by c a general constant that may vary on different occasions, even within the same line of estimates. Relevant dependencies on parameters and special constants will be suitably emphasized using parentheses or subscripts. All the norms we use on $\mathbb{R}^n$, $\mathbb{R}^N$ and $\mathbb{R}^{N\times n}$ will be the standard euclidean ones and denoted by $|\cdot|$ in all cases. In particular, for matrices ξ, $\eta \in \mathbb{R}^{N\times n}$ we write $\langle \xi, \eta \rangle := \mathrm{trace}(\xi^T \eta)$ for the usual inner product of ξ and η, and $|\xi| := \langle \xi, \xi \rangle^{\frac{1}{2}}$ for the corresponding euclidean norm. When $a \in \mathbb{R}^N$ and $b \in \mathbb{R}^n$ we write $a \otimes b \in \mathbb{R}^{N\times n}$ for the tensor product defined as the matrix that has the element $a_r b_s$ in its r-th row and s-th column. Observe that $(a \otimes b)x = (b \cdot x)a$ for $x \in \mathbb{R}^n$, and $|a \otimes b| = |a||b|$.
For a C^1 function $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ we shall write

$$F'(\xi)[\eta] := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F(\xi + t\eta)$$

for ξ, $\eta \in \mathbb{R}^{N\times n}$. Hereby we think of $F'(\xi)$ both as an $N \times n$ matrix and as the corresponding linear form on $\mathbb{R}^{N\times n}$, though $|F'(\xi)|$ will always denote the euclidean norm of the matrix $F'(\xi)$.
We shall require some further elementary notions from convex analysis, all of which are discussed in the scalar case $N = 1$ in [7]. However, as we shall briefly recall below, the relevant parts easily extend to the vectorial case $N > 1$ too. Given $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$, its polar (or Fenchel conjugate) integrand is defined by

$$F^*(\zeta) := \sup_{\xi \in \mathbb{R}^{N\times n}} \left(\langle \zeta, \xi \rangle - F(\xi) \right), \quad \zeta \in \mathbb{R}^{N\times n}, \tag{2.1}$$

which is convex. One can check that the bipolar integrand $F^{**} := (F^*)^*$ equals F at ξ if and only if F is lower semicontinuous and convex at ξ, and more generally, that it is the convex envelope of F. In particular, $F^{**} = F$ precisely when F is convex and lower semicontinuous (the latter being a consequence of the former when, as here, F is real–valued).
The definition of polar integrand means that we have the Young–type inequality

$$\langle \zeta, \xi \rangle \leq F^*(\zeta) + F^{**}(\xi), \tag{2.2}$$

for all ζ, $\xi \in \mathbb{R}^{N\times n}$. Notice that for a given ξ we have equality in (2.2) precisely for $\zeta \in \partial F^{**}(\xi)$, the subgradient for F^{**} at ξ. Furthermore, we record that F is strictly convex precisely when F^* is C^1, and that in this case we also have

$$(F^*)'(F'(\xi)) = \xi, \tag{2.3}$$

for all $\xi \in \mathbb{R}^{N\times n}$.
In order to establish the higher differentiability result for the minimizers, we shall use the difference quotient method. To this aim, we recall the following

Definition 3. *For every vector valued function $w\colon \mathbb{R}^n \to \mathbb{R}^N$ the finite difference operator is defined by*

$$\tau_{s,h}w(x) = w(x + he_s) - w(x)$$

where $h \in \mathbb{R}$, e_s is the unit vector in the x_s direction and $s \in \{1, \dots, n\}$.
The difference quotient is defined for $h \in \mathbb{R} \setminus \{0\}$ as

$$\Delta_{s,h}w(x) = \frac{\tau_{s,h}w(x)}{h}.$$

The following proposition describes some elementary properties of the finite difference operator and can be found, for example, in [11].

Proposition 1. *Let f and g be two functions such that $f, g \in \mathrm{W}^{1,p}(\Omega)$, with $p \geq 1$, and let us consider the set*

$$\Omega_{|h|} := \{x \in \Omega : dist(x, \partial\Omega) > |h|\}.$$

Then

$(d1)$ $\tau_{s,h}f \in \mathrm{W}^{1,p}(\Omega)$ *and*

$$D_i(\tau_{s,h}f) = \tau_{s,h}(D_i f).$$

$(d2)$ *If at least one of the functions f or g has support contained in $\Omega_{|h|}$ then*

$$\int_\Omega f\, \tau_{s,h}g \,\mathrm{d}x = -\int_\Omega g\, \tau_{s,-h}f \,\mathrm{d}x.$$

$(d3)$ *We have*

$$\tau_{s,h}(fg)(x) = f(x + he_s)\tau_{s,h}g(x) + g(x)\tau_{s,h}f(x).$$

We also recall fundamental embedding properties for fractional Sobolev spaces (for the proof see, for example, [19]).

Lemma 1. *If* $w\colon \mathbb{R}^n \to \mathbb{R}^N$, $w \in \mathrm{L}^p(B_R)$ *with* $1 < p < n$ *and for some* $\rho \in (0,R)$, $\beta \in (0,1]$, $M > 0$,

$$\sum_{s=1}^{n} \int_{B_\rho} |\tau_{s,h} w(x)|^p \,\mathrm{d}x \le M^p |h|^{p\beta} \tag{2.4}$$

for every h *with* $|h| < \frac{R-\rho}{2}$, *then* $w \in \mathrm{W}^{k,p}(B_\rho;\mathbb{R}^N) \cap \mathrm{L}^{\frac{np}{n-kp}}(B_\rho;\mathbb{R}^N)$ *for every* $k \in (0,\beta)$ *and*

$$\|w\|_{\mathrm{L}^{\frac{np}{n-kp}}(B_\rho)} \le c\left(M + \|w\|_{\mathrm{L}^p(B_R)}\right),$$

with $c \equiv c(n,N,R,\rho,\beta,k)$.

The assumption (2.4) in previous Lemma can be given equivalently by using the notion of Besov space. Therefore, we recall the following

Definition 4. *Let* $A \subset \mathbb{R}^n$ *be an open set,* $k \in \mathbb{N}$, $\alpha \in (0,1)$ *and* $q \in [1,\infty)$. *For a mapping* $w \in \mathrm{L}^q_{\mathrm{loc}}(A,\mathbb{R}^k)$ *we say that* w *is locally in* $\mathrm{B}^{\alpha,q}_\infty$ *on* A *provided for each ball* $B \Subset A$ *there exist* $d \in (0, \mathrm{dist}(B,\partial A))$, $M > 0$ *such that*

$$\int_B |\tau_{s,h} w(x)|^q \,\mathrm{d}x \le M|h|^{\alpha q}$$

for every $s \in \{1,\dots,n\}$ *and* $h \in \mathbb{R}$ *satisfying* $|h| \le d$.

The embedding properties of Besov spaces, that are a reformulation of Lemma 1, are contained in the following

Theorem 3. *On any domain* $\Omega \subset \mathbb{R}^n$ *we have the continuous embeddings:*

(i) $\mathrm{B}^{\alpha,q}_\infty \hookrightarrow \mathrm{L}^r_{\mathrm{loc}}$ *for all* $r < \frac{nq}{n-\alpha q}$ *provided* $\alpha \in (0,1)$, $q > 1$ *and* $\alpha q < n$;

(ii) $\mathrm{W}^{1,p}_{\mathrm{loc}} \hookrightarrow \mathrm{B}^{\alpha,q}_\infty$ *provided* $\alpha = 1 - n(\frac{1}{p} - \frac{1}{q})$, *where* $1 < p \le q < \infty$.

We refer to sections 30–32 in [19] for a proof of this theorem. In fact, the above statements follow by localizing the corresponding results proved for functions defined on $\mathbb{R}^n$ in [19], by simply using a smooth cut–off function.
To simplify the notations, we shall use the following auxiliary function defined for $\xi \in \mathbb{R}^k$

$$V(\xi) = (1 + |\xi|^2)^{\frac{\beta-2}{4}}\xi,$$

for any exponent $\beta \ge 1$. Next result, proved in [1] (Lemma 2.2), describes one of the fundamental properties of the function V.

Lemma 2. *For every $\gamma \in (-1/2, 0)$ and $\mu \geq 0$ we have*

$$(2\gamma+1)|\xi-\eta| \leq \frac{|(\mu^2+|\xi|^2)^\gamma \xi - (\mu^2+|\eta|^2)^\gamma \eta|}{(\mu^2+|\xi|^2+|\eta|^2)^\gamma} \leq \frac{c(k)}{2\gamma+1}|\xi-\eta|$$

for every $\xi, \eta \in \mathbb{R}^k$.

3 Proof of Theorem 1

This section is devoted to the proof of our main result. We start by constructing a class of auxiliary problems whose solutions u_k, on the one hand approximate the minimizer u, and on the other can be dealt with by standard means. To those solutions we associate row–wise solenoidal matrix fields σ_k and, for the pairs (Du_k, σ_k) we shall establish suitable estimates. Finally we show that these estimates are preserved in passing to the limit.

Proof of Theorem 1. Recall that the polar of F,

$$F^*(z) := \sup_{\xi \in \mathbb{R}^{N\times n}} \Big(\langle \xi, z\rangle - F(\xi) \Big)$$

is a real–valued convex function.
For each $k > 0$ define

$$\overline{F}_k^{**}(\xi) := \max_{|z|\leq k} \left(\langle \xi, z\rangle - F^*(z) \right).$$

which is a real–valued convex, globally k–Lipschitz function. Since F is lower semicontinuous and convex, we have that

$$\overline{F}_k^{**}(\xi) \nearrow F^{**}(\xi) = F(\xi) \quad \text{as} \quad k \nearrow \infty, \tag{3.1}$$

pointwise in $\xi \in \mathbb{R}^{N\times n}$.
Define

$$\widetilde{F}_k(\xi) := \max \left\{ \overline{F}_k^{**}(\xi), \theta(|\xi|) - c \right\},$$

where $\theta(t)$ is the function appearing in assumption (H1). In view of (3.1), we still have that $\widetilde{F}_k(\xi) \nearrow F(\xi)$ as $k \nearrow \infty$. Since $\overline{F}_k^{**}$ is k–Lipschitz, there exist numbers $r_k > 0$ such that $r_k \nearrow \infty$ as $k \nearrow \infty$ and

$$\widetilde{F}_k(\xi) = \theta(|\xi|) - c,$$

for $|\xi| \geq r_k - 1$. Define

$$H_k(\xi) := \begin{cases} \widetilde{F}_k(\xi) & \text{when } |\xi| \leq r_k \\ \frac{\theta(r_k)}{r_k}|\xi| - c & \text{when } |\xi| > r_k. \end{cases}$$

It is not hard to check that H_k is convex and globally m_k–Lipschitz, we may take any $m_k \geq \frac{\theta(r_k)}{r_k}$. Moreover,

$$H_k(\xi) \nearrow F^{**}(\xi) = F(\xi) \quad \text{as} \quad k \nearrow \infty, \tag{3.2}$$

pointwise in $\xi \in \mathbb{R}^{N\times n}$. Next we regularize H_k by the use of the following standard radially symmetric and smooth convolution kernel

$$\Phi(\xi) := \begin{cases} c\exp\left(\frac{1}{|\xi|^2-1}\right) & \text{for } |\xi| < 1 \\ 0 & \text{for } |\xi| \geq 1, \end{cases}$$

where the constant $c = c(n,N)$ is chosen such that $\int_{\mathbb{R}^{N\times n}}\Phi = 1$. For each $\varepsilon > 0$ we put $\Phi_\varepsilon(\xi) := \varepsilon^{-nN}\Phi(\varepsilon^{-1}\xi)$. It is routine to check that the mollified function $\Phi_\varepsilon * H_k$ (as usual defined by convolution) is convex and C^∞, and, since H_k is convex and m_k–Lipschitz, also

$$H_k(\xi) \leq (\Phi_\varepsilon * H_k)(\xi) \leq H_k(\xi) + m_k\varepsilon \tag{3.3}$$

holds for all $\xi \in \mathbb{R}^{N\times n}$. For integers $k > 1$ and sequences (δ_k), $(\mu_k) \subset (0,\infty)$ (specified at (3.5) below), define

$$F_k(\xi) := (\Phi_{\delta_k} * H_k)(\xi) - \mu_k. \tag{3.4}$$

Then we have for all $\xi \in \mathbb{R}^{N\times n}$ and $k > 1$:

$$\begin{aligned} F_k(\xi) &\leq H_k(\xi) + m_k\delta_k - \mu_k \\ &\leq H_{k+1}(\xi) + m_k\delta_k - \mu_k \\ &\leq (\Phi_{\delta_{k+1}} * H_{k+1})(\xi) + m_k\delta_k - \mu_k \\ &= F_{k+1}(\xi) + \mu_{k+1} + m_k\delta_k - \mu_k. \\ &\leq F_{k+1}(\xi) \end{aligned}$$

by taking

$$\delta_k := \frac{1}{k^2 m_k} \qquad \text{and} \qquad \mu_k := \frac{1}{k-1}. \tag{3.5}$$

Hence, we have that $F_k(\xi) \nearrow F(\xi)$ as $k \nearrow \infty$ pointwise in ξ. It follows in particular from Dini's Lemma that the convergence is locally uniform in ξ. We record that F_k is C^1 on $\mathbb{R}^{N\times n}$, C^∞ on $\mathbb{R}^{N\times n}\setminus\{0\}$. Next we check that also $F_k'(\xi)\to F'(\xi)$ locally uniformly in ξ as $k\to\infty$. To that end assume that $\xi_k\to\xi$ and consider $(F_k'(\xi_k))$. Because difference–quotients of convex functions are increasing in the increment, we have for all $\eta\in\mathbb{R}^{N\times n}$ and $0<|t|\leq 1$:

$$
\begin{aligned}
\left|\langle F_k'(\xi_k)-F'(\xi),\eta\rangle\right| &\leq \left|\frac{F_k(\xi_k+t\eta)-F_k(\xi_k)-\langle F'(\xi),t\eta\rangle}{t}\right|\\
&\leq \left|F_k(\xi_k+\eta)-F_k(\xi_k)-\langle F'(\xi),\eta\rangle\right|.
\end{aligned}
$$

Consequently, we get

$$
\limsup_{k\to\infty}\left|\langle F_k'(\xi_k)-F'(\xi),\eta\rangle\right|\leq\left|F(\xi+\eta)-F(\xi)-\langle F'(\xi),\eta\rangle\right|,
$$

for all $\eta\in\mathbb{R}^{N\times n}$. Hence, for all $0<s\leq 1$, we have that

$$
\limsup_{k\to\infty}\frac{|\langle F_k'(\xi_k)-F'(\xi),s\eta\rangle|}{s}\leq\frac{|F(\xi+s\eta)-F(\xi)-\langle F'(\xi),s\eta\rangle|}{s},
$$

that is

$$
\limsup_{k\to\infty}\left|\langle F_k'(\xi_k)-F'(\xi),\eta\rangle\right|\leq\left|\frac{F(\xi+s\eta)-F(\xi)}{s}-\langle F'(\xi),\eta\rangle\right|.
$$

Since F is differentiable at ξ, we conclude that the left–hand side must vanish, as s tends to 0. This proves the asserted local uniform convergence of derivatives.

Let $u_k\in\mathrm{W}^{1,1}_g(\Omega,\mathbb{R}^N)$ denote the unique F_k–minimizer, and recall from above that setting $\sigma_k:=F_k'(Du_k)$ we have that $F_k^*(\sigma_k)\in\mathrm{L}^1(\Omega,\mathbb{R}^{N\times n})$ is a solution to the dual problem that consists in maximizing the functional

$$
\int_\Omega\left(\langle\sigma,Dg\rangle-F_k^*(\sigma)\right),
$$

over row–wise solenoidal matrix fields σ such that $F_k^*(\sigma)\in\mathrm{L}^1(\Omega,\mathbb{R}^{N\times n})$, where F_k^* denotes the polar of F_k. It is not difficult to check that $F_k^*(\zeta)\searrow F^*(\zeta)$ as $k\nearrow\infty$, pointwise in ζ. Furthermore, we record the extremality relation

$$
\langle\sigma_k,Du_k\rangle=F_k^*(\sigma_k)+F_k(Du_k)\quad\text{a.e. on }\Omega \tag{3.6}
$$

that holds for all $k>1$. Since $F_k^*(\sigma_k)\in\mathrm{L}^1(\Omega,\mathbb{R}^{N\times n})$, $F(Dg)\in\mathrm{L}^1(\Omega)$, σ_k is row–wise solenoidal and $u_k-g\in\mathrm{W}^{1,1}_0(\Omega,\mathbb{R}^N)$, we have that

$$
\int_\Omega\langle\sigma_k,Du_k\rangle=\int_\Omega\langle\sigma_k,Dg\rangle. \tag{3.7}
$$

Our next goal is to show that $u_k \to u$ strongly in $\mathrm{W}^{1,1}(\Omega,\mathbb{R}^N)$. To that end, we start by observing that by the definition of F_k at (3.4), the right inequality in (3.3) and the definition of H_k, we have

$$F_k(\xi) \geq \theta(|\xi|) - c - \mu_k \geq \theta(|\xi|) - (c+1),$$

with the choice of μ_k given by (3.5). Therefore, since $F_k \nearrow F$, we have that

$$\int_\Omega (\theta(|Du_k|) - \tilde{c}) \leq \int_\Omega F_k(Du_k) \leq \int_\Omega F(Dg) < \infty.$$

Hence Du_k are equi–integrable and by the De La Vallée–Poussin and Dunford–Pettis Theorems there exists a not relabeled subsequence (u_k) that converges weakly to some v in $\mathrm{W}^{1,1}(\Omega,\mathbb{R}^N)$. By Mazur's Lemma, we get that $v \in \mathrm{W}^{1,1}_g(\Omega,\mathbb{R}^N)$ and, for each $k > 1$, that

$$\liminf_{j\to\infty} \int_\Omega F_k(Du_j) \geq \int_\Omega F_k(Dv). \tag{3.8}$$

Since $F_k \nearrow F$, by monotone convergence Theorem, the minimality of u and (3.8) we find that

$$\liminf_{k\to\infty} \int_\Omega F_k(Du_k) \geq \int_\Omega F(Dv) \geq \int_\Omega F(Du). \tag{3.9}$$

Using first that u_k is F_k–minimizing and then the monotone convergence Theorem, yield

$$\limsup_{k\to\infty} \int_\Omega F_k(Du_k) \leq \limsup_{k\to\infty} \int_\Omega F_k(Du) = \int_\Omega F(Du),$$

and by comparing this with the inequality (3.9), we deduce that

$$\int_\Omega F_k(Du_k) \to \int_\Omega F(Du) = \int_\Omega F(Dv). \tag{3.10}$$

By the uniqueness of F–minimizers, the equality in (3.10) implies that $v = u$. To deduce that the convergence is actually strong we use the uniform convexity of the F_k, i.e. we use that $F_k - \theta(|\cdot|)$ is convex for all $k > 1$, where θ is the function appearing in assumption (H1). As F_k is C^1 and by virtue of (H0) and (H2), we have that there exists a constant $c > 0$ such that

$$\begin{aligned} c\int_\Omega |Du - Du_k| &\leq \int_\Omega \big(F_k(Du) - F_k(Du_k) - \langle F_k'(Du_k), D(u-u_k)\rangle\big) \\ &= \int_\Omega (F_k(Du) - F_k(Du_k)) \to 0, \end{aligned}$$

as $k \to \infty$. It follows that $Du_k \to Du$ in measure on Ω and since $|Du_k|$ is equi–integrable on Ω, by Vitali's convergence theorem, also that $Du_k \to Du$ strongly in L^1. Since $u_k - u \in \mathrm{W}_0^{1,1}(\Omega, \mathbb{R}^N)$ we have shown that the (relabelled) subsequence (u_k) converges strongly to u in $\mathrm{W}^{1,1}(\Omega, \mathbb{R}^N)$. By the uniqueness of limit we conclude by a standard argument that the full sequence (u_k) converges strongly in $\mathrm{W}^{1,1}$ to u. It follows in particular that $\sigma_k = F_k'(Du_k) \to F'(Du)$ in measure on Ω, and so passing to the limit in (3.6) we recover the pointwise extremality relation

$$\langle \sigma^*, Du \rangle = F^*(\sigma^*) + F(Du) \quad \text{a.e. on } \Omega, \tag{3.11}$$

with

$$\sigma^* := F'(Du).$$

Now, using (3.6) and (3.7) and Young's inequality, we have that

$$\begin{aligned}
\int_\Omega F_k^*(\sigma_k) &= \int_\Omega \langle \sigma_k, Du_k \rangle - \int_\Omega F_k(Du_k) \\
&= \int_\Omega \langle \sigma_k, Dg \rangle - \int_\Omega F_k(Du_k) \\
&\leqslant \frac{1}{2} \int_\Omega F_k^*(\sigma_k) + \int_\Omega F(2Dg) + \int_\Omega F_k(Du_k)
\end{aligned} \tag{3.12}$$

where we also used the convexity of F_k^*, the fact that $F_k(\xi) \leq F(\xi)$, for all $\xi \in \mathbb{R}^{N \times n}$ and the Young's inequality. Reabsorbing the first integral in the right hand side by the left hand side we get for all $k > 1$

$$\int_\Omega F_k^*(\sigma_k) \leq 2 \int_\Omega F(2Dg) + 2 \int_\Omega F_k(Du_k)$$

Since $F(2Dg) \in \mathrm{L}^1(\Omega)$ and by virtue of (3.10), from previous relation it follows

$$\begin{aligned}
\int_\Omega F^*(F'(Du)) &\leq \liminf_{k \to \infty} \int_\Omega F_k^*(\sigma_k) \\
&\leq c \int_\Omega F(Du) + c \int_\Omega F(2Dg),
\end{aligned} \tag{3.13}$$

where we used Fatou's lemma. $\square$

4 Higher differentiability

This section is devoted to the proof of the higher differentiability result stated in Theorem 2. We apply the difference quotient method to the minimizers of the auxiliary problems constructed in the proof of Theorem 1 to deduce that they belong to a suitable Besov space . Then we conclude by passing to the limit.

Proof of Theorem 2. Define the integrands F_k and corresponding F_k–minimizers u_k of class $\mathrm{W}^{1,p}_g(\Omega,\mathbb{R}^N)$ as in the proof of Theorem 1 (see in particular (3.4)). We have shown there that $u_k \to u$ strongly in $\mathrm{W}^{1,1}$ and that $\sigma_k := F_k'(Du_k) \to F'(Du)$ in measure on Ω. Fix $B_{3R} = B(x_0, 3R) \subset \Omega$, an integer $1 \le s \le n$, a cut off function $\eta \in C_0^\infty(B_{2R})$, $\eta = 1$ on B_R and an increment $0 \neq h \in (-R, R)$. Since $F'(Du_k)$ is row-wise solenoidal, it follows that

$$\int_{B_{2R}} \langle \tau_{s,h} F_k'(Du_k), D(\eta \tau_{s,h} u_k)\rangle = 0 \tag{4.1}$$

and therefore

$$\begin{aligned}\int_{B_{2R}} \langle \tau_{s,h} F_k'(Du_k), \tau_{s,h} Du_k\rangle \eta &= -\int_{B_{2R}} \langle \tau_{s,h} F_k'(Du_k), \tau_{s,h} u_k \otimes D\eta\rangle \\ &\le \int_{B_{2R}} \left|\langle \tau_{s,h} F_k'(Du_k), \tau_{s,h} u_k \otimes D\eta\rangle\right|. \end{aligned}\tag{4.2}$$

Since F_k is C^1, by virtue of (H0) and (H2), we have

$$\begin{aligned}\Big\langle F_k'(\xi_1) - F_k'(\xi_2), \xi_1 - \xi_2 \Big\rangle &\ge \Big\langle \theta'(|\xi_1|)\frac{\xi_1}{|\xi_1|} - \theta'(|\xi_2|)\frac{\xi_2}{|\xi_2|}, \xi_1 - \xi_2 \Big\rangle \\ &\ge |\xi_1 - \xi_2| \ge c\frac{|\xi_1 - \xi_2|^2}{(1 + |\xi_1|^2 + |\xi_2|^2)^{\frac{1}{2}}} \end{aligned}\tag{4.3}$$

Estimate (4.3) yields that the left hand side of (4.2) can be controlled from below as follows

$$\int_{B_{2R}} \langle \tau_{s,h} F_k'(Du_k), \tau_{s,h} Du_k\rangle \eta \ge c\int_{B_R} \frac{|\tau_{s,h}(Du_k)|^2}{(1 + |Du(x + he_s)|^2 + |Du(x)|^2)^{\frac{1}{2}}}. \tag{4.4}$$

Therefore, combining (4.2) and (4.4), we get

$$\int_{B_R} \frac{|\tau_{s,h}(Du_k)|^2}{(1 + |Du(x + he_s)|^2 + |Du(x)|^2)^{\frac{1}{2}}} \le \int_{B_{2R}} \Big|\langle \tau_{s,h} F_k'(Du_k), \tau_{s,h} u_k \otimes D\eta\rangle\Big|.$$

By Lemma 2, applied with $\gamma = -\frac{1}{4}$, we deduce that

$$\begin{aligned}
\int_{B_R} |\tau_{s,h} V(Du_k)|^2 &\leq c \int_{B_{2R}} \left| \langle \tau_{s,h} F_k'(Du_k), \tau_{s,h} u_k \otimes D\eta \rangle \right| \\
&\leq c|h| \int_{B_{2R}} |\langle \Delta_{s,h} F_k'(Du_k), \Delta_{s,h} u_k \otimes D\eta \rangle| \\
&\leq c|h| \int_{B_{2R}} |\langle F_k'(Du_k(x+he_s)), \Delta_{s,h} u_k \otimes D\eta \rangle| \\
&\quad + c|h| \int_{B_{2R}} |\langle F_k'(Du_k(x)), \Delta_{s,h} u_k \otimes D\eta \rangle| \\
&\leq c|h| \int_{B_{2R}} F_k^*(F_k'(Du_k(x+he_s))) + F_k^*(F_k'(Du_k(x))) \\
&\quad + c|h| \int_{B_{2R}} F_k \left(\Delta_{s,h} u_k \otimes D\eta \right). \qquad (4.5)
\end{aligned}$$

In order to estimate last integral in (4.5), recalling that $F_k \nearrow F$ as $k \to \infty$, we can use the structure assumption (1.4) thus obtaining

$$\begin{aligned}
&\int_{B_R} |\tau_{s,h} V(Du_k)|^2 \\
\leq\ & |h| \int_{B_{2R}} F_k^*(F_k'(Du_k(x+he_s))) + F_k^*(F_k'(Du_k(x))) \\
& + c|h| \int_{B_{2R}} \Phi \left(|\Delta_{s,h} u_k| |D\eta| \right) \\
\leq\ & |h| \int_{B_{2R}} F_k^*(F_k'(Du_k(x+he_s))) + F_k^*(F_k'(Du_k(x))) \\
& + c(R)|h| \int_{B_{2R}} \Phi \left(|\Delta_{s,h} u_k| \right), \qquad (4.6)
\end{aligned}$$

where we also used the monotonicity of the Young function Φ. Now, by the convexity of Φ and Jensen's inequality it follows that

$$\begin{aligned}
\int_{B_{2R}} \Phi \left(|\Delta_{s,h} u_k| \right) dx &= \int_{B_{2R}} \Phi \left(\Big| \int_0^1 \frac{d}{ds} u(x + the_s)\, dt \Big| \right) dx \\
&\leq \int_{B_{2R}} \int_0^1 \Phi \left(\Big| \frac{d}{ds} u(x + the_s) \Big| \right) dt\, dx \\
&\leq \int_{B_{3R}} \Phi \left(|D_s u| \right) dx \leq \int_\Omega F(Du). \qquad (4.7)
\end{aligned}$$

Moreover, by a simple change of variable and by virtue of the estimate (3.13), we have that

$$\begin{aligned} \int_{B_{2R}} F_k^*(F_k'(Du_k(x+he_s))) &\leq \int_{B_{3R}} F_k^*(F_k'(Du_k(x))) \\ &\leq 2\int_\Omega F(2Dg) + c\int_\Omega F(Du) \end{aligned} \tag{4.8}$$

Inserting (4.7) and (4.8) in (4.6), we get

$$\int_{B_R} |\tau_{s,h} V(Du_k)|^2 \leq c|h| \left(\int_\Omega F(2Dg) + c\int_\Omega F(Du) \right) \tag{4.9}$$

Therefore, by Fatou's Lemma, taking the limit as $k \to \infty$, we have for every $s \in \{1, \ldots, n\}$

$$\int_{B_R} |\tau_{s,h} V(Du)|^2 \leq c|h| \left(\int_\Omega F(2Dg) + c\int_\Omega F(Du) \right)$$

From previous inequality, by virtue of Lemma 1, we obtain that $V(Du) \in \mathrm{W}^{\vartheta,2}_{loc}(\Omega)$ for every $\vartheta \in (0, \frac{1}{2})$ and $V(Du) \in \mathrm{L}^{\frac{2n}{n-\alpha}}_{loc}(\Omega)$ for every $\alpha < 1$. This concludes the proof. $\square$

Let us finally note that the structure assumption that the integrand be radial is needed only to control last integral in (4.5), which is of the form

$$F_k(\Delta_{s,h} u_k \otimes D\eta) = F_k\left(\Delta_{s,h} u_k^1 D_1\eta, \ldots, \Delta_{s,h} u_k^1 D_n\eta, \ldots, \Delta_{s,h} u_k^N D_1\eta, \ldots, \Delta_{s,h} u_k^N D_n\eta\right)$$

While it is possible that this can be controlled with a milder structure condition, we decided to use it anyway in order to simplify the presentation.

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