

**STABILITY OF TRANSONIC SHOCKS IN STEADY
SUPERSONIC FLOW PAST MULTIDIMENSIONAL
WEDGES**

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ABSTRACT. We are concerned with the stability of multidimensional (M-D) transonic shocks in steady supersonic flow past multidimensional wedges. One of our motivations is that the global stability issue for the M-D case is much more sensitive than that for the 2-D case, which requires more careful rigorous mathematical analysis. In this paper, we develop a nonlinear approach and employ it to establish the stability of weak shock solutions containing a transonic shock-front for potential flow with respect to the M-D perturbation of the wedge boundary in appropriate function spaces. To achieve this, we first formulate the stability problem as a free boundary problem for nonlinear elliptic equations. Then we introduce the partial hodograph transformation to reduce the free boundary problem into a fixed boundary value problem near a background solution with fully nonlinear boundary conditions for second-order nonlinear elliptic equations in an unbounded domain. To solve this reduced problem, we linearize the nonlinear problem on the background shock solution and then, after solving this linearized elliptic problem, develop a nonlinear iteration scheme that is proved to be contractive.

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1. INTRODUCTION

We are concerned with the stability of multidimensional (M-D) transonic shocks in steady supersonic flow past M-D wedges. In this paper, we focus on the fluid flow governed by the potential flow equation:

$$\operatorname{div}(\rho(|D\varphi|^2)D\varphi) = 0, \quad (1.1)$$

where $\varphi = \varphi(\mathbf{x})$ is the potential of the velocity field in $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, ρ is the density with

$$\rho(q^2) = \left(1 - \frac{\gamma - 1}{2}q^2\right)^{\frac{1}{\gamma-1}}$$

from Bernoulli's law for polytropic gases of adiabatic exponent $\gamma > 1$ by scaling, and $D := (\partial_{x_1}, \dots, \partial_{x_n})$ is the gradient in \mathbf{x} .

Then the potential flow equation (1.1) can be written as

$$\sum_{i,j=1}^n a_{ij}(D\varphi)\partial_{x_i x_j}\varphi = 0,$$

where

$$a_{ij}(D\varphi) = \begin{cases} c^2(|D\varphi|^2) - |\partial_{x_i}\varphi|^2, & i = j, \\ -\partial_{x_i}\varphi\partial_{x_j}\varphi, & i \neq j, \end{cases}$$

with $c(q^2) = (1 - \frac{\gamma-1}{2}q^2)^{1/2}$ being the sonic speed. Denote $A(D\varphi) := [a_{ij}(D\varphi)]_{n \times n}$.

For an upstream supersonic flow past a straight wedge, a flat shock-front is formed in the flow (see Fig. 1.1). When the wedge angle is less than the critical angle, the shock-front may be attached to the wedge edge. There exist shock-fronts of two different types depending on the downstream flow behind them: Transonic (supersonic-subsonic) shock-fronts and supersonic-supersonic shock-fronts. For a given two-dimensional (2-D) wedge which produces an attached shock-front, there are two admissible shock solutions that satisfy both the Rankine-Hugoniot conditions and the entropy condition. The weaker one may be a supersonic-supersonic shock-front or a transonic shock-front, while the stronger one is always a transonic shock-front (*cf.* [11, 19]). It is analogous for the M-D case (see §2). The non-uniqueness and related stability issues of such M-D steady shock waves have been longstanding open problems in mathematical fluid mechanics, which have attracted many mathematical scientists including Busemann [2], Meyer [32], Prandtl [34], Courant-Friedrichs [11], and von Neumann [38]; also see [1], [4]–[10], [12, 15, 18, 24, 36, 37, 41], and the references cited therein. In this paper, we are interested in the stability problem of the M-D transonic shock-fronts, behind which the flow is fully subsonic.

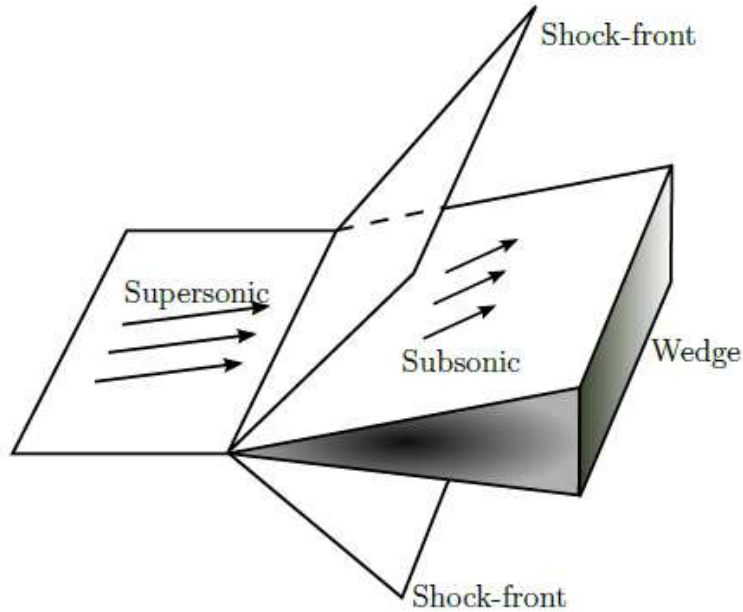


FIGURE 1.1. The shock-front in steady supersonic flow past an M-D wedge

For the 2-D case, local solutions involving a supersonic-supersonic shock around the curved wedge vertex were first constructed by Gu [18], Li [24], Schaeffer [36], and the references cited therein. Global potential solutions are constructed in [7, 8, 11, 40, 41] when the wedge has certain convexity, or the wedge is a small perturbation of the straight-sided wedge with fast decay in the flow direction. In Chen-Zhang-Zhu [6], two-dimensional steady supersonic flows governed by the full Euler equations past Lipschitz wedges were systematically analyzed, and the existence and stability of supersonic Euler flows in BV were established via a modified Glimm difference scheme (*cf.* [16]), when the total variation of the tangent angle function along the wedge boundary is suitably small. Furthermore, the L^1 -stability and uniqueness of entropy solutions in BV containing the strong supersonic-supersonic shock were established in Chen-Li [5]. The stability of transonic shocks under a perturbation of the upstream flow, or a perturbation of wedge boundary, has been studied in Chen-Fang [10] for the potential flow and in Fang [15] for the Euler flow with a uniform Bernoulli constant. In particular, the stability of transonic shocks in the steady Euler flows with a uniform Bernoulli constant was first established in the weighted Sobolev norms in Fang [15], while the downstream asymptotic decay rate of the shock speed at infinity for the weak transonic shock solution for the full Euler equations has been achieved in Chen-Chen-Feldman [4]. Also see Yin-Zhou [39] for the stability of strong transonic shock solutions.

For the M-D case, local solutions involving a supersonic-supersonic shock past a 3-D wing were first constructed by Chen [9]. One of our motivations in this paper is that the global stability issue for the M-D case is much more sensitive than that for the 2-D case, which requires more careful rigorous mathematical analysis. In this paper, we develop a nonlinear approach and employ it to establish the stability of weak shock solutions

containing a transonic shock-front with respect to the M-D perturbation of the wedge boundary in appropriate function spaces.

To achieve this, we first formulate the stability problem as a free boundary problem for nonlinear elliptic equations. Then we introduce the partial hodograph transformation to reduce the free boundary problem into a fixed boundary value problem near a background solution with fully nonlinear boundary conditions for second-order nonlinear elliptic equations in an unbounded domain. To solve this reduced problem, we linearize the nonlinear problem on the background shock solution and then, after solving this linearized elliptic problem, employ a nonlinear iteration scheme that is proved to be contractive. For this, the well-posedness theory for the corresponding linearized elliptic problem also plays an important role in this stability analysis of the transonic shocks.

The linearized problem here is a boundary value problem of elliptic equations in an unbounded domain of a dihedral angle. The singularities of the solution near the edge with the dihedral angle and the asymptotic behavior at infinity are two important aspects for such problems. As far as we have known, there have been plenty of literature for the elliptic problems in a domain with conical or/and edge singularities; see [3, 13, 14, 17], [20]–[23], [26]–[31], [33, 35], and the references cited therein. In this paper, the well-posedness of the linearized problem can be obtained by directly applying the results established by Maz'ya, Plamenevskij, and others in [23], [26]–[31], and [35]. According to the theory, the linearized elliptic problem can be well-posed in weighted Sobolev spaces or weighted Hölder spaces, whose weights describe the singularity of the solution near the edge and the asymptotic behavior at infinity simultaneously. It is shown that the admissible weights are essentially associated with the eigenvalues of the deduced elliptic boundary value problem in an angular domain; see [23, 28, 31] for the rigorous definitions and related details. We calculate an example of the eigenvalues for oblique derivative boundary value problems of the Poisson equation in an angular domain in the appendix, which is used in this paper. It turns out that, for these problems, there are countable many eigenvalues and the admissible weights are separated into countable many intervals according to these eigenvalues. Then there arises an interesting and important difference between an M-D ($n \geq 3$) dihedral-angled wedge, whose edge is a straight line or a hyperplane, and a 2-D one whose edge shrinks to a point. Roughly speaking, only one interval of admissible weights was proved to be valid in [26, 35] for the M-D edge singularity of the domain for the linear theory, while there are countable many intervals of admissible weights that are valid for the 2-D corner singularity; *cf.* [21, 23, 28]. That is, there are much more admissible weights that are valid for the 2-D case than for the M-D case. It is this difference that will lead us to different stability consequences for the M-D case from the 2-D case: The M-D stability result is established in this paper only for the weak transonic shocks, while the 2-D stability results can be established for both the weak and strong ones.

For our stability problem, the distribution of eigenvalues for the linearized problem is closely related to the angle between the velocity vector behind the shock-front and the outer normal of the shock polar in the (u, v) -plane, whose tangent value, according to the shock polar, is positive for weak transonic shocks, while negative for strong ones; see §6 and

§8. This fact will result in different stability consequences for weak transonic shocks and strong ones for the M-D case. In fact, the only valid admissible interval of weights for the weak transonic shocks satisfies the property that the solution is physically reasonable, that is, the velocity should be bounded; while the weights for the strong transonic shocks fail to satisfy this property. Therefore, for the M-D transonic shocks, the stability of weak ones can be established in this paper, while the stability of the strong ones cannot be established via this analysis regime; see §6 for more details. However, for the 2-D transonic shocks, since there are countable many valid admissible intervals of weights, we can choose one of them, accordingly for weak and strong ones, such that the solution is physically reasonable; see §8 for more details.

Therefore, it is also interesting to question whether the stability of the strong transonic shocks for the M-D case is still valid. For the stability of strong transonic shocks, the nature of the boundary condition is significantly different from the weak transonic shock case. Such a difference may affect the regularity of solutions, as well as the asymptotic behavior, in general. It requires further understanding of some special features of the problem along the wedge edge to ensure that there exists a smooth solution. A different approach may be required to handle this case, which is currently under investigation. In this regard, we notice that an instability result has been observed recently in Li-Xu-Yin [25].

The organization of this paper is as follows. In §2, we establish the shock polar for the M-D shock-fronts for the potential flow equation (1.1). In §3, we formulate the stability problem as a free boundary problem and describe our main theorem. In §4, we introduce the weighted norms applied in this paper measuring the perturbations and provide the well-established theory for boundary value problems of the Poisson equation in a dihedral angle. In §5, we introduce the partial hodograph transformation to reduce the free boundary problem into a fixed boundary value problem and describe the theorem which will be proved in §6–7. In §6, we analyze the regularity of solutions near the wedge edge by linearizing the nonlinear stability problem. In §7, we develop an iteration scheme and establish its convergence, which completes the proof of our main theorem. In §8, different from the M-D case, we show that all the weak and strong transonic shock solutions are conditionally stable in the 2-D case, for which the strong one has even better regularity near the wedge vertex. For self-containedness, in the appendix, we give a sketch of the proof of Theorem 4.4.

2. THE SHOCK POLAR FOR MULTIDIMENSIONAL SHOCK-FRONTS

Assume that the velocity of the uniform supersonic flow ahead of a shock-front \mathcal{S} is $\mathbf{v}^- = (q_0, 0, 0, \dots, 0)^\top$, and the velocity of the uniform flow behind \mathcal{S} is $\mathbf{v} = (v_1, v_2, \mathbf{v}')^\top$ with $\mathbf{v}' = (v_3, \dots, v_n)$. Then the corresponding potential functions are

$$\varphi^-(\mathbf{x}) = q_0 x_1, \quad \varphi^+(\mathbf{x}) = v_1 x_1 + v_2 x_2 + \mathbf{v}' \cdot \mathbf{x}',$$

respectively, where $\mathbf{x} = (x_1, x_2, \dots, x_n)^\top = (x_1, x_2, \mathbf{x}')^\top$ with $\mathbf{x}' = (x_3, \dots, x_n)^\top$. Let

$$\varphi(\mathbf{x}) = \varphi^-(\mathbf{x}) - \varphi^+(\mathbf{x}) = (q_0 - v_1)x_1 - v_2 x_2 - \mathbf{v}' \cdot \mathbf{x}'.$$

Then the Rankine-Hugoniot conditions on \mathcal{S} can be written as

$$D\varphi \cdot (\rho(|D\varphi^+|^2)D\varphi^+ - \rho(|D\varphi^-|^2)D\varphi^-) = 0, \quad (2.1)$$

$$\varphi(\mathbf{x}) = 0. \quad (2.2)$$

Condition (2.1) indicates the conservation of mass across the shock-front, and condition (2.2) implies that the tangential components of the velocity are continuous across the shock-front.

Now we determine the position of the shock-front and velocity \mathbf{v} behind it, for the given wedge and the uniform incoming supersonic flow \mathbf{v}^- . To this end, the rigidity assumption is imposed on the flow along the wedge surface:

$$\mathbf{v} \cdot \boldsymbol{\nu} = 0, \quad (2.3)$$

where $\boldsymbol{\nu}$ is the unit normal of the wedge.

Condition (2.1) can be rewritten as

$$(\rho + \rho^-) q_0 v_1 - \rho q^2 - \rho^- q_0^2 = 0, \quad (2.4)$$

where $\rho^- = \rho(q_0^2)$, $q = |\mathbf{v}|$, and $\rho = \rho(q^2)$. Then the admissible solution \mathbf{v} to equation (2.4) can be described by a shock balloon rotating the 2-D shock polar around the v_1 -axis; see Fig. 2.1(a).

Since the shock-front is attached to the wedge edge, which is assumed to be the hyperplane spanned by the unit vectors $\boldsymbol{\tau}_j = (\tau_1, 0, \dots, \tau_j, \dots, 0)^\top$, $j = 3, \dots, n$, with τ_j the j -th component, we can differentiate condition (2.2) along the edge to obtain

$$q_0 \tau_1 = v_1 \tau_1 + v_j \tau_j, \quad j = 3, \dots, n, \quad (2.5)$$

which implies that $\mathbf{v}^- - \mathbf{v}$ is orthogonal to $\boldsymbol{\tau}_j$. Thus, the M-D shock polar determined by the Rankine-Hugoniot conditions (2.1)–(2.2) is the intersection curve of the shock balloon determined by (2.4) and the hyperplanes in (2.5), which is similar to the 2-D shock polar, when such an intersection curve exists; see loop QS_*N in Fig. 2.1(b).

Finally, for a given wedge, the rigidity assumption (2.3) yields that \mathbf{v} should also be tangential to the wedge plane, plane O_1O_2W , which intersects with the shock balloon at loop P_1P_2 when the dihedral wedge angle is less than the critical value; see Fig. 2.1(c). Therefore, the velocity behind the shock-front must be determined by the intersection points A and B of loop P_1P_2 and the shock polar QS_*N ; see Fig. 2.1(d). Each intersection point represents a shock solution, which is called the background solution, to our problem for supersonic potential flow past a straight M-D wedge. Notice that, as the wedge angle increases to the critical value, the intersection points A and B coincide with S_* ; and when it is larger than the critical value, there is no intersection point, which implies that the shock-front cannot attach the wedge edge.

Both shock solutions determined by A and B satisfy the entropy condition, and the shock strength of the solution represented by A is stronger than B . In addition, A must correspond to a transonic shock solution, while B may correspond to a transonic or supersonic shock solution. The critical shock solution S_* must be transonic. We are interested

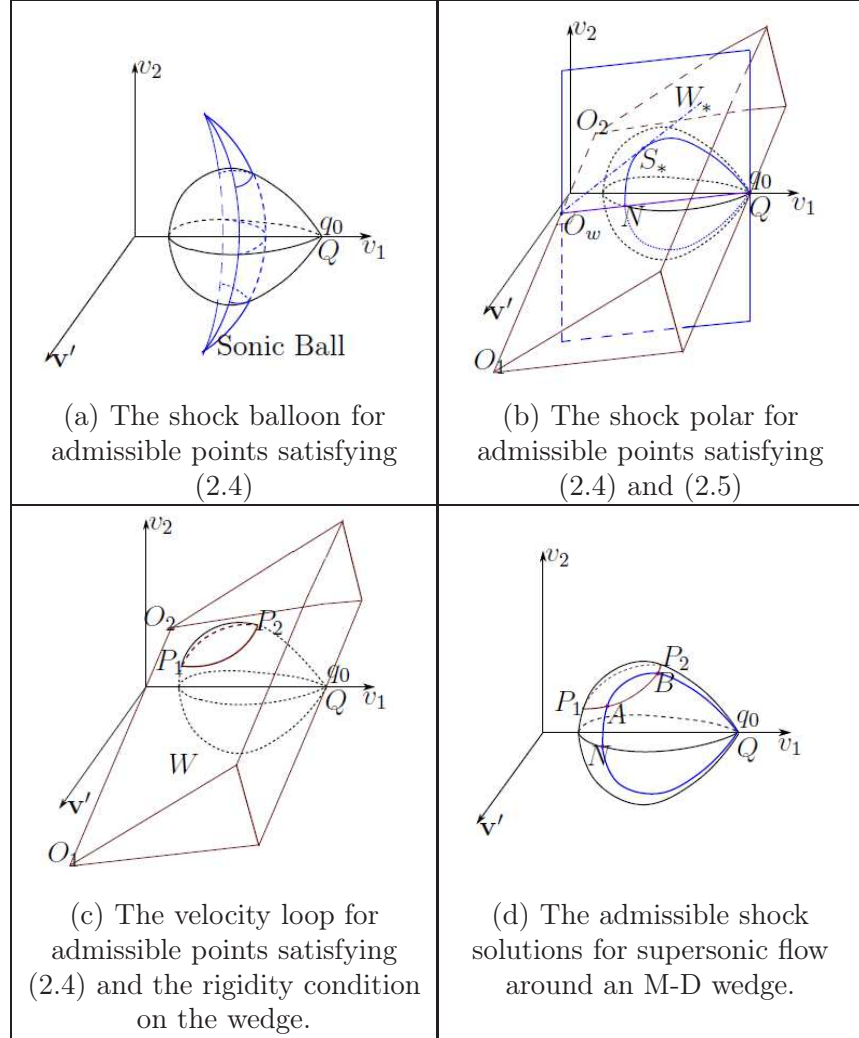


FIGURE 2.1. The shock polar and shock solutions for a given M-D wedge

in the stability of transonic shocks, including the weak and strong transonic shocks on the shock polar.

3. FORMULATION OF THE STABILITY PROBLEM AND MAIN THEOREM

In this section, we formulate the stability problem as a free boundary problem for non-linear elliptic equations and describe our main theorem for the stability results.

We first reformulate the coordinate system, for simplicity of presentation of the computation. Fix the x_1 -axis to be in the surface of the straight wedge and perpendicular to the wedge edge, the x_2 -axis to be perpendicular to the wedge surface, and the x_3 -axis to be parallel with the component of the velocity vector behind the shock-front on the $(n-2)$ -D hyperplane $\{x_1 = 0, x_2 = 0\}$; see Fig. 3.1.

Assume that the wedge angle is α_w . Then, for the background shock solution, the velocity of the incoming flow ahead of the shock-front is

$$U_0^- = (q_0^- \cos \alpha_w, -q_0^- \sin \alpha_w, U_{03}^-, 0, \dots, 0)^\top$$

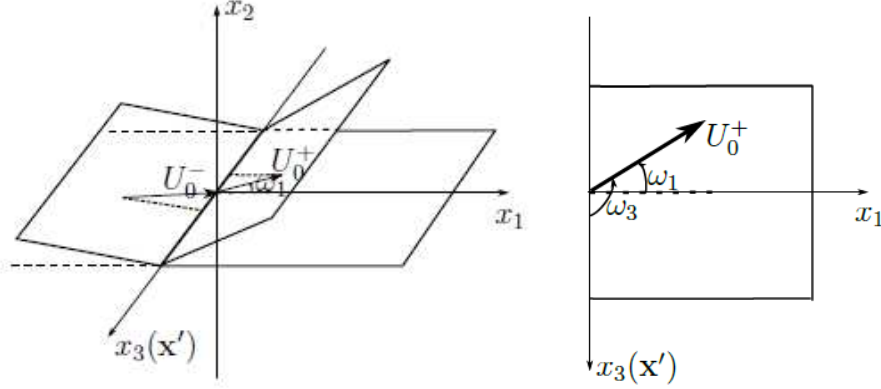


FIGURE 3.1. The flow field under the reformulated coordinate system

with $|U_0^-| = \sqrt{(q_0^-)^2 + (U_{03}^-)^2}$, and the velocity behind the shock-front is

$$U_0^+ = q_0^+ (\cos \omega_1, 0, \cos \omega_3, 0, \dots, 0)^\top,$$

where ω_j is the angle between U_0^+ and the x_j -axis for $j = 1, 3$; see Fig. 3.1.

By the Rankine-Hugoniot conditions, we have

$$\begin{aligned} U_{03}^- &= q_0^+ \cos \omega_3, \\ \cos^2 \omega_1 + \cos^2 \omega_3 &= 1. \end{aligned}$$

Thus, the corresponding potential functions are

$$\varphi_0^-(\mathbf{x}) = x_1 q_0^- \cos \alpha_w - x_2 q_0^- \sin \alpha_w + x_3 U_{03}^-, \quad (3.1)$$

$$\varphi_0^+(\mathbf{x}) = x_1 q_0^+ \cos \omega_1 + x_3 q_0^+ \cos \omega_3, \quad (3.2)$$

and the location of the shock-front \mathcal{S}_0 is determined by

$$\varphi_0(\mathbf{x}) := \varphi_0^-(\mathbf{x}) - \varphi_0^+(\mathbf{x}) = 0, \quad (3.3)$$

that is,

$$x_1 (q_0^- \cos \alpha_w - q_0^+ \cos \omega_1) - x_2 q_0^- \sin \alpha_w = 0.$$

Now assume that the wedge surface is perturbed by the perturbed surface:

$$\Gamma_w := \{\mathbf{x} \in \mathbb{R}^n : x_2 = \varphi_w(x_1, \mathbf{x}'), x_1 > \varphi_w^e(\mathbf{x}'), \mathbf{x}' \in \mathbb{R}^{n-2}\};$$

see Fig. 3.2. We investigate whether the background transonic shock solution (3.1)–(3.2) with the position of shock-front determined by (3.3) is stable.

If the shock solution is stable, then Γ_s is denoted as the shock-front, \mathcal{D}^- as the supersonic flow field ahead Γ_s , and \mathcal{D}^+ the subsonic flow field between Γ_s and Γ_w ; see Fig. 3.2. Let $\varphi^\pm(\mathbf{x})$ be the potential functions of the perturbed steady flow in \mathcal{D}^\pm , respectively. Then we have

$$\sum_{i,j=1}^n a_{ij} (D\varphi^\pm) \partial_{x_i x_j} \varphi^\pm = 0 \quad \text{in } \mathcal{D}^\pm. \quad (3.4)$$

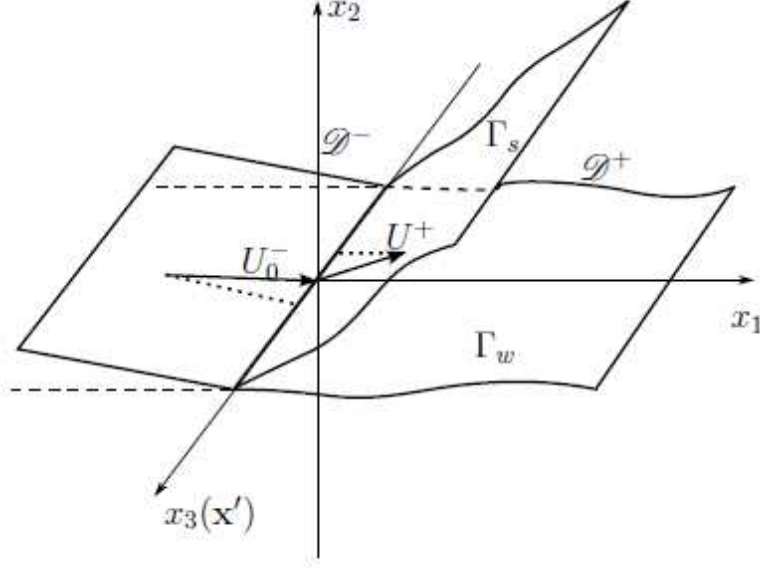


FIGURE 3.2. The perturbed wedge and resulting perturbed shock-front

Let

$$\varphi(\mathbf{x}) := \varphi^-(\mathbf{x}) - \varphi^+(\mathbf{x}). \quad (3.5)$$

Then $\varphi(\mathbf{x})$ in \mathcal{D}^+ is governed by

$$\sum_{i,j=1}^n a_{ij}(D\varphi - D\varphi^-)\partial_{x_i x_j} \varphi = \sum_{i,j=1}^n a_{ij}(D\varphi - D\varphi^-)\partial_{x_i x_j} \varphi^- \quad \text{in } \mathcal{D}^+. \quad (3.6)$$

We assume that the fluid satisfies the rigidity condition on the wedge boundary Γ_w :

$$H_w(D\varphi; D\varphi_w) = 0 \quad \text{on } \Gamma_w, \quad (3.7)$$

where, with $(D_{\mathbf{x}'}\varphi_w)^\top = (\partial_{x_3}\varphi_w, \dots, \partial_{x_n}\varphi_w)$,

$$H_w(D\varphi; D\varphi_w) := (-\partial_{x_1}\varphi_w, 1, -(D_{\mathbf{x}'}\varphi_w)^\top)^\top \cdot (D\varphi - D\varphi^-).$$

On the shock front Γ_s , the Rankine-Hugoniot conditions hold:

$$\varphi(\mathbf{x}) = 0 \quad \text{on } \Gamma_s, \quad (3.8)$$

$$H_s(D\varphi; D\varphi^-) = 0 \quad \text{on } \Gamma_s, \quad (3.9)$$

where

$$H_s(D\varphi; D\varphi^-) := D\varphi \cdot (\rho(|D\varphi^- - D\varphi|^2)(D\varphi^- - D\varphi) - \rho(|D\varphi^-|^2)D\varphi^-).$$

Then the stability problem can be formulated as

Problem 3.1 (Free boundary problem): For the given perturbation of the wedge surface $\varphi_w(x_1, \mathbf{x}')$ and the given incoming supersonic flow $\varphi^-(\mathbf{x}) := \varphi_0^-(\mathbf{x})$, determine $\varphi(\mathbf{x})$ and the free boundary Γ_s of domain \mathcal{D}^+ such that (3.6)–(3.9) hold. Moreover, $\varphi^-(\mathbf{x}) - \varphi(\mathbf{x})$ describes a subsonic flow behind the shock-front.

The main purpose of this paper is to establish the following stability theorem for the weak transonic shock solutions:

Theorem 3.1. *Let $(\varphi_0^-(\mathbf{x}); \varphi_0^+(\mathbf{x}))$ be the weak transonic shock solution that is represented by B on the shock polar; see Fig. 2.1. If the wedge edge is not perturbed, that is,*

$$\varphi_w^e(\mathbf{x}') \equiv 0, \quad \varphi_w(0, \mathbf{x}') \equiv 0 \quad \text{for all } \mathbf{x}' \in \mathbb{R}^{n-2}, \quad (3.10)$$

and the perturbation $\varphi_w(x_1, \mathbf{x}')$ of the wedge surface is sufficiently small, then there exists a unique $\varphi^+(\mathbf{x})$, which is also a small perturbation of $\varphi_0^+(\mathbf{x})$, such that $(\varphi_0^-(\mathbf{x}); \varphi^+(\mathbf{x}))$ solves Problem 3.1, i.e., the free boundary problem (3.6)–(3.9), with the perturbed shock-front Γ_s determined by

$$\varphi(\mathbf{x}) := \varphi_0^-(\mathbf{x}) - \varphi^+(\mathbf{x}) = 0.$$

This indicates that the weak transonic shock solution is conditionally stable.

We remark that the same results hold if $\varphi^-(\mathbf{x})$ is replaced by any smooth incoming supersonic flow near the background potential function $\varphi_0^-(\mathbf{x})$. This can be achieved by the same arguments below without difficulties. For simplicity of presentation, we focus our proof on Problem 3.1.

4. A WELL-POSEDNESS THEOREM FOR BOUNDARY VALUE PROBLEMS OF THE POISSON EQUATION IN A DIHEDRAL ANGLE

We now present here a well-established theory on boundary value problems of the Poisson equation in a dihedral angle established by Maz'ya, Plamenevskij, Reisman, and others in [23], [26]–[31], [35], and the references therein, which will be employed for solving the free boundary problem, Problem 3.1.

4.1. Weighted norms. As before, denote $\mathbf{x} = (x_1, x_2, \mathbf{x}') \in \mathbb{R}^n$ with $\mathbf{x}' = (x_3, \dots, x_n) \in \mathbb{R}^{n-2}$. Let (r, ω) be the polar coordinates for $(x_1, x_2) \in \mathbb{R}^2$ and $\omega_* \in (0, 2\pi)$. Define an angular domain K in \mathbb{R}^2 with its boundaries γ^\pm as in Fig. 4.1:

$$K = \left\{ (x_1, x_2) \in \mathbb{R}^2 : |\omega| < \frac{\omega_*}{2} \right\},$$

$$\gamma^\pm = \left\{ (x_1, x_2) \in \mathbb{R}^2 : |\omega| = \frac{\omega_*}{2} := \pm \frac{\omega_*}{2} \right\}.$$

Then $\mathcal{D} = K \times \mathbb{R}^{n-2}$ is a domain of dihedral angles in \mathbb{R}^n , and $\Gamma^\pm = \gamma^\pm \times \mathbb{R}^{n-2}$ are its two faces intersecting at edge $\mathcal{E} = \{\mathbf{x} : x_1 = x_2 = 0, \mathbf{x}' \in \mathbb{R}^{n-2}\}$.

Definition 4.1. Define the following *weighted Hölder norms*:

$$\begin{aligned} \|u\|_{\mathcal{C}_\beta^{\ell, \alpha}(\mathcal{D})} &:= \sup_{\mathbf{x} \in \mathcal{D}} \sum_{|\mathbf{k}|=0}^{\ell} r_{\mathbf{x}}^{\beta-\ell-\alpha+|\mathbf{k}|} |D^{\mathbf{k}}u(\mathbf{x})| \\ &+ \sup_{\mathbf{x}, \mathbf{y} \in \mathcal{D}} |\mathbf{x} - \mathbf{y}|^{-\alpha} \sum_{|\mathbf{k}|=0}^{\ell} |r_{\mathbf{x}}^{\beta-\ell+|\mathbf{k}|} D^{\mathbf{k}}u(\mathbf{x}) - r_{\mathbf{y}}^{\beta-\ell+|\mathbf{k}|} D^{\mathbf{k}}u(\mathbf{y})|, \quad (4.1) \end{aligned}$$

where $0 < \alpha < 1$, $\ell = 0, 1, \dots$, $\beta \in \mathbb{R}$, $r_{\mathbf{x}} = \sqrt{x_1^2 + x_2^2}$, $r_{\mathbf{y}} = \sqrt{y_1^2 + y_2^2}$, and $D^{\mathbf{k}} = \partial_{x_1}^{k_1} \cdots \partial_{x_n}^{k_n}$ for multi-index $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$. Denote by $\mathcal{C}_\beta^{\ell, \alpha}(\mathcal{D})$ the completion of set $\mathcal{C}_c^\infty(\overline{\mathcal{D}} \setminus \mathcal{E})$ under norm (4.1).

Remark. The weight β in (4.1) has simultaneous control for both the regularity of u near edge \mathcal{E} and the asymptotic behavior as $r_{\mathbf{x}} \rightarrow \infty$. For our later use of the weighted Hölder norms, we will employ double weights for different control for the regularity of u near edge \mathcal{E} and the asymptotic behavior. Let $\beta_0, \beta_\infty \in \mathbb{R}$. Set

$$\mathcal{C}_{\beta_0, \beta_\infty}^{\ell, \alpha}(\mathcal{D}) := \mathcal{C}_{\beta_0}^{\ell, \alpha}(\mathcal{D}) \cap \mathcal{C}_{\beta_\infty}^{\ell, \alpha}(\mathcal{D})$$

with the weighted norm as

$$\|u\|_{(\ell, \alpha; \mathcal{D})}^{(\beta_0, \beta_\infty)} := \|u\|_{\mathcal{C}_{\beta_0}^{\ell, \alpha}(\mathcal{D})} + \|u\|_{\mathcal{C}_{\beta_\infty}^{\ell, \alpha}(\mathcal{D})}.$$

Definition 4.2. A multiplier in $\mathcal{C}_\beta^{\ell, \alpha}(\mathcal{D})$ is a function φ such that

$$\varphi u \in \mathcal{C}_\beta^{\ell, \alpha}(\mathcal{D}) \quad \text{for any } u \in \mathcal{C}_\beta^{\ell, \alpha}(\mathcal{D}).$$

We denote the set of all the multipliers in $\mathcal{C}_\beta^{\ell, \alpha}(\mathcal{D})$ by $\mathcal{MC}_\beta^{\ell, \alpha}(\mathcal{D})$.

In fact, the multiplier space is independent of the weight power β as shown in Maz'ya–Plamenevskij [30]:

Proposition 4.3. $\mathcal{MC}_\beta^{\ell, \alpha}(\mathcal{D}) = \mathcal{C}_{\ell+\alpha}^{\ell, \alpha}(\mathcal{D})$.

4.2. The well-posedness theorem. Consider the elliptic boundary value problem in the dihedral angle \mathcal{D} :

$$\Delta_{\mathbf{x}} u = f \quad \text{in } \mathcal{D}, \quad (4.2)$$

$$\partial \mathcal{D}^\pm u = g^\pm \quad \text{on } \Gamma^\pm, \quad (4.3)$$

where $\Delta_{\mathbf{x}} := \partial_{x_1 x_1} + \partial_{x_2 x_2} + \Delta_{\mathbf{x}'}$ with $\Delta_{\mathbf{x}'} := \sum_{j=3}^n \partial_{x_j x_j}$, and $\partial \mathcal{D}^\pm := \partial_{\boldsymbol{\nu}^\pm} + \alpha^\pm \partial_{\boldsymbol{\tau}^\pm} + \mathbf{c}^\pm \cdot D_{\mathbf{x}'}$ with $\alpha^\pm \in \mathbb{R}$, $\mathbf{c}^\pm \in \mathbb{R}^{n-2}$, $\boldsymbol{\nu}^\pm$ the inward normal of Γ^\pm , and $\boldsymbol{\tau}^\pm$ tangent vector to Γ^\pm , perpendicular to \mathcal{E} and directed from \mathcal{E} into \mathcal{D} ; see Fig. 4.1.

Directly applying the results in [27, 29, 30, 35], we obtain the following theorem for the boundary value problem (4.2)–(4.3). For completeness, we will describe the main steps of the proof in the appendix.

Theorem 4.4. Let $\Phi = \arctan \alpha^- + \arctan \alpha^+$. Suppose that

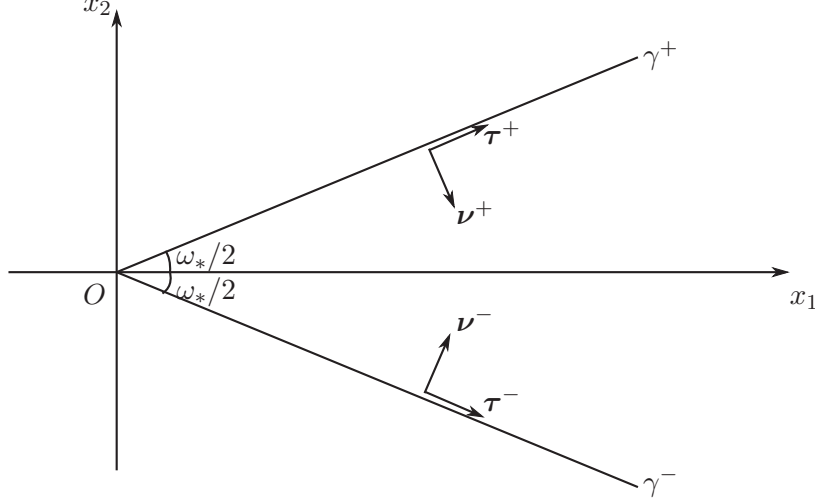
$$-\frac{\Phi}{\omega_*} < \sigma < 0 \quad \text{or} \quad 0 < \sigma < -\frac{\Phi}{\omega_*}, \quad (4.4)$$

and $\beta = 2 + \alpha - \sigma$. Then the operator of problem (4.2)–(4.3) induces the isomorphism

$$\mathcal{C}_\beta^{2, \alpha}(\mathcal{D}) \approx \mathcal{C}_\beta^{0, \alpha}(\mathcal{D}) \times \prod_{\pm} \mathcal{C}_\beta^{1, \alpha}(\Gamma^\pm).$$

Moreover, suppose that both σ_1 and σ_2 satisfy (4.4), and $\beta_j = 2 + \alpha - \sigma_j$. Assume that

$$f \in \mathcal{C}_{\beta_1, \beta_2}^{0, \alpha}(\mathcal{D}), \quad g^\pm \in \mathcal{C}_{\beta_1, \beta_2}^{1, \alpha}(\Gamma^\pm).$$

FIGURE 4.1. The angular domain K

Then solution $u \in \mathcal{C}_{\beta_1}^{2,\alpha}(\mathcal{D})$ of problem (4.2)–(4.3) is also in $\mathcal{C}_{\beta_2}^{2,\alpha}(\mathcal{D})$, that is, $u \in \mathcal{C}_{\beta_1, \beta_2}^{2,\alpha}(\mathcal{D})$ with the estimate:

$$\|u\|_{(2,\alpha;\mathcal{D})}^{(\beta_1,\beta_2)} \leq C \left(\|f\|_{(0,\alpha;\mathcal{D})}^{(\beta_1,\beta_2)} + \sum_{\pm} \|g^{\pm}\|_{(1,\alpha;\Gamma_j)}^{(\beta_1,\beta_2)} \right). \quad (4.5)$$

5. THE PARTIAL HODOGRAPH TRANSFORMATION

To solve Problem 3.1, the free boundary problem (3.6)–(3.9), our strategy is to fix first the free boundary Γ_s . To achieve this, we introduce the following partial hodograph transformation:

$$\mathcal{P}\mathbf{x} = \mathbf{y} = (y_1, y_2, \mathbf{y}')^{\top} := (\varphi(\mathbf{x}), x_2 - \varphi_w(x_1, \mathbf{x}'), \mathbf{x}')^{\top},$$

which is invertible as $\partial_{x_1}\varphi \neq 0$, and we denote its inverse by

$$\mathcal{P}^{-1}\mathbf{y} = \mathbf{x} = (x_1, x_2, \mathbf{x}')^{\top} := (u(\mathbf{y}), y_2 + \varphi_w(u(\mathbf{y}), \mathbf{y}'), \mathbf{y}')^{\top}.$$

Taking the partial derivatives to the equation:

$$y_1 = \varphi \circ \mathcal{P}^{-1}(\mathbf{y})$$

with respect to $y_j, j = 1, \dots, n$, we have

$$\begin{aligned} \partial_{x_1}\varphi &= \frac{1}{\partial_{y_1}u} (1 + \partial_{x_1}\varphi_w(u, \mathbf{y}') \partial_{y_2}u), & \partial_{x_2}\varphi &= -\frac{\partial_{y_2}u}{\partial_{y_1}u}, \\ \partial_{x_j}\varphi &= -\frac{1}{\partial_{y_1}u} (\partial_{y_j}u - \partial_{x_j}\varphi_w(u, \mathbf{y}') \partial_{y_2}u), & j &= 3, \dots, n, \end{aligned}$$

that is,

$$D\varphi = \frac{1}{\partial_{y_1}u} (1 + \partial_{x_1}\varphi_w \partial_{y_2}u, -\partial_{y_2}u, -\partial_{y_3}u + \partial_{x_3}\varphi_w \partial_{y_2}u, \dots, -\partial_{y_n}u + \partial_{x_n}\varphi_w \partial_{y_2}u)^{\top}.$$

Thus, the Jacobi matrix of transformation \mathcal{P} is

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \begin{bmatrix} \partial_{x_1} \varphi & \partial_{x_2} \varphi & \partial_{x_3} \varphi & \cdots & \partial_{x_n} \varphi \\ -\partial_{x_1} \varphi_w & 1 & -\partial_{x_3} \varphi_w & \cdots & -\partial_{x_n} \varphi_w \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} := \frac{1}{\partial_{y_1} u} J^\top,$$

where

$$J := \begin{bmatrix} 1 + \partial_{x_1} \varphi_w \partial_{y_2} u & -\partial_{x_1} \varphi_w \partial_{y_1} u & 0 & \cdots & 0 \\ -\partial_{y_2} u & \partial_{y_1} u & 0 & \cdots & 0 \\ -\partial_{y_3} u + \partial_{x_3} \varphi_w \partial_{y_2} u & -\partial_{x_3} \varphi_w \partial_{y_1} u & \partial_{y_1} u & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\partial_{y_n} u + \partial_{x_n} \varphi_w \partial_{y_2} u & -\partial_{x_n} \varphi_w \partial_{y_1} u & 0 & \cdots & \partial_{y_1} u \end{bmatrix}.$$

After a direct computation, we also obtain

$$\begin{aligned} \frac{\partial(D_{\mathbf{x}} \varphi)}{\partial(D_{\mathbf{y}} u, u, \mathbf{y}')} &:= \begin{bmatrix} \frac{\partial(\partial_{x_1} \varphi)}{\partial(\partial_{y_1} u)} & \cdots & \frac{\partial(\partial_{x_1} \varphi)}{\partial(\partial_{y_n} u)} & \frac{\partial(\partial_{x_1} \varphi)}{\partial u} & \frac{\partial(\partial_{x_1} \varphi)}{\partial y_3} & \cdots & \frac{\partial(\partial_{x_1} \varphi)}{\partial y_n} \\ \frac{\partial(\partial_{x_2} \varphi)}{\partial(\partial_{y_1} u)} & \cdots & \frac{\partial(\partial_{x_2} \varphi)}{\partial(\partial_{y_n} u)} & \frac{\partial(\partial_{x_2} \varphi)}{\partial u} & \frac{\partial(\partial_{x_2} \varphi)}{\partial y_3} & \cdots & \frac{\partial(\partial_{x_2} \varphi)}{\partial y_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial(\partial_{x_n} \varphi)}{\partial(\partial_{y_1} u)} & \cdots & \frac{\partial(\partial_{x_n} \varphi)}{\partial(\partial_{y_n} u)} & \frac{\partial(\partial_{x_n} \varphi)}{\partial u} & \frac{\partial(\partial_{x_n} \varphi)}{\partial y_3} & \cdots & \frac{\partial(\partial_{x_n} \varphi)}{\partial y_n} \end{bmatrix}_{n \times (2n-1)} \\ &= \begin{bmatrix} -\frac{1}{(\partial_{y_1} u)^2} J & \frac{\partial_{y_2} u}{\partial_{y_1} u} W_1 & \frac{\partial_{y_2} u}{\partial_{y_1} u} W_3 & \cdots & \frac{\partial_{y_2} u}{\partial_{y_1} u} W_n \end{bmatrix}_{n \times (2n-1)}, \end{aligned}$$

where $W_j := (\partial_{x_j x_1} \varphi_w(u, \mathbf{y}'), 0, \partial_{x_j x_3} \varphi_w(u, \mathbf{y}'), \dots, \partial_{x_j x_n} \varphi_w(u, \mathbf{y}'))^\top$, with $j = 1, 3, \dots, n$.

Notice that

$$\begin{aligned} D_{\mathbf{x}}^2 \varphi &= \frac{\partial(D_{\mathbf{x}} \varphi)}{\partial(D_{\mathbf{y}} u, u, \mathbf{y}')} \begin{bmatrix} D_{\mathbf{y}}^2 u \\ (D_{\mathbf{y}} u)^\top \\ \frac{\partial \mathbf{y}'}{\partial \mathbf{y}} \end{bmatrix}_{(2n-1) \times n} \frac{\partial \mathbf{y}}{\partial \mathbf{x}} \\ &= -\frac{1}{(\partial_{y_1} u)^3} J D_{\mathbf{y}}^2 u J^\top + \frac{\partial_{y_2} u}{(\partial_{y_1} u)^2} W_1 (D_{\mathbf{y}} u)^\top J^\top + \frac{\partial_{y_2} u}{(\partial_{y_1} u)^2} W' \frac{\partial \mathbf{y}'}{\partial \mathbf{y}} J^\top \\ &:= -\frac{1}{(\partial_{y_1} u)^3} J D_{\mathbf{y}}^2 u J^\top + J_w, \end{aligned}$$

where $\frac{\partial \mathbf{y}'}{\partial \mathbf{y}} = \left[\frac{\partial y_i}{\partial y_j} \right]_{(n-2) \times n}$ ($i = 3, \dots, n$ and $j = 1, \dots, n$), $W' := [W_3, \dots, W_n]$ and

$$J_w = J_w(D^2 \varphi_w; Du, D\varphi_w(u, \mathbf{y}')) := \frac{\partial_{y_2} u}{(\partial_{y_1} u)^2} W_1 (D_{\mathbf{y}} u)^\top J^\top + \frac{\partial_{y_2} u}{(\partial_{y_1} u)^2} W' \frac{\partial \mathbf{y}'}{\partial \mathbf{y}} J^\top.$$

Then we have

$$\begin{aligned} \sum_{i,j=1}^n a_{ij} \partial_{x_i x_j} \varphi &= \text{Tr}(A^\top D_{\mathbf{x}}^2 \varphi) \\ &= -\frac{1}{(\partial_{y_1} u)^3} \text{Tr}(A J D_{\mathbf{y}}^2 u J^\top) + \text{Tr}(A J_w) \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{(\partial_{y_1} u)^3} \text{Tr}(J^\top A J D_{\mathbf{y}}^2 u) + \text{Tr}(A J_{\mathbf{w}}) \\
&= -\frac{1}{(\partial_{y_1} u)^3} \sum_{i,j=1}^n \tilde{a}_{ij} \partial_{y_i y_j} u + \Phi_{\mathbf{w}},
\end{aligned}$$

where $\tilde{A} = \tilde{A}^\top = J^\top A J := [\tilde{a}_{ij}]_{n \times n}$ with $\tilde{a}_{ij} = \tilde{a}_{ij}(Du; D\varphi^-(u, y_2, \mathbf{y}'); D\varphi_{\mathbf{w}}(u, \mathbf{y}'))$, and

$$\Phi_{\mathbf{w}} = \Phi_{\mathbf{w}}(D^2 \varphi_{\mathbf{w}}; Du; D\varphi_{\mathbf{w}}(u, \mathbf{y}')) := \text{Tr}(A J_{\mathbf{w}}).$$

Thus, under the partial hodograph transformation, the potential flow equation (3.6) becomes

$$-\frac{1}{(\partial_{y_1} u)^3} \sum_{i,j=1}^n \tilde{a}_{ij} \partial_{y_i y_j} u + \Phi_{\mathbf{w}} = \sum_{i,j=1}^n a_{ij} \partial_{x_i x_j} \varphi_0^- = 0. \quad (5.1)$$

Under the partial hodograph transformation, $\Gamma_{\mathbf{w}}$ becomes

$$\Gamma_1 = \{y_2 = 0, y_1 > 0, \mathbf{y}' \in \mathbb{R}^{n-2}\},$$

as shown in Fig. 5.1, and the boundary condition (3.7) on the wedge becomes

$$G_1(Du; D\varphi_{\mathbf{w}}(u, \mathbf{y}')) = 0, \quad (5.2)$$

where

$$\begin{aligned}
G_1(Du; D\varphi_{\mathbf{w}}) &:= \left(\partial_{x_1} \varphi_{\mathbf{w}} \partial_{x_1} \varphi^- - \partial_{x_2} \varphi^- + \sum_{j=3}^n \partial_{x_j} \varphi_{\mathbf{w}} \partial_{x_j} \varphi^- \right) \partial_{y_1} u \\
&\quad - (1 + |D\varphi_{\mathbf{w}}|^2) \partial_{y_2} u + \sum_{j=3}^n \partial_{x_j} \varphi_{\mathbf{w}} \partial_{y_j} u - \partial_{x_1} \varphi_{\mathbf{w}}.
\end{aligned}$$

The shock front Γ_s becomes a fixed boundary $\Gamma_2 = \{y_1 = 0, y_2 > 0\}$, and the boundary condition (3.8) becomes

$$G_2(Du; D\varphi_{\mathbf{w}}) = 0, \quad (5.3)$$

where

$$G_2(Du; D\varphi_{\mathbf{w}}) := H_s(D\varphi(Du, D\varphi_{\mathbf{w}}); D\varphi_0^-).$$

Finally, since the wedge edge:

$$\{\mathbf{x} \in \mathbb{R}^n : x_1 = \varphi_{\mathbf{w}}^e(\mathbf{x}'), x_2 = \varphi(x_1, \mathbf{x}'), \mathbf{x}' \in \mathbb{R}^{n-2}\}$$

is the intersection of the wedge surface and the shock-front, which yields that $\varphi \equiv 0$ on the edge, by condition (3.8). Thus, the tangential derivatives of φ on the edge should be 0. Then, under the partial hodograph transformation,

$$\partial_{y_j} u = \partial_{x_j} \varphi_{\mathbf{w}}^e(\mathbf{y}') \quad (5.4)$$

on edge $\{\mathbf{y} \in \mathbb{R}^n : y_1 = y_2 = 0, \mathbf{y}' \in \mathbb{R}^{n-2}\}$. Therefore, on the edge, $u(0, 0, \mathbf{y}') = \varphi_{\mathbf{w}}^e(\mathbf{y}')$.

Remark 5.1. For the background transonic shock solution $(\varphi_0^-(\mathbf{x}); \varphi_0^+(\mathbf{x}))$, we have

$$\varphi_0(\mathbf{x}) = \varphi_0^-(\mathbf{x}) - \varphi_0^+(\mathbf{x}) = x_1 (q_0^- \cos \alpha_{\mathbf{w}} - q_0^+ \cos \omega_1) - x_2 q_0^- \sin \alpha_{\mathbf{w}},$$

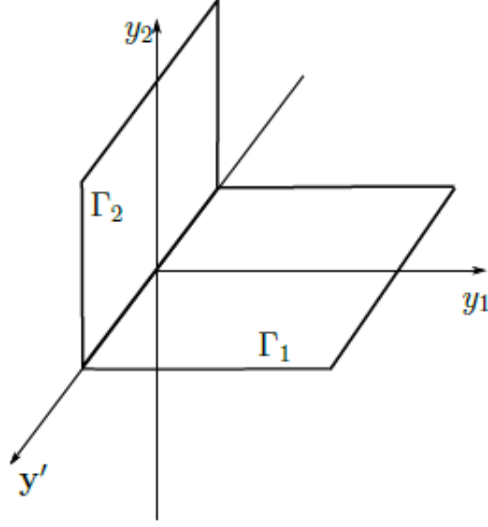


FIGURE 5.1. The domain after the partial hodograph transformation

and the corresponding partial hodograph transformation is

$$\mathcal{P}_0 \mathbf{x} = \mathbf{y} = (y_1, y_2, \mathbf{y}')^\top := (\varphi_0(\mathbf{x}), x_2, \mathbf{x}')^\top.$$

It is invertible since $\partial_{x_1} \varphi_0 = q_0^- \cos \alpha_w - q_0^+ \cos \omega_1 > 0$, and its inverse is

$$\mathcal{P}_0^{-1} \mathbf{y} = \mathbf{x} = (x_1, x_2, \mathbf{x}')^\top := (u_0(\mathbf{y}), y_2, \mathbf{y}')^\top,$$

where

$$u_0(\mathbf{y}) = \frac{y_1 + y_2 q_0^- \sin \alpha_w}{q_0^- \cos \alpha_w - q_0^+ \cos \omega_1}.$$

Then we have

$$\partial_{y_1} u_0 = \frac{1}{q_0^- \cos \alpha_w - q_0^+ \cos \omega_1} > 0.$$

Therefore, the partial hodograph transformation is still invertible in the case that $u(y)$ is a small perturbation of $u_0(y)$ such that $|\partial_{y_1} u - \partial_{y_1} u_0|$ is small enough.

We now solve the deduced fixed boundary value problem (5.1)–(5.4) near $u_0(\mathbf{y})$ and prove the following theorem which implies our main theorem, Theorem 3.1.

Theorem 5.2. *Let $(\varphi_0^-(\mathbf{x}); \varphi_0^+(\mathbf{x}))$ be the weak transonic shock solution that is represented by B on the shock polar in Fig. 2.1. Then there exist constants $\delta_0 > 0$ and $\sigma_s > 0$ depending on the background solution such that, for any $-1 < \sigma_\infty \leq 0 < \sigma_0 < \sigma_s$, if the wedge edge is not perturbed, that is, (3.10) holds, and the perturbation of the wedge surface Γ_w satisfies*

$$\|\varphi_w\|_{(2, \alpha; \mathbb{R}_+ \times \mathbb{R}^{n-2})}^{(\beta_0, \beta_\infty)} \leq \delta \leq \delta_0 \quad (5.5)$$

for $\beta_0 = 1 + \alpha - \sigma_0$ and $\beta_\infty = 1 + \alpha - \sigma_\infty$, then there exists a unique solution $u(\mathbf{y})$ to the boundary value problem (5.1)–(5.4) satisfying

$$\|u - u_0\|_{(2, \alpha; \mathcal{D})}^{(\beta_0, \beta_\infty)} \leq K\delta, \quad (5.6)$$

where $K > 0$ depends on the background solution, but is independent of $\delta_0 > 0$.

Remark 5.3. Theorem 5.2 will be proved via a nonlinear iteration scheme, in which the linearized problem plays an important role. The linearized problem can be reformulated as an oblique derivative boundary value problem of the Poisson equation, which can be solved without conditions (5.4) on the edge, according to Theorem 4.4. Thus, it looks like that problem (5.1)–(5.4) is over-determined, which is exactly the instability mechanism for strong transonic solutions shown in [25]. In Theorem 5.2, since $\sigma_0 > 0$, estimate (5.6) yields that $Du - Du_0 \equiv 0$ on edge $\{y_1 = 0, y_2 = 0, \mathbf{y}' \in \mathbb{R}^{n-2}\}$, which indicates that, as the wedge edge is not perturbed such that (3.10) holds, conditions (5.4) hold automatically, and solution $u(y)$ to problem (5.1)–(5.3) is indeed a solution to problem (5.1)–(5.4). Thus, the instability mechanism for strong transonic shocks shown in [25] may not happen for weak transonic shocks.

6. THE LINEARIZED PROBLEM ON THE BACKGROUND SHOCK SOLUTION

To prove Theorem 5.2, we first linearize the nonlinear problem (5.1)–(5.3) on the background shock solution, then solve this corresponding linearized elliptic problem, and finally develop a nonlinear iteration scheme that is proved to be contractive. Therefore, the well-posedness theory for the linearized problem also plays an important role in our approach for the stability analysis of the transonic shocks.

Let

$$u(\mathbf{y}) = u_0(\mathbf{y}) + \dot{u}(\mathbf{y}).$$

Then the linearized problem for the nonlinear problem (5.1)–(5.3) on the background solution $u_0(\mathbf{y})$ reads

$$\begin{aligned} \sum_{i,j=1}^n \tilde{a}_{ij}^0 \partial_{y_i y_j} \dot{u} &= -(\partial_{y_1} u_0)^3 f(\dot{u}; \varphi_w) && \text{in } \mathcal{D}, \\ \nabla_{Du} G_j(Du_0; 0) \cdot D\dot{u} &= g_j(\dot{u}; \varphi_w) && \text{on } \Gamma_j, j = 1, 2, \end{aligned}$$

where

$$\tilde{a}_{ij}^0 = \tilde{a}_{ij}(Du_0; D\varphi_0^-; 0),$$

and the iteration terms $f(\dot{u}; \varphi_w)$ and $g_j(\dot{u}; \varphi_w)$, $j = 1, 2$, will be specified later in §7. Note that

$$\begin{aligned} \nabla_{Du} G_1(Du; D\varphi_w) &= \left(\frac{\partial(D\varphi)}{\partial(Du)} \right)^\top \nabla_{D\varphi} H_w(D\varphi; D\varphi_w) = -\frac{1}{(\partial_{y_1} u)^2} J^\top \nabla_{D\varphi} H_w, \\ \nabla_{Du} G_2(Du; D\varphi_w) &= \left(\frac{\partial(D\varphi)}{\partial(Du)} \right)^\top \nabla_{D\varphi} H_s(D\varphi; D\varphi_0^-) = -\frac{1}{(\partial_{y_1} u)^2} J^\top \nabla_{D\varphi} H_s. \end{aligned}$$

We have

$$\nabla_{Du} G_1(Du_0; 0) = -\frac{1}{(\partial_{y_1} u_0)^2} J_0^\top \nabla_{D\varphi} H_w(D\varphi_0; 0),$$

$$\nabla_{Du} G_2(Du_0; 0) = -\frac{1}{(\partial_{y_1} u_0)^2} J_0^\top \nabla_{D\varphi} H_s(D\varphi_0; D\varphi_0^-),$$

where $\nabla_{D\varphi} H_w(D\varphi_0; 0) = (0, 1, 0, \dots, 0)^\top$, $\nabla_{D\varphi} H_s(D\varphi_0; D\varphi_0^-) := \boldsymbol{\nu} = (\nu_1, \dots, \nu_n)^\top$ which is exactly the outer unit normal of the shock balloon, and

$$J_0 := J(Du_0; 0) = \frac{1}{q_0^- \cos \alpha_w - q_0^+ \cos \omega_1} \begin{bmatrix} q_0^- \cos \alpha_w - q_0^+ \cos \omega_1 & 0 & 0 & \cdots & 0 \\ -q_0^- \sin \alpha_w & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Therefore, in this section, we deal with the linear boundary value problem of elliptic type (u is still denoted as the unknown function):

$$\sum_{i,j=1}^n \tilde{a}_{ij}^0 \partial_{y_i y_j} u = \hat{f} \quad \text{in } \mathcal{D}, \quad (6.1)$$

$$-q_0^- \sin \alpha_w \partial_{y_1} u + \partial_{y_2} u = \hat{g}_1 \quad \text{on } \Gamma_1, \quad (6.2)$$

$$(Du)^\top \cdot J_0^\top \boldsymbol{\nu} = \hat{g}_2 \quad \text{on } \Gamma_2. \quad (6.3)$$

Then Theorem 4.4 can be employed to establish the following well-posedness theorem for problem (6.1)–(6.3).

Theorem 6.1. *Assume that*

$$\frac{\nu_1}{\nu_2} > 0. \quad (6.4)$$

Then there exists a constant $\sigma_s > 0$ depending only on the parameters of the unperturbed background transonic shock solution such that, for any

$$-1 < \sigma_\infty \leq 0 < \sigma_0 < \sigma_s, \quad (6.5)$$

if

$$\hat{f} \in \mathcal{C}_{\beta_0, \beta_\infty}^{0, \alpha}(\mathcal{D}), \quad \hat{g}_j \in \mathcal{C}_{\beta_0, \beta_\infty}^{1, \alpha}(\Gamma_j), \quad j = 1, 2,$$

with $\beta_0 = 1 + \alpha - \sigma_0$ and $\beta_\infty = 1 + \alpha - \sigma_\infty$, there exists a unique solution $u \in \mathcal{C}_{\beta_0, \beta_\infty}^{2, \alpha}(\mathcal{D})$ to the boundary value problem (6.1)–(6.3) satisfying the following estimate:

$$\|u\|_{(2, \alpha; \mathcal{D})}^{(\beta_0, \beta_\infty)} \leq C \left(\|\hat{f}\|_{(0, \alpha; \mathcal{D})}^{(\beta_0, \beta_\infty)} + \sum_{j=1}^2 \|\hat{g}_j\|_{(1, \alpha; \Gamma_j)}^{(\beta_0, \beta_\infty)} \right). \quad (6.6)$$

Remark 6.2. For the weak transonic shock solution represented by point B on the shock polar, condition (6.4) holds. However, $\frac{\nu_1}{\nu_2} < 0$ for the strong transonic shock solution represented by point A , and $\frac{\nu_1}{\nu_2} = 0$ when point A coincides with point B . That is, condition (6.4) does not hold for these two cases. See Figures 6.2–6.3.

We remark that, for the M-D case, if the incoming supersonic flow is perpendicular to the edge, the background shock solution is the same as the shock solution for the 2-D flow. However, there are differences between the M-D flow and 2-D flow, so that it is worth of

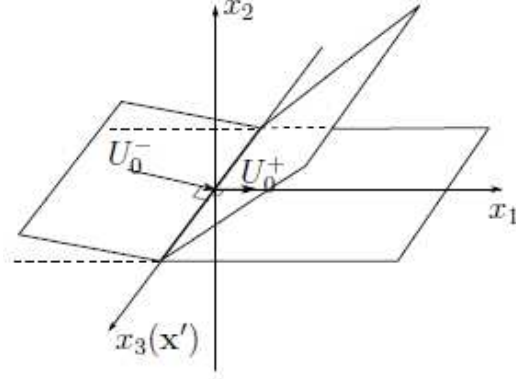


FIGURE 6.1. The uniform incoming flow is perpendicular to the edge

dealing with the perpendicular case independently, for later comparison between these two cases.

6.1. The case that the incoming supersonic flow is perpendicular to the edge.

In this case, $\omega_1 = 0$ (see Fig. 6.1), and the corresponding potential functions are

$$\begin{aligned}\varphi_0^-(\mathbf{x}) &= x_1 q_0^- \cos \alpha_w - x_2 q_0^- \sin \alpha_w, \\ \varphi_0^+(\mathbf{x}) &= x_1 q_0^+.\end{aligned}$$

Then we have

$$\varphi_0(\mathbf{x}) = \varphi_0^-(\mathbf{x}) - \varphi_0^+(\mathbf{x}) = x_1 (q_0^- \cos \alpha_w - q_0^+) - x_2 q_0^- \sin \alpha_w,$$

and the corresponding partial hodograph transformation becomes

$$\mathcal{P}_0 \mathbf{x} = \mathbf{y} = (y_1, y_2, \mathbf{y}')^\top := (\varphi_0(\mathbf{x}), x_2, \mathbf{x}')^\top,$$

with its inverse

$$\mathcal{P}_0^{-1} \mathbf{y} = \mathbf{x} = (x_1, x_2, \mathbf{x}')^\top := (u_0(\mathbf{y}), y_2, \mathbf{y}')^\top,$$

where

$$u_0(\mathbf{y}) = \frac{1}{q_0^- \cos \alpha_w - q_0^+} (y_1 + y_2 q_0^- \sin \alpha_w).$$

Let

$$A_0 = A(D\varphi_0^+) = [a_{ij}^0]_{n \times n} = (c_0^+)^2 \begin{bmatrix} 1 - (M_0^+)^2 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix},$$

and

$$J_0 = J(Du_0; 0) = \frac{1}{q_0^- \cos \alpha_w - q_0^+} \begin{bmatrix} q_0^- \cos \alpha_w - q_0^+ & 0 & \cdots & 0 \\ -q_0^- \sin \alpha_w & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Then, in equation (6.1),

$$\tilde{A}_0 := [\tilde{a}_{ij}^0]_{n \times n} = J_0^\top A_0 J_0.$$

Moreover, in the boundary condition (6.3), the unit normal $\boldsymbol{\nu} = (\nu_1, \nu_2, 0, \dots, 0)^\top$.

Let

$$Y = P\mathbf{y},$$

where $P = A_0^{-\frac{1}{2}}(J_0^{-1})^\top$ is a nonsingular matrix, with

$$A_0^{-\frac{1}{2}} = \frac{1}{c_0^+} \begin{bmatrix} \frac{1}{\sqrt{1-(M_0^+)^2}} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix},$$

$$J_0^{-1} = (q_0^- \cos \alpha_w - q_0^+) \begin{bmatrix} \frac{1}{q_0^- \cos \alpha_w - q_0^+} & 0 & 0 & \cdots & 0 \\ \frac{q_0^- \sin \alpha_w}{q_0^- \cos \alpha_w - q_0^+} & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Then

$$\frac{\partial Y}{\partial \mathbf{y}} = [\partial_{y_j} Y_i]_{n \times n} = P.$$

For $u(\mathbf{y}) = u(Y(\mathbf{y}))$,

$$D_{\mathbf{y}} u = [\partial_{y_i} u]_{n \times 1} = P^\top [\partial_{Y_i} u]_{n \times 1} = P^\top D_Y u,$$

and

$$D_{\mathbf{y}}^2 u = [\partial_{y_i y_j} u]_{n \times n} = P^\top D_Y^2 u P.$$

Thus we have

$$\begin{aligned} \sum_{i,j=1}^n \tilde{a}_{ij}^0 \partial_{y_i y_j} u &= \text{Tr}(\tilde{A}_0^\top D_{\mathbf{y}}^2 u) = \text{Tr}(J_0^\top A_0 J_0 \cdot P^\top D_Y^2 u P) \\ &= \text{Tr}(P J_0^\top A_0 J_0 P^\top \cdot D_Y^2 u) \\ &= \Delta_Y u. \end{aligned}$$

Then equation (6.1) becomes

$$\Delta_Y u = \hat{f}. \quad (6.7)$$

The boundaries Γ_1 and Γ_2 become

$$\begin{aligned} \bar{\Gamma}_1 &= \{Y_2 = 0, Y_1 > 0, Y' \in \mathbb{R}^{n-2}\}, \\ \bar{\Gamma}_2 &= \{Y_2 = \tan \omega_s Y_1, Y_1 > 0, Y' \in \mathbb{R}^{n-2}\}, \end{aligned}$$

where

$$\tan \omega_s = \frac{q_0^- \cos \alpha_w - q_0^+}{q_0^- \sin \alpha_w} \sqrt{1 - (M_0^+)^2}.$$

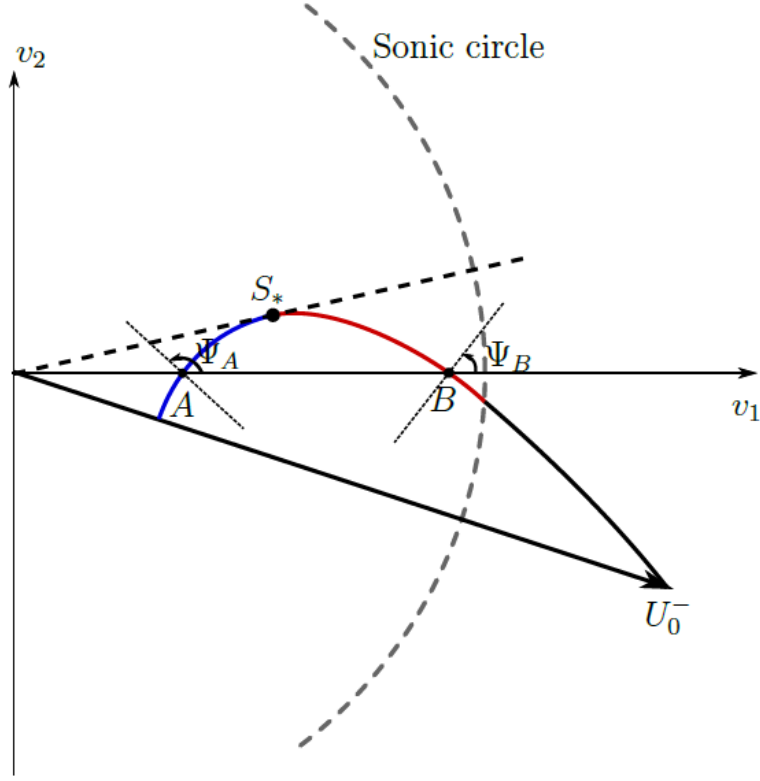


FIGURE 6.2. Condition (6.11) and the shock polar: Perpendicular cases

The boundary condition (6.2) becomes

$$\partial_{Y_2} u = \bar{g}_1 := \frac{c_0^+}{q_0^- \cos \alpha_w - q_0^+} \hat{g}_1, \quad (6.8)$$

and the boundary condition (6.3) becomes

$$\frac{\nu_1}{\sqrt{1 - (M_0^+)^2}} \partial_{Y_1} u + \nu_2 \partial_{Y_2} u = \bar{g}_2 := c_0^+ \hat{g}_2, \quad (6.9)$$

which can be rewritten under the polar coordinates for (Y_1, Y_2) as

$$\frac{1}{r} \partial_\omega u + \tan(\omega_s + \Phi_s) \partial_r u = \frac{\cos \Phi_s}{\cos(\omega_s + \Phi_s)} \frac{\bar{g}_2}{\nu_2},$$

where

$$\tan \Phi_s = \frac{1}{\sqrt{1 - (M_0^+)^2}} \frac{\nu_1}{\nu_2}.$$

Remark 6.3. Applying Theorem 4.4, we conclude that problem (6.7)–(6.9) can be well-posed in the weighted Hölder space $\mathcal{C}_\beta^{2,\alpha}(\mathcal{D})$ for any admissible weight $\beta := 1 + \alpha - \sigma$ with σ satisfying

$$-1 < \sigma < \frac{\Phi_s}{\omega_s}.$$

On the other hand, we also need solution u to be physically reasonable such that the velocity is bounded. Then Du should be bounded in \mathcal{D} . Therefore, there exist valid admissible weights β which are further applicable to our stability problem, only when constant $\sigma_s := \frac{\Phi_s}{\omega_s}$ satisfies the condition:

$$\sigma_s > 0, \quad (6.10)$$

that is,

$$\Phi_s > 0, \quad \text{or equivalently} \quad \frac{\nu_1}{\nu_2} > 0. \quad (6.11)$$

Let $\Psi = \operatorname{arccot}(\frac{\nu_1}{\nu_2})$. Then condition (6.11) yields that

$$0 < \Psi < \frac{\pi}{2}. \quad (6.12)$$

We remark that Ψ equals the angle between the velocity vector and the outer normal of the shock polar (see Fig. 6.2). Then we can observe that, for the strong transonic shock solution represented by point A on the shock polar,

$$\frac{\pi}{2} < \Psi_A < \pi;$$

while, for the weak transonic shock solution represented by point B ,

$$0 < \Psi_B < \frac{\pi}{2}.$$

This means that, via this analysis, the following well-posedness theorem, which is a direct consequence of Theorem 4.4, can be established only for weak transonic shocks, that is, the shock solution represented by point B .

Theorem 6.4. *Suppose that (6.10) holds. Let*

$$-1 < \sigma_\infty \leq 0 < \sigma_0 < \sigma_s. \quad (6.13)$$

Assume that

$$\hat{f} \in \mathcal{C}_{\beta_0, \beta_\infty}^{0, \alpha}(\mathcal{D}), \quad \bar{g}_j \in \mathcal{C}_{\beta_0, \beta_\infty}^{1, \alpha}(\Gamma_j), \quad j = 1, 2,$$

where $\beta_0 = 1 + \alpha - \sigma_0$ and $\beta_\infty = 1 + \alpha - \sigma_\infty$. Then there exists a unique solution $u \in \mathcal{C}_{\beta_0, \beta_\infty}^{2, \alpha}(\mathcal{D})$ to the boundary value problem (6.7)–(6.9) satisfying the following estimate:

$$\|u\|_{(2, \alpha; \mathcal{D})}^{(\beta_0, \beta_\infty)} \leq C \left(\|\hat{f}\|_{(0, \alpha; \mathcal{D})}^{(\beta_0, \beta_\infty)} + \sum_{j=1}^2 \|\bar{g}_j\|_{(1, \alpha; \Gamma_j)}^{(\beta_0, \beta_\infty)} \right). \quad (6.14)$$

Then Theorem 6.1 is a direct consequence of Theorem 6.4.

6.2. The general case that the incoming supersonic flow may not be perpendicular to the edge. In this case, $|\omega_1| < \frac{\pi}{2}$ (see Fig. 3.1). Taking $P = (J_0^{-1})^\top$, equation (6.1) becomes

$$\begin{aligned} & (1 - M^2 \cos^2 \omega_1) \partial_{Y_1 Y_1} u - 2M^2 \cos \omega_1 \cos \omega_3 \partial_{Y_1 Y_3} u \\ & + (1 - M^2 \cos^2 \omega_3) \partial_{Y_3 Y_3} u + \partial_{Y_2 Y_2} u + \sum_{i=4}^n \partial_{Y_i Y_i} u = \frac{\hat{f}}{(c_0^+)^2}, \end{aligned} \quad (6.15)$$

where $M = \frac{q_0^+}{c_0^+}$. The boundaries Γ_1 and Γ_2 become

$$\begin{aligned}\bar{\Gamma}_1 &= \{Y_2 = 0, Y_1 > 0, Y_3 \in \mathbb{R}\}, \\ \bar{\Gamma}_2 &= \{Y_2 = \tan \omega_s Y_1, Y_1 > 0, Y_3 \in \mathbb{R}\},\end{aligned}$$

where

$$\tan \omega_s = \frac{q_0^- \cos \alpha_w - q_0^+ \cos \omega_1}{q_0^- \sin \alpha_w}.$$

The boundary condition (6.2) becomes

$$\partial_{Y_2} u = \bar{g}_1 := \frac{\hat{g}_1}{q_0^- \cos \alpha_w - q_0^+ \cos \omega_1}, \quad (6.16)$$

and the boundary condition (6.3) becomes

$$\nu_1 \partial_{Y_1} u + \nu_2 \partial_{Y_2} u + \nu_3 \partial_{Y_3} u = \bar{g}_2 := \hat{g}_2. \quad (6.17)$$

Now we rewrite the operator in equation (6.15) into the Laplacian. Let

$$P_0 = \begin{bmatrix} \frac{1}{\sqrt{1-M^2 \cos^2 \omega_1}} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{M^2 \cos \omega_1 \cos \omega_3}{\sqrt{(1-M^2)(1-M^2 \cos^2 \omega_1)}} & 0 & \sqrt{\frac{1-M^2 \cos^2 \omega_1}{1-M^2}} & 0 \\ 0 & 0 & 0 & I_{n-3} \end{bmatrix}_{n \times n},$$

and

$$X = P_0 Y.$$

Then equation (6.15) becomes

$$\Delta_X u = \hat{f}, \quad (6.18)$$

and the boundaries $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ become

$$\begin{aligned}\tilde{\Gamma}_1 &= \{X_2 = 0, X_1 > 0, X' \in \mathbb{R}^{n-2}\}, \\ \tilde{\Gamma}_2 &= \{X_2 = \tan \tilde{\omega}_s X_1, X_1 > 0, X' \in \mathbb{R}^{n-2}\}\end{aligned}$$

with

$$\tan \tilde{\omega}_s = \frac{q_0^- \cos \alpha_w - q_0^+ \cos \omega_1}{q_0^- \sin \alpha_w} \sqrt{1 - M^2 \cos^2 \omega_1},$$

the boundary condition (6.16) becomes

$$\partial_{X_2} u = \bar{g}_1, \quad (6.19)$$

and the boundary condition (6.17) becomes

$$\tilde{\nu}_1 \partial_{X_1} u + \tilde{\nu}_2 \partial_{X_2} u + \tilde{\nu}_3 \partial_{X_3} u = \bar{g}_2, \quad (6.20)$$

where

$$\tilde{\nu} = \begin{bmatrix} \tilde{\nu}_1 \\ \tilde{\nu}_2 \\ \tilde{\nu}_3 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = P_0 \nu = \begin{bmatrix} \frac{\nu_1}{\sqrt{1 - M^2 \cos^2 \omega_1}} \\ \nu_2 \\ \frac{\sqrt{1 - M^2 \cos^2 \omega_1}}{\sqrt{1 - M^2}} \left(\frac{M^2 \cos \omega_1 \cos \omega_3}{1 - M^2 \cos^2 \omega_1} \nu_1 + \nu_3 \right) \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The boundary condition can be rewritten under the polar coordinates for (X_1, X_2) as

$$\frac{1}{r} \partial_\omega u + \tan(\tilde{\omega}_s + \tilde{\Phi}_s) \partial_r u + \frac{\tilde{\nu}_3}{\nu_2} \partial_{X_3} u = \frac{\tilde{\Phi}_s}{\cos(\tilde{\omega}_s + \tilde{\Phi}_s)} \frac{\tilde{g}_2}{\tilde{\nu}_2}, \quad (6.21)$$

where

$$\tan \tilde{\Phi}_s = \frac{1}{\sqrt{1 - M^2 \cos^2 \omega_1}} \frac{\nu_1}{\nu_2}.$$

Remark 6.5. By Theorem 4.4, we conclude that problem (6.18)–(6.21) can be well-posed in the weighted Hölder space $\mathcal{C}_\beta^{2,\alpha}(\mathcal{D})$ for any admissible weight $\beta := 1 + \alpha - \sigma$ with σ satisfying

$$-1 < \sigma < \frac{\tilde{\Phi}_s}{\tilde{\omega}_s}.$$

On the other hand, we also need solution u to be physically reasonable such that the velocity is bounded. Then Du should be bounded in \mathcal{D} . Therefore, there exist valid admissible weights β which are further applicable to our stability problem, only when constant $\tilde{\sigma}_s := \frac{\tilde{\Phi}_s}{\tilde{\omega}_s}$ satisfies the condition:

$$\tilde{\sigma}_s > 0, \quad (6.22)$$

that is,

$$\tilde{\Phi}_s > 0, \quad \text{or equivalently,} \quad \frac{\nu_1}{\nu_2} > 0. \quad (6.23)$$

Let

$$\Psi := \operatorname{arccot}\left(\frac{\nu_1}{\nu_2}\right).$$

Then (6.23) yields that

$$0 < \Psi < \frac{\pi}{2}. \quad (6.24)$$

If the shock polar is projected onto the (v_1, v_2) -plane (see Fig. 6.3), then Ψ is the exact angle between the projection of the velocity behind the shock-front and the projection of the outer normal of the shock balloon. Moreover, we can observe that, for the strong transonic shock solution represented by point A on the shock polar,

$$\frac{\pi}{2} < \Psi_A < \pi;$$

while, for the weak transonic shock solution represented by point B ,

$$0 < \Psi_B < \frac{\pi}{2}.$$

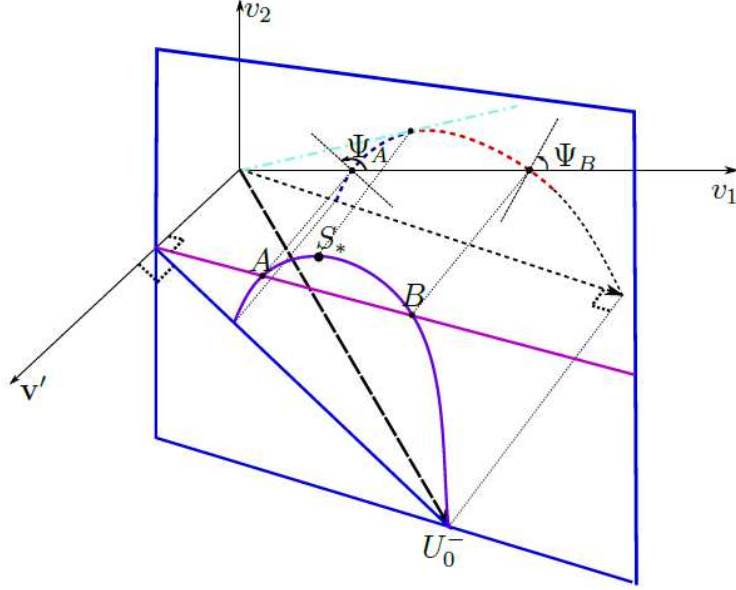


FIGURE 6.3. Condition (6.23) and the shock polar: General cases

This also means that, via this analysis, the following well-posedness theorem, which is a direct consequence of Theorem 4.4, can only be established for weak transonic shocks, that is, the shock solution represented by point B . Therefore, we have the following similar theorem to Theorem 6.4.

Theorem 6.6. *Suppose that (6.23) holds. Let*

$$-1 < \sigma_\infty \leq 0 < \sigma_0 < \tilde{\sigma}_s. \quad (6.25)$$

Assume that

$$\hat{f} \in \mathcal{C}_{\beta_0, \beta_\infty}^{0, \alpha}(\mathcal{D}), \quad \bar{g}_j \in \mathcal{C}_{\beta_0, \beta_\infty}^{1, \alpha}(\Gamma_j), \quad j = 1, 2,$$

where $\beta_0 = 1 + \alpha - \sigma_0$ and $\beta_\infty = 1 + \alpha - \sigma_\infty$. Then there exists a unique solution $u \in \mathcal{C}_{\beta_0, \beta_\infty}^{2, \alpha}(\mathcal{D})$ to the boundary value problem (6.1)–(6.3) satisfying the following estimate:

$$\|u\|_{(2, \alpha; \mathcal{D})}^{(\beta_0, \beta_\infty)} \leq C \left(\|\hat{f}\|_{(0, \alpha; \mathcal{D})}^{(\beta_0, \beta_\infty)} + \sum_{j=1}^2 \|\bar{g}_j\|_{(1, \alpha; \Gamma_j)}^{(\beta_0, \beta_\infty)} \right). \quad (6.26)$$

It can be seen that Theorem 6.1 is a direct consequence of Theorem 6.6.

7. THE ITERATION SCHEME

Now we develop the iteration scheme to solve the nonlinear problem (5.1)–(5.3) to establish Theorem 5.2.

Let $-1 < \sigma_\infty \leq 0 < \sigma_0$ be the constants defined in (6.13) or (6.25). Define

$$O_\epsilon^{(\sigma_0, \sigma_\infty)} = \left\{ u \in \mathcal{C}_{\beta_0, \beta_\infty}^{2, \alpha}(\mathcal{D}) : \|u\|_{(2, \alpha; \mathcal{D})}^{(\beta_0, \beta_\infty)} \leq \epsilon \right\},$$

where $\beta_0 = 1 + \alpha - \sigma_0$ and $\beta_\infty = 1 + \alpha - \sigma_\infty$.

Let

$$\begin{aligned}\mathcal{D} &= \left\{ \mathbf{y} = (y_1, y_2, \mathbf{y}')^\top \in \mathbb{R}^n : y_1 > 0, y_2 > 0, \mathbf{y}' \in \mathbb{R}^{n-2} \right\}, \\ \Gamma_1 &= \left\{ \mathbf{y} = (y_1, y_2, \mathbf{y}')^\top \in \mathbb{R}^n : y_1 > 0, y_2 = 0, \mathbf{y}' \in \mathbb{R}^{n-2} \right\}, \\ \Gamma_2 &= \left\{ \mathbf{y} = (y_1, y_2, \mathbf{y}')^\top \in \mathbb{R}^n : y_1 = 0, y_2 > 0, \mathbf{y}' \in \mathbb{R}^{n-2} \right\}.\end{aligned}$$

Let

$$\begin{aligned}u(\mathbf{y}) &= u_0(\mathbf{y}) + \dot{u}(\mathbf{y}), \\ v(\mathbf{y}) &= u_0(\mathbf{y}) + \dot{v}(\mathbf{y}).\end{aligned}$$

Assume that $\dot{v}(\mathbf{y}) \in O_{K\delta}^{(\sigma_0, \sigma_\infty)}$ with $K > 0$ and $0 < \delta \ll 1$ to be determined later. The iteration scheme is determined by solving the linearized elliptic boundary value problem:

$$\sum_{i,j=1}^n \tilde{a}_{ij}^0 \partial_{y_i y_j} \dot{u} = f(\dot{v}; \varphi_w) \quad \text{in } \mathcal{D}, \quad (7.1)$$

$$\nabla_{Du} G_j(Du_0; 0) \cdot D\dot{u} = g_j(\dot{v}; \varphi_w) \quad \text{on } \Gamma_j, j = 1, 2, \quad (7.2)$$

where

$$\begin{aligned}f(\dot{v}; \varphi_w) &= \sum_{i,j=1}^n \tilde{a}_{ij}^0 \partial_{y_i y_j} \dot{v} - \frac{(\partial_{y_1} u_0)^3}{(\partial_{y_1} v)^3} \left(\sum_{i,j=1}^n \tilde{a}_{ij} \partial_{y_i y_j} v - (\partial_{y_1} v)^3 \Phi_w(D^2 \varphi_w; Dv; D\varphi_w(v, \mathbf{y}')) \right), \\ g_j(\dot{v}; \varphi_w) &= \nabla_{Du} G_j(Du_0; 0) \cdot D\dot{v} - G_j(Dv; D\varphi_w(v, \mathbf{y}')), \quad j = 1, 2.\end{aligned}$$

Lemma 7.1. *There exist a sufficiently small constant $\delta_1 > 0$ and a constant $K > 1$ that is independent of δ_1 such that, for any $0 < \delta \leq \delta_1$ and $\dot{v}(\mathbf{y}) \in O_{K\delta}^{(\sigma_0, \sigma_\infty)}$, there exists a unique solution $\dot{u}(\mathbf{y}) \in O_{K\delta}^{(\sigma_0, \sigma_\infty)}$ to the boundary value problem (7.1)–(7.2). That is, the mapping*

$$\mathcal{J} : \dot{v} \mapsto \dot{u}$$

is well-defined in $O_{K\delta}^{(\sigma_0, \sigma_\infty)}$.

Proof. Notice that

$$\begin{aligned}\tilde{a}_{ij} - \tilde{a}_{ij}^0 &= \tilde{a}_{ij}(Dv; D\varphi_0^-; D\varphi_w(v, \mathbf{y}')) - \tilde{a}_{ij}(Du_0; D\varphi_0^-; 0) \\ &= \int_0^1 \nabla_{Du} \tilde{a}_{ij}(t) dt \cdot D\dot{v} + \int_0^1 \nabla_{D\varphi_w} \tilde{a}_{ij}(t) dt \cdot D\varphi_w(v, \mathbf{y}'),\end{aligned}$$

where

$$\begin{aligned}\nabla_{Du} \tilde{a}_{ij}(t) &:= \nabla_{Du} \tilde{a}_{ij}(Du_0 + tD\dot{v}; D\varphi_0^-; tD\varphi_w(v, \mathbf{y}')), \\ \nabla_{D\varphi_w} \tilde{a}_{ij}(t) &:= \nabla_{D\varphi_w} \tilde{a}_{ij}(Du_0 + tD\dot{v}; D\varphi_0^-; tD\varphi_w(v, \mathbf{y}')).\end{aligned}$$

Since

$$\|D\varphi_w(u_0, \mathbf{y}')\|_{C_\alpha^{0,\alpha}(\mathcal{D})} \leq \delta,$$

and \tilde{a}_{ij} is a smooth function with respect to all of its parameters, we have

$$\|\tilde{a}_{ij} - \tilde{a}_{ij}^0\|_{C_\alpha^{0,\alpha}(\mathcal{D})} \leq CK\delta.$$

Since

$$\left\| \frac{(\partial_{y_1} u_0)^3}{(\partial_{y_1} v)^3} - 1 \right\|_{C_\alpha^{0,\alpha}(\mathcal{D})} \leq CK\delta,$$

we obtain via a direct computation and employing Proposition 4.3 that

$$\left\| \sum_{i,j=1}^n \tilde{a}_{ij}^0 \partial_{y_i y_j} \dot{v} - \frac{(\partial_{y_1} u_0)^3}{(\partial_{y_1} v)^3} \sum_{i,j=1}^n \tilde{a}_{ij} \partial_{y_i y_j} v \right\|_{(0,\alpha;\mathcal{D})}^{(\beta_0,\beta_\infty)} \leq CK^2 \delta^2.$$

We can also analogously verify that

$$\|\Phi_w\|_{(0,\alpha;\mathcal{D})}^{(\beta_0,\beta_\infty)} \leq C(1 + K\delta)\delta,$$

since it is easy to check that

$$\Phi_w = \sum_{i,j \neq 2} \partial_{x_i x_j} \varphi_w(u, \mathbf{y}') \Phi_w^{ij}(Du; D\varphi_w),$$

with Φ_w^{ij} being some smooth functions of Du and $D\varphi_w$.

Thus, we obtain that

$$\|f(\dot{v}; \varphi_w)\|_{(0,\alpha;\mathcal{D})}^{(\beta_0,\beta_\infty)} \leq CK^2 \delta^2.$$

Notice that

$$\begin{aligned} \nabla_{Du} G_1(Du_0; 0) \cdot D\dot{v} - G_1(Dv; 0) &= -\frac{1}{2} (D\dot{v})^\top \int_0^1 \nabla_{Du}^2 G_1(t) dt D\dot{v}, \\ G_1(Dv; D\varphi_w(v, \mathbf{y}')) - G_1(Dv; 0) &= \int_0^1 \nabla_{D\varphi_w} G_1(t) dt \cdot D\varphi_w(v, \mathbf{y}'), \end{aligned}$$

where

$$\begin{aligned} \nabla_{Du}^2 G_1(t) &:= \nabla_{Du}^2 G_1(Du_0 + tD\dot{v}; 0), \\ \nabla_{D\varphi_w} G_1(t) &:= \nabla_{D\varphi_w} G_1(Dv; tD\varphi_w(v, \mathbf{y}')). \end{aligned}$$

Thus, we also obtain

$$\|g_1(\dot{v}; \varphi_w)\|_{(1,\alpha;\Gamma_1)}^{(\beta_0,\beta_\infty)} \leq C(1 + K^2\delta)\delta.$$

Similarly, we have

$$\|g_2(\dot{v}; \varphi_w)\|_{(1,\alpha;\Gamma_2)}^{(\beta_0,\beta_\infty)} \leq C(1 + K^2\delta)\delta.$$

Therefore, there exists a unique solution $\dot{u} \in \mathcal{C}_{\beta_0,\beta_\infty}^{2,\alpha}(\mathcal{D})$ with the following estimate:

$$\begin{aligned} \|\dot{u}\|_{(2,\alpha;\mathcal{D})}^{(\beta_0,\beta_\infty)} &\leq C \left(\|f(\dot{v}; \varphi_w)\|_{(0,\alpha;\mathcal{D})}^{(\beta_0,\beta_\infty)} + \sum_{j=1,2} \|g_j(\dot{v}; \varphi_w)\|_{(1,\alpha;\Gamma_j)}^{(\beta_0,\beta_\infty)} \right) \\ &\leq C(1 + K^2\delta)\delta \leq C_1\delta \end{aligned}$$

for given K and sufficiently small δ .

Fix $K = C_1$ from now on. Then we find that $\dot{u}(\mathbf{y}) \in O_{K\delta}^{(\sigma_0,\sigma_\infty)}$, and the mapping

$$\mathcal{J} : \dot{v} \mapsto \dot{u}$$

is well-defined in $O_{K\delta}^{(\sigma_0,\sigma_\infty)}$. This completes the proof. \square

Lemma 7.2. *There exists a sufficiently small constant $\delta_0 > 0$ such that, for any $0 < \delta \leq \delta_0$, \mathcal{J} is a contraction mapping in $O_{K\delta}^{(\sigma_0,\sigma_\infty)}$.*

Proof. Denote that $(\dot{u}_1, \dot{u}_2) := \mathcal{J}(\dot{v}_1, \dot{v}_2)$. Then we have

$$\sum_{i,j=1}^n \tilde{a}_{ij}^0 \partial_{y_i y_j} (\dot{u}_1 - \dot{u}_2) = f(\dot{v}_1; \varphi_w) - f(\dot{v}_2; \varphi_w) \quad \text{in } \mathcal{D}, \quad (7.3)$$

$$\nabla_{Du} G_j(Du_0; 0) \cdot D(\dot{u}_1 - \dot{u}_2) = g_j(\dot{v}_1; \varphi_w) - g_j(\dot{v}_2; \varphi_w) \quad \text{on } \Gamma_j. \quad (7.4)$$

For the right-hand side of equation (7.3),

$$\begin{aligned} f(\dot{v}_1; \varphi_w) - f(\dot{v}_2; \varphi_w) &= \sum_{i,j=1}^n \left(\tilde{a}_{ij}^0 - \frac{(\partial_{y_1} u_0)^3}{(\partial_{y_1} v_1)^3} \tilde{a}_{ij}(\dot{v}_1) \right) \partial_{y_i y_j} (\dot{v}_1 - \dot{v}_2) \\ &\quad + \sum_{i,j=1}^n \left(\frac{(\partial_{y_1} u_0)^3}{(\partial_{y_1} v_1)^3} \tilde{a}_{ij}(\dot{v}_1) - \frac{(\partial_{y_1} u_0)^3}{(\partial_{y_1} v_2)^3} \tilde{a}_{ij}(\dot{v}_2) \right) \partial_{y_i y_j} \dot{v}_2 \\ &\quad + \sum_{i,j \neq 2} (\partial_{x_i x_j} \varphi_w(v_1, \mathbf{y}') - \partial_{x_i x_j} \varphi_w(v_2, \mathbf{y}')) \Phi_w^{ij}(Dv_1; D\varphi_w) \\ &\quad + \sum_{i,j \neq 2} \partial_{x_i x_j} \varphi_w(v_2, \mathbf{y}') (\Phi_w^{ij}(Dv_1; D\varphi_w) - \Phi_w^{ij}(Dv_2; D\varphi_w)), \end{aligned}$$

which, with analogous computations as in Lemma 7.1, implies

$$\|f(\dot{v}_1; \varphi_w) - f(\dot{v}_2; \varphi_w)\|_{(0,\alpha;\mathcal{D})}^{(\beta_0, \beta_\infty)} \leq CK\delta \|\dot{v}_1 - \dot{v}_2\|_{(2,\alpha;\mathcal{D})}^{(\beta_0, \beta_\infty)}.$$

For the right-hand side of the boundary condition (7.4) on $\Gamma_j, j = 1, 2$,

$$\begin{aligned} &g_j(\dot{v}_1; \varphi_w) - g_j(\dot{v}_2; \varphi_w) \\ &= \nabla_{Du} G_j(Du_0; 0) \cdot D(\dot{v}_1 - \dot{v}_2) - (G_j(Dv_1; 0) - G_j(Dv_2; 0)) \\ &\quad + (G_j(Dv_1; 0) - G_j(Dv_2; 0)) \\ &\quad - (G_j(Dv_1; D\varphi_w(v_1, \mathbf{y}')) - G_j(Dv_2; D\varphi_w(v_1, \mathbf{y}'))) \\ &\quad - (G_j(Dv_2; D\varphi_w(v_1, \mathbf{y}')) - G_j(Dv_2; D\varphi_w(v_2, \mathbf{y}'))) \\ &= \int_0^1 (\nabla_{Du} G_j(Du_0; 0) - \nabla_{Du} G_j(Dv_t; 0)) dt \cdot D(\dot{v}_1 - \dot{v}_2) \\ &\quad + \int_0^1 (\nabla_{Du} G_j(Dv_t; 0) - \nabla_{Du} G_j(Dv_t; D\varphi_w(v_1, \mathbf{y}'))) dt \cdot D(\dot{v}_1 - \dot{v}_2) \\ &\quad - \int_0^1 \nabla_{D\varphi_w} G_j(Dv_2; D\varphi_w^t) dt \cdot D(\varphi_w(v_1, \mathbf{y}') - \varphi_w(v_2, \mathbf{y}')), \end{aligned}$$

which implies

$$\|g_j(\dot{v}_1; \varphi_w) - g_j(\dot{v}_2; \varphi_w)\|_{(1,\alpha;\Gamma_j)}^{(\beta_0, \beta_\infty)} \leq CK\delta \|\dot{v}_1 - \dot{v}_2\|_{(2,\alpha;\mathcal{D})}^{(\beta_0, \beta_\infty)}.$$

Thus, we have

$$\begin{aligned} &\|\dot{u}_1 - \dot{u}_2\|_{(2,\alpha;\mathcal{D})}^{(\beta_0, \beta_\infty)} \\ &\leq C \left(\|f(\dot{v}_1; \varphi_w) - f(\dot{v}_2; \varphi_w)\|_{(0,\alpha;\mathcal{D})}^{(\beta_0, \beta_\infty)} + \sum_{j=1,2} \|g_j(\dot{v}_1; \varphi_w) - g_j(\dot{v}_2; \varphi_w)\|_{(1,\alpha;\Gamma_j)}^{(\beta_0, \beta_\infty)} \right) \\ &\leq CK\delta \|\dot{v}_1 - \dot{v}_2\|_{(2,\alpha;\mathcal{D})}^{(\beta_0, \beta_\infty)}. \end{aligned}$$

Then, choosing $0 < \delta \leq \delta_0$ such that $CK\delta_0 = \frac{1}{2}$, we have

$$\|\dot{u}_1 - \dot{u}_2\|_{(2,\alpha;\mathcal{D})}^{(\beta_0,\beta_\infty)} \leq \frac{1}{2} \|\dot{v}_1 - \dot{v}_2\|_{(2,\alpha;\mathcal{D})}^{(\beta_0,\beta_\infty)},$$

which implies that the mapping \mathcal{J} is a contraction mapping in $O_{K\delta}^{(\sigma_0,\sigma_\infty)}$. \square

Combining Lemma 7.1 with Lemma 7.2, we obtain Theorem 5.2.

8. THE TWO-DIMENSIONAL CASE

From Theorem 3.1 and Remarks 6.3 and 6.5, we can see that, for an M-D wedge ($n \geq 3$), the weak transonic shock solution, which is represented by point B on the shock polar (see Fig. 2.1), is conditionally stable when the wedge edge is not perturbed and the perturbation of the wedge surface is within some weighted Hölder spaces. However, the stability of the strong transonic shock solution, represented by point A on the shock polar, may require a different approach, since condition (4.4) for the admissible weights cannot be improved. This fact is indeed interesting since, for the 2-D wedge, both the weak and strong transonic shock solutions are conditionally stable, except the critical point S_* (see [10, 15]). Moreover, for the strong case, we can even have better regularity at the wedge vertex. We now show these facts in this section.

For a 2-D wedge, its edge shrinks to a point. Thus, we can consider the stability problem as a special situation for the case that the incoming supersonic flow is perpendicular to the wedge edge with the perturbation of the whole fluid, independent of \mathbf{x}' or \mathbf{y}' . Therefore, the partial hodograph transformation and the nonlinear iteration scheme are still valid. On the other hand, this yields the differences for the linearized elliptic problem (6.1)–(6.3), since the singularity of \mathcal{D} is a straight line for $n = 3$ and a hyperplane for $n \geq 4$, while it is only a point for $n = 2$ for which the better results can be achieved.

For $n = 2$, equation (6.1) becomes

$$\sum_{i,j=1}^2 \tilde{a}_{ij}^0 \partial_{y_i y_j} u = \hat{f}, \quad (8.1)$$

where

$$\begin{aligned} \tilde{A}_0 &:= [\tilde{a}_{ij}^0]_{2 \times 2} = J_0^\top A_0 J_0, \\ A_0 &= A(D\varphi_0^+) = [a_{ij}^0]_{2 \times 2} = (c_0^+)^2 \begin{bmatrix} 1 - (M_0^+)^2 & 0 \\ 0 & 1 \end{bmatrix}, \\ J_0 &= J(Du_0; 0) = \frac{1}{q_0^- \cos \alpha_w - q_0^+} \begin{bmatrix} q_0^- \cos \alpha_w - q_0^+ & 0 \\ -q_0^- \sin \alpha_w & 1 \end{bmatrix}. \end{aligned}$$

The boundary condition (6.2) on Γ_1 remains unchanged:

$$-q_0^- \sin \alpha_w \partial_{y_1} u + \partial_{y_2} u = \hat{g}_1, \quad (8.2)$$

and condition (6.3) on Γ_2 is

$$(Du)^\top \cdot J_0 \nu = \hat{g}_2, \quad (8.3)$$

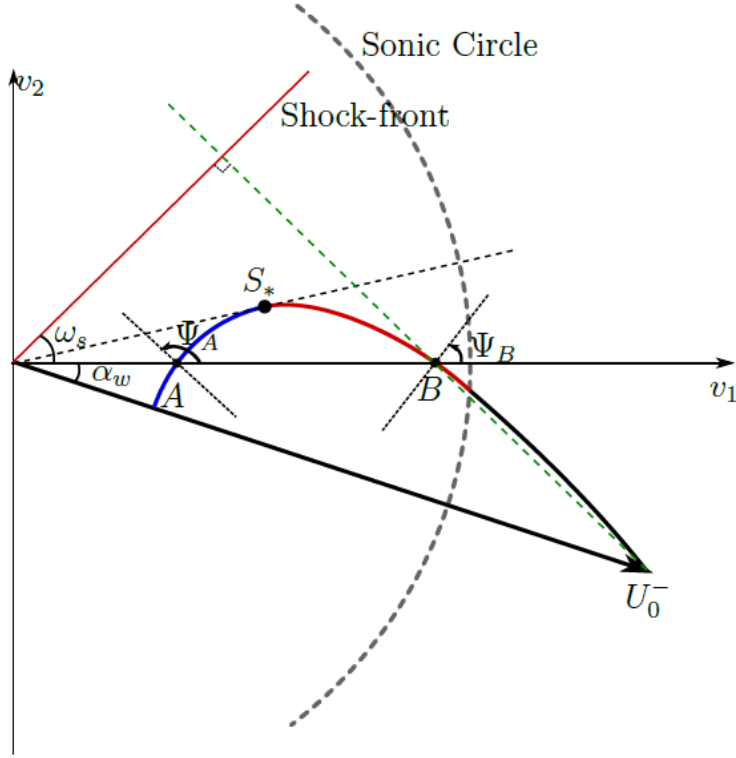


FIGURE 8.1. The two-dimensional case

where $\boldsymbol{\nu} = (\nu_1, \nu_2)^\top = \nabla_{D\varphi} H_s(D\varphi_0)$, the unit outer normal of the 2-D shock polar.

Let

$$Y = P\mathbf{y},$$

where

$$P = A_0^{-\frac{1}{2}}(J_0^{-1})^\top$$

with

$$A_0^{-\frac{1}{2}} = \frac{1}{c_0^+} \begin{bmatrix} \frac{1}{\sqrt{1-(M_0^+)^2}} & 0 \\ 0 & 1 \end{bmatrix},$$

$$J_0^{-1} = (q_0^- \cos \alpha_w - q_0^+) \begin{bmatrix} \frac{1}{q_0^- \cos \alpha_w - q_0^+} & 0 \\ \frac{q_0^- \sin \alpha_w}{q_0^- \cos \alpha_w - q_0^+} & 1 \end{bmatrix}.$$

Then equation (8.1) becomes

$$\partial_{Y_1 Y_1} u + \partial_{Y_2 Y_2} u = \hat{f}. \quad (8.4)$$

The boundaries Γ_1 and Γ_2 become

$$\bar{\Gamma}_1 = \{Y_2 = 0, Y_1 > 0\},$$

$$\bar{\Gamma}_2 = \{Y_2 = \tan \omega_s Y_1, Y_1 > 0\},$$

where

$$\tan \omega_s = \frac{q_0^- \cos \alpha_w - q_0^+}{q_0^- \sin \alpha_w} \sqrt{1 - (M_0^+)^2}.$$

The boundary condition (8.2) becomes

$$\partial_{Y_2} u = \bar{g}_1 := \frac{c_0^+}{q_0^- \cos \alpha_w - q_0^+} \hat{g}_1, \quad (8.5)$$

and the boundary condition (8.3) becomes

$$\frac{\nu_1}{\sqrt{1 - (M_0^+)^2}} \partial_{Y_1} u + \nu_2 \partial_{Y_2} u = \bar{g}_2 := c_0^+ \hat{g}_2, \quad (8.6)$$

which can be rewritten as

$$\frac{1}{r} \partial_\omega u + \tan(\omega_s + \Phi_s) \partial_r u = \frac{\cos \Phi_s}{\cos(\omega_s + \Phi_s)} \frac{\bar{g}_2}{\nu_2},$$

under the polar coordinates for (Y_1, Y_2) , where

$$\tan \Phi_s = \frac{1}{\sqrt{1 - (M_0^+)^2}} \frac{\nu_1}{\nu_2}.$$

In the stability analysis of 2-D transonic shocks, problem (8.4)–(8.6) plays the same role as problem (6.7)–(6.9) for the M-D case with the incoming supersonic flow orthogonal to the edge. Notice that both problems have the same formulation with the only difference of the dimension of the domain between them. Thus, it is Theorem A.2, rather than Theorem 4.4, that will be employed to establish the well-posedness of problem (8.4)–(8.6) so that the following lemma can be concluded, which is better than Theorem 6.4.

Lemma 8.1. *Let Λ be the set of eigenvalues λ satisfying (A.13):*

$$\Lambda = \{0\} \cup \left\{ 1 + \frac{m\pi + \Phi_s}{\omega_s} : m \in \mathbb{Z} \right\}.$$

Let $\sigma_1 < \sigma_2$ and $\beta_j = 1 + \alpha - \sigma_j$. If

$$[1 + \sigma_1, 1 + \sigma_2] \cap \Lambda = \emptyset,$$

and

$$f \in \mathcal{C}_{\beta_1, \beta_2}^{0, \alpha}(\mathcal{D}), \quad \bar{g}_j \in \mathcal{C}_{\beta_1, \beta_2}^{1, \alpha}(\Gamma_j), \quad j = 1, 2,$$

then there exists a unique solution $u \in \mathcal{C}_{\beta_1, \beta_2}^{2, \alpha}(\mathcal{D})$ to the boundary value problem (8.4)–(8.6) with the following estimate:

$$\|u\|_{(2, \alpha; \mathcal{D})}^{(\beta_1, \beta_2)} \leq C \left(\|f\|_{(0, \alpha; \mathcal{D})}^{(\beta_1, \beta_2)} + \sum_{j=1}^2 \|\bar{g}_j\|_{(1, \alpha; \Gamma_j)}^{(\beta_1, \beta_2)} \right). \quad (8.7)$$

Remark 8.2. From the definition of the weighted Hölder norms, we can see that the weight power σ_1 describes the asymptotic behavior of solution u with the property that $Du = O(r^{\sigma_1})$ as $r \rightarrow \infty$, while the weight power σ_2 describes the regularity of u at the origin with the property that $Du = O(r^{\sigma_2})$ as $r \rightarrow 0$. Since, in the stability analysis of the wedge shocks, Du relates with the velocity field of the flow, we expect that Du should be bounded

in \mathcal{D} . Therefore, we need that

$$\sigma_1 \leq 0 \leq \sigma_2.$$

Then, in order to decide the applicable admissible weights, we need to calculate the eigenvalues corresponding to $m = -1, 0, 1$ in set Λ . By definition, $-\frac{\pi}{2} < \Phi_s < \frac{\pi}{2}$ and $0 < \omega_s < \frac{\pi}{2}$. Thus, for $m = -1$,

$$\lambda_{-1} := 1 + \frac{-\pi + \Phi_s}{\omega_s} < 0;$$

for $m = 0$,

$$\lambda_0 := 1 + \frac{\Phi_s}{\omega_s} := 1 + \sigma_s;$$

and for $m = 1$,

$$\lambda_1 := 1 + \frac{\pi + \Phi_s}{\omega_s} > 2.$$

That is, we obtain the following inequality for the eigenvalues $\{\lambda_{-1}, 0, \lambda_0, \lambda_1\} \subset \Lambda$:

$$\lambda_{-1} < 0, \quad \lambda_0 < \lambda_1, \quad \text{or} \quad \lambda_{-1} - 1 < -1, \quad \sigma_s < \lambda_1 - 1.$$

Therefore, in order to decide the applicable admissible weights, we need to compare λ_0 with 1, or equivalently, to compare σ_s with 0. Notice that σ_s is determined by the background shock solution. One can verify that $\sigma_s < 0$ for the strong transonic shock represented by A , $\sigma_s > 0$ for the weak transonic shock represented by B , and $\sigma_s = 0$ for the critical shock solution represented by S_* (see Fig. 8.1).

For the strong transonic shock solution represented by A on the shock polar, we have

$$\sigma_s = \frac{\Phi_s}{\omega_s} < 0.$$

Then any σ_1 and σ_2 satisfying

$$\max(-1, \sigma_s) < \sigma_1 \leq 0 \leq \sigma_2 < \lambda_1 - 1$$

can be applicable weights. Since $\lambda_1 > 2$, the regularity of velocity Du near the origin (the wedge vertex) can be \mathcal{C}^1 , or even better. Velocity Du decays slower than r^{-1} as $r \rightarrow \infty$, while Du decays slower than r^{σ_s} in case $\sigma_s > -1$.

For the weak transonic shock solution represented by B , we have

$$\sigma_s = \frac{\Phi_s}{\omega_s} > 0.$$

Then any σ_1 and σ_2 satisfying

$$-1 < \sigma_1 \leq 0 \leq \sigma_2 < \sigma_s$$

can be applicable weights, and solution u can be $\mathcal{C}^{1+\sigma_2}$ near the origin (the wedge vertex), while Du decays slower than r^{σ_1} as $r \rightarrow \infty$.

Concluding the above argument, we obtain the following theorem for the linearized problem (8.1)–(8.3).

Theorem 8.3. *Let $(U_0^-; U_0^+)$ be a transonic shock solution on the shock polar (see Fig. 8.1).*

(i). If $(U_0^-; U_0^+)$ is the strong transonic shock solution represented by A, which implies that

$$\frac{\nu_1}{\nu_2} < 0,$$

then, for any σ_0^A and σ_∞^A with

$$\max \left\{ -1, \frac{\Phi_s}{\omega_s} \right\} < \sigma_\infty^A \leq 0 \leq \sigma_0^A < \frac{\pi + \Phi_s}{\omega_s},$$

when

$$f \in \mathcal{C}_{\beta_0^A, \beta_\infty^A}^{0, \alpha}(\mathcal{D}), \quad g_j \in \mathcal{C}_{\beta_0^A, \beta_\infty^A}^{1, \alpha}(\Gamma_j), \quad j = 1, 2,$$

with $\beta_0^A = 1 + \alpha - \sigma_0^A$ and $\beta_\infty^A = 1 + \alpha - \sigma_\infty^A$, there exists a unique solution $u \in \mathcal{C}_{\beta_0^A, \beta_\infty^A}^{2, \alpha}(\mathcal{D})$ to the boundary value problem (8.1)–(8.3) satisfying the following estimate:

$$\|u\|_{(2, \alpha; \mathcal{D})}^{(\beta_0^A, \beta_\infty^A)} \leq C \left(\|f\|_{(0, \alpha; \mathcal{D})}^{(\beta_0^A, \beta_\infty^A)} + \sum_{j=1}^2 \|g_j\|_{(1, \alpha; \Gamma_j)}^{(\beta_0^A, \beta_\infty^A)} \right). \quad (8.8)$$

(ii). If $(U_0^-; U_0^+)$ is the weak transonic shock solution represented by B, which implies that

$$\frac{\nu_1}{\nu_2} > 0,$$

then, for any σ_0^B and σ_∞^B with

$$-1 < \sigma_\infty^B \leq 0 \leq \sigma_0^B < \frac{\Phi_s}{\omega_s},$$

when

$$f \in \mathcal{C}_{\beta_0^B, \beta_\infty^B}^{0, \alpha}(\mathcal{D}), \quad g_j \in \mathcal{C}_{\beta_0^B, \beta_\infty^B}^{1, \alpha}(\Gamma_j), \quad j = 1, 2,$$

with $\beta_0^B = 1 + \alpha - \sigma_0^B$ and $\beta_\infty^B = 1 + \alpha - \sigma_\infty^B$, there exists a unique solution $u \in \mathcal{C}_{\beta_0^B, \beta_\infty^B}^{2, \alpha}(\mathcal{D})$ to the boundary value problem (8.1)–(8.3) satisfying the following estimate:

$$\|u\|_{(2, \alpha; \mathcal{D})}^{(\beta_0^B, \beta_\infty^B)} \leq C \left(\|f\|_{(0, \alpha; \mathcal{D})}^{(\beta_0^B, \beta_\infty^B)} + \sum_{j=1}^2 \|g_j\|_{(1, \alpha; \Gamma_j)}^{(\beta_0^B, \beta_\infty^B)} \right). \quad (8.9)$$

Then, with an analogous nonlinear iteration argument as in §7 for $n \geq 3$, we can obtain the following stability theorem for both the weak transonic shock solution and the strong one.

Theorem 8.4. Let $(\varphi_0^-(\mathbf{x}); \varphi_0^+(\mathbf{x}))$ be a transonic shock solution.

(i). If $(\varphi_0^-(\mathbf{x}); \varphi_0^+(\mathbf{x}))$ is the strong transonic shock solution that is represented by A on the shock polar (see Fig. 8.1), then there exist $\delta_0^A > 0$ sufficiently small and $\sigma_s := \frac{\Phi_s}{\omega_s} < 0$, depending on the background solution, such that, for any

$$\max \{-1, \sigma_s\} < \sigma_\infty^A \leq 0 \leq \sigma_0^A < \frac{\pi}{\omega_s} + \sigma_s,$$

when the perturbation of the wedge surface Γ_w is small in the sense that

$$\|\varphi_w(x_1)\|_{(2, \alpha; \mathbb{R}_+)}^{(\beta_0^A, \beta_\infty^A)} \leq \delta \leq \delta_0^A,$$

with $\beta_0^A = 1 + \alpha - \sigma_0^A$ and $\beta_\infty^A = 1 + \alpha - \sigma_\infty^A$, there exists a unique solution $u(\mathbf{y})$ to the boundary value problem (5.1)–(5.3) satisfying

$$\|u - u_0\|_{(2,\alpha;\mathcal{D})}^{(\beta_0^A, \beta_\infty^A)} \leq K\delta,$$

where $K > 0$ depends on the background solution, but is independent of δ_0^A .

- (ii). If $(\varphi_0^-(\mathbf{x}); \varphi_0^+(\mathbf{x}))$ is the weak transonic shock solution, that is, the one represented by B on the shock polar (see Fig. 8.1), then there exist $\delta_0^B > 0$ sufficiently small and $\sigma_s := \frac{\Phi_s}{\omega_s} > 0$, depending on the background solution, such that, for any

$$-1 < \sigma_\infty^B \leq 0 \leq \sigma_0^B < \sigma_s,$$

when the perturbation of the wedge surface Γ_w is small in the sense that

$$\|\varphi_w(x_1)\|_{(2,\alpha;\mathbb{R}_+)}^{(\beta_0^B, \beta_\infty^B)} \leq \delta \leq \delta_0^B,$$

with $\beta_0^B = 1 + \alpha - \sigma_0^B$ and $\beta_\infty^B = 1 + \alpha - \sigma_\infty^B$, there exists a unique solution $u(\mathbf{y})$ to the boundary value problem (5.1)–(5.3) satisfying

$$\|u - u_0\|_{(2,\alpha;\mathcal{D})}^{(\beta_0^B, \beta_\infty^B)} \leq K\delta,$$

where $K > 0$ depends on the background solution, but is independent of δ_0^B .

Remark 8.5. When $n = 2$, we have the stability property for both the strong transonic shock solution represented by A and the weak one represented by B , that is, for all the transonic shock solutions on the shock polar except the critical one S_* . This result is better than the result we have obtained for $n \geq 3$ in §7, where only the stability property for the weak transonic shock solution represented by B is obtained.

APPENDIX A. PROOF OF THEOREM 4.4

For self-containedness, in this appendix, we give a sketch of the proof of Theorem 4.4, based mainly on the results in Maz'ya, Plamenevskij, Reisman, and others in [23], [26]–[31], and [35], and the references therein.

A.1. Function spaces and the equipped norms. We first quote the weighted norms used in Maz'ya-Plamenevskij in [27]–[30].

A.1.1. Weighted Sobolev spaces in the dihedral angle $\mathcal{D} = K \times \mathbb{R}^{n-2}$. Let $\beta \in \mathbb{R}$, $1 < p < \infty$, $\ell = 0, 1, 2, \dots$, and $D^\ell = \{D_{\mathbf{x}}^{\mathbf{k}}u : |\mathbf{k}| = \ell\}$. Define the weighted Sobolev norms:

$$\|u\|_{V_{p,\beta}^\ell(\mathcal{D})}^p := \sum_{|\mathbf{k}|=0}^{\ell} \int_{\mathcal{D}} r^{p(\beta-\ell+|\mathbf{k}|)} |D_{\mathbf{x}}^{\mathbf{k}}u|^p dx, \quad (\text{A.1})$$

where $r = \sqrt{x_1^2 + x_2^2}$ and $D = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3}, \dots, \partial_{x_n})$.

Denote by $V_{p,\beta}^\ell(\mathcal{D})$ the completion of set $C_c^\infty(\overline{\mathcal{D}} \setminus \mathcal{E})$ under norm (A.1). Denote by $V_{p,\beta}^{\circ\ell}(\mathcal{D}, \Gamma^\pm)$ the completion of set $C_c^\infty(\mathcal{D})$ under norm (A.1).

Denote by $V_{p,\beta}^{\ell-1/p}(\Gamma^\pm)$ the space of traces on Γ^\pm of the functions in $V_{p,\beta}^\ell(\mathcal{D})$, that is,

$$V_{p,\beta}^{\ell-1/p}(\Gamma^\pm) = V_{p,\beta}^\ell(\mathcal{D}) / \mathring{V}_{p,\beta}^\ell(\mathcal{D}, \Gamma^\pm).$$

The corresponding trace norm is defined as

$$\|u\|_{V_{p,\beta}^{\ell-1/p}(\Gamma^\pm)} := \inf \left\{ \|v\|_{V_{p,\beta}^\ell(\mathcal{D})} : v - u \in \mathring{V}_{p,\beta}^\ell(\mathcal{D}, \Gamma^\pm) \right\}. \quad (\text{A.2})$$

A.1.2. *The first type of weighted Sobolev spaces in the angular domain K .* If $n = 2$, the dihedral angle \mathcal{D} becomes an angular domain K , and the edge \mathbb{R}^{n-2} shrinks to a point. In this case, we can also define analogous weighted Sobolev spaces and norms in the angular domain K , with $\mathbf{y} = (y_1, y_2)^\top \in K$.

Define

$$\|u\|_{V_{p,\beta}^\ell(K)}^p := \sum_{|\mathbf{k}|=0}^{\ell} \int_K r^{p(\beta-\ell+|\mathbf{k}|)} |D_{\mathbf{y}}^{\mathbf{k}} u|^p d\mathbf{y}, \quad (\text{A.3})$$

where $r^2 = y_1^2 + y_2^2 = |\mathbf{y}|^2$ and $D_{\mathbf{y}}^{\mathbf{k}} = \partial_{y_1}^{k_1} \partial_{y_2}^{k_2}$. Note that, by applying the blow-up transformation $\mathcal{B} : t = \ln r$, K becomes an infinite strip:

$$\mathcal{S} := \left\{ (t, \omega) : t \in \mathbb{R}, -\frac{\omega_*}{2} < \omega < \frac{\omega_*}{2} \right\},$$

and

$$\|u\|_{V_{p,\beta}^\ell(K)} \approx \|e^{-\sigma t} u(e^t, \omega)\|_{W_{p,\beta}^\ell(\mathcal{S})},$$

where $-\sigma = \beta - \ell + \frac{2}{p}$.

Denote by $V_{p,\beta}^\ell(K)$ the completion of set $\mathcal{C}_c^\infty(\overline{K} \setminus \{O\})$ under norm (A.3). Denote by $\mathring{V}_{p,\beta}^\ell(K, \gamma^\pm)$ the completion of set $\mathcal{C}_c^\infty(K)$ under norm (A.3).

Denote by $V_{p,\beta}^{\ell-1/p}(\gamma^\pm)$ the space of traces on γ^\pm of the functions in $V_{p,\beta}^\ell(K)$, that is,

$$V_{p,\beta}^{\ell-1/p}(\gamma^\pm) = V_{p,\beta}^\ell(K) / \mathring{V}_{p,\beta}^\ell(K, \gamma^\pm).$$

The corresponding trace norm is defined as

$$\|u\|_{V_{p,\beta}^{\ell-1/p}(\gamma^\pm)} := \inf \left\{ \|v\|_{V_{p,\beta}^\ell(K)} : v - u \in \mathring{V}_{p,\beta}^\ell(K, \gamma^\pm) \right\}. \quad (\text{A.4})$$

A.1.3. *The second type of weighted Sobolev spaces in the angular domain K .* In the analysis, we also need the following weighted norms in K :

$$\|u\|_{E_{p,\beta}^\ell(K)}^p := \sum_{|\mathbf{k}|=0}^{\ell} \int_K r^{p\beta} (1 + r^{p(|\mathbf{k}|-\ell)}) |D_{\mathbf{y}}^{\mathbf{k}} u|^p d\mathbf{y}. \quad (\text{A.5})$$

Then we also define the related function spaces and traces. Denote by $E_{p,\beta}^\ell(K)$ the completion of set $\mathcal{C}_c^\infty(\overline{K} \setminus \{O\})$ under norm (A.5). Denote by $\mathring{E}_{p,\beta}^\ell(K, \gamma^\pm)$ the completion of set $\mathcal{C}_c^\infty(K)$ under norm (A.5).

Denote by $E_{p,\beta}^{\ell-1/p}(\gamma^\pm)$ the space of traces on γ^\pm of the functions in $E_{p,\beta}^\ell(K)$, that is,

$$E_{p,\beta}^{\ell-1/p}(\gamma^\pm) = E_{p,\beta}^\ell(K) / \mathring{E}_{p,\beta}^\ell(K, \gamma^\pm).$$

The corresponding trace norm is defined as

$$\|u\|_{E_{p,\beta}^{\ell-1/p}(\gamma^\pm)} := \inf \left\{ \|v\|_{E_{p,\beta}^\ell(K)} : v - u \in \mathring{E}_{p,\beta}^\ell(K, \gamma^\pm) \right\}. \quad (\text{A.6})$$

A.2. Sketch of the proof of Theorem 4.4. Now we specify the major steps of the proof of Theorem 4.4 and the main consequences of each step.

A.2.1. Step 1: The well-posed theory for the homogeneous operator $(-\Delta_Y, \mathcal{P}^\pm(D_Y; 0))$ in the angular domain K . Consider the boundary value problem of the homogeneous operator $(-\Delta_Y, \mathcal{P}^\pm(D_Y; 0))$ in K , with $Y = (Y_1, Y_2)^\top \in K$:

$$\Delta_Y U(Y) = F(Y) \quad \text{in } K, \quad (\text{A.7})$$

$$\mathcal{P}^\pm(D_Y; 0) U(Y) = G^\pm(Y) \quad \text{on } \gamma^\pm, \quad (\text{A.8})$$

where $\mathcal{P}^\pm(D_Y; 0) = \partial_{\nu^\pm} + \alpha^\pm \partial_{\tau^\pm}$.

The L^2 well-posedness theory for boundary value problems for elliptic equations in an angular or conical domain was established in Kondrat'ev [21], and later it was improved to the L^p and Hölder well-posedness in Maz'ya-Plamenevskij [28]. We now employ their results to obtain the well-posedness of the solutions to problem (A.7)–(A.8). Here we only introduce the key conditions and theorems.

Let (r, ω) be the polar coordinates on K . Then we have

$$\Delta_Y := \frac{1}{r^2} P(\partial_\omega, r\partial_r) = \frac{1}{r^2} \left\{ (r\partial_r)^2 + (\partial_\omega)^2 \right\},$$

$$\mathcal{P}^\pm(D_Y; 0) := \frac{1}{r} P^\pm(\partial_\omega, r\partial_r) = \frac{1}{r} \left\{ \mp \partial_\omega + \alpha^\pm (r\partial_r) \right\}.$$

Set $v = r^{-\sigma} U$, and apply transformation \mathcal{B} . Then the boundary value problem (A.7)–(A.8) becomes

$$\partial_{\omega\omega} v + \partial_{tt} v + 2\sigma \partial_t v c + \sigma^2 v = e^{(2-\sigma)t} F := f \quad \text{in } \mathcal{S}, \quad (\text{A.9})$$

$$\mp \partial_\omega v + \alpha^\pm \partial_t v + \sigma \alpha^\pm v = e^{(1-\sigma)t} G^\pm := g^\pm \quad \text{on } \mathcal{B}\gamma^\pm. \quad (\text{A.10})$$

Applying the Fourier transform with respect to $t \rightarrow \xi$ to problem (A.9)–(A.10), we obtain a boundary value problem of an ordinary differential equation with parameter $\lambda = \sigma + i\xi$:

$$\partial_{\omega\omega} \hat{v} + (\sigma + i\xi)^2 \hat{v} = \hat{f} \quad \text{in } \omega^- < \omega < \omega^+, \quad (\text{A.11})$$

$$\mp \partial_\omega \hat{v} + (\sigma + i\xi) \alpha^\pm \hat{v} = \hat{g}^\pm \quad \text{on } \omega = \omega^\pm, \quad (\text{A.12})$$

for given σ and any $\xi \in \mathbb{R}$. In order to apply the inverse Fourier transform, we need the existence and uniqueness of solutions to the boundary value problems (A.11)–(A.12) for any $\xi \in \mathbb{R}$. Thus, in the case that the homogeneous problem of (A.11)–(A.12) does not have nontrivial solutions, we can employ the inverse Fourier transform to obtain a solution v to problem (A.9)–(A.10). Then $U = r^\sigma v$ is the solution to problem (A.7)–(A.8).

The complex number $\lambda = \sigma + i\xi$ is called an *eigenvalue* for problem (A.11)–(A.12) if the homogeneous problem of (A.11)–(A.12) has nontrivial solutions. It can be checked that a complex number λ is an eigenvalue for problem (A.11)–(A.12) if and only if

$$\lambda = 0, \quad \text{or} \quad \lambda_m = \frac{m\pi - \Phi}{\omega_*}, \quad m \in \mathbb{Z},$$

where $\Phi = \arctan \alpha^- + \arctan \alpha^+$. Define the following set Λ to be the collection of the above eigenvalues:

$$\Lambda := \{0\} \cup \left\{ \frac{m\pi - \Phi}{\omega_*} : m \in \mathbb{Z} \right\}. \quad (\text{A.13})$$

Therefore, we can fulfill the above argument for σ satisfying

$$\sigma \notin \Lambda. \quad (\text{A.14})$$

With the above calculations, Theorems 4.1–4.2 in Maz'ya-Plamenevskij [28] directly lead to the following theorem.

Theorem A.1. *Let $1 < p < \infty$, $\sigma \in \mathbb{R}$, $\ell = 0, 1, 2, \dots$, and $\beta = \ell + 2 - \frac{2}{p} - \sigma$. Let $F \in V_{p,\beta}^\ell(K)$ and $G^\pm \in V_{p,\beta}^{\ell+1-1/p}(\gamma^\pm)$. Then the boundary value problem (A.7)–(A.8) has a unique solution $u \in V_{p,\beta}^{\ell+2}(K)$ for all F and G^\pm if and only if the line:*

$$\Re \lambda = \sigma$$

contains no eigenvalues of problem (A.11)–(A.12), that is, σ satisfies (A.14). Moreover, when (A.14) is satisfied, the following estimate holds:

$$\|e^{-\sigma t} u\|_{W_p^{\ell+2}(\mathcal{D})} \approx \|u\|_{V_{p,\beta}^{\ell+2}(K)} \leq C \left(\|F\|_{V_{p,\beta}^\ell(K)} + \sum_{\pm} \|G^\pm\|_{V_{p,\beta}^{\ell+1-1/p}(\gamma^\pm)} \right). \quad (\text{A.15})$$

That is, operator $(-\Delta_Y, \mathcal{P}^\pm(D_Y; 0))$ of problem (A.7)–(A.8) induces an isomorphism:

$$V_{p,\beta}^{\ell+2}(K) \approx V_{p,\beta}^\ell(K) \times \prod_{\pm} V_{p,\beta}^{\ell+1-1/p}(\gamma^\pm).$$

Theorem A.1 presents the L^p well-posedness for operator $(-\Delta_Y, \mathcal{P}^\pm(D_Y; 0))$ of problem (A.7)–(A.8). The Schauder well-posedness for this problem has also been established in [28, Theorems 5.1–5.2], which leads to the following theorem.

Theorem A.2. *Suppose that σ satisfies (A.14). Then operator $(-\Delta_Y, \mathcal{P}^\pm(D_Y; 0))$ of problem (A.7)–(A.8) induces an isomorphism $\mathcal{C}_\beta^{\ell+2,\alpha}(K) \approx \mathcal{C}_\beta^{\ell,\alpha}(K) \times \prod_{\pm} \mathcal{C}_\beta^{\ell+1,\alpha}(\gamma^\pm)$ for $\beta = \ell + 2 + \alpha - \sigma$.*

Moreover, let $\underline{\sigma} < \bar{\sigma}$ be two real numbers satisfying that strip $\underline{\sigma} < \Re \lambda < \bar{\sigma}$ in the complex plane \mathbb{C} contains no eigenvalues of problem (A.11)–(A.12). Assume $\sigma_1, \sigma_2 \in (\underline{\sigma}, \bar{\sigma})$, $\beta_j = \ell + 2 + \alpha - \sigma_j$, and

$$f \in \mathcal{C}_{\beta_1, \beta_2}^{\ell, \alpha}(K), \quad g^\pm \in \mathcal{C}_{\beta_1, \beta_2}^{\ell+1, \alpha}(\gamma^\pm).$$

Then there exists a unique solution $u \in \mathcal{C}_{\beta_1, \beta_2}^{\ell+2, \alpha}(K)$ of problem (A.7)–(A.8) with the following estimate:

$$\|u\|_{(2, \alpha; K)}^{(\beta_1, \beta_2)} \leq C \left(\|f\|_{(0, \alpha; K)}^{(\beta_1, \beta_2)} + \sum_{\pm} \|g^\pm\|_{(1, \alpha; \gamma^\pm)}^{(\beta_1, \beta_2)} \right).$$

A.2.2. *Step 2: Fredholm property of the nonhomogeneous operator $(-\Delta_Y + 1, \mathcal{P}^\pm(D_Y; \boldsymbol{\theta}))$ in the angular domain K .* In this step, we consider the boundary value problem for the nonhomogeneous operator $\mathcal{A}(\boldsymbol{\theta}) := (-\Delta_Y + 1, \mathcal{P}^\pm(D_Y; \boldsymbol{\theta}))$:

$$-\Delta_Y U(Y) + U(Y) = F(Y) \quad \text{in } K, \quad (\text{A.16})$$

$$\mathcal{P}^\pm(D_Y; \boldsymbol{\theta})U(Y) = G^\pm(Y) \quad \text{on } \gamma^\pm, \quad (\text{A.17})$$

where $\mathcal{P}^\pm(D_Y; \boldsymbol{\theta}) := \partial_{\nu^\pm} + \alpha^\pm \partial_{\tau^\pm} + \mathbf{i}\boldsymbol{\theta} \cdot \mathbf{c}^\pm$ with $\boldsymbol{\theta} \in S^{n-3}$, the unit sphere in \mathbb{R}^{n-2} .

Different from the homogeneous operator $(-\Delta_Y, \mathcal{P}^\pm(D_Y; 0))$, $\mathcal{A}(\boldsymbol{\theta})$ does not induce an isomorphism in general, unlike in Theorem A.1 or Theorem A.2 under the condition that σ satisfies (A.14). In fact, operator $\mathcal{A}(\boldsymbol{\theta})$ induces a Fredholm operator, as indicated in the following theorem shown in Maz'ya-Plamenevskij [27]:

Theorem A.3. *Suppose that σ satisfies (A.14), that is, the line:*

$$\Re \lambda = \sigma$$

contains no eigenvalues of problem (A.11)–(A.12). Then the operator induced by the boundary value problem (A.16)–(A.17) with $\beta = \ell + 2 - \frac{2}{p} - \sigma$:

$$\mathcal{A}(\boldsymbol{\theta}) : E_{p,\beta}^2(K) \rightarrow E_{p,\beta}^0(K) \times \prod_{\pm} E_{p,\beta}^{1-1/p}(\gamma^\pm) \quad (\text{A.18})$$

is a Fredholm operator for all $\boldsymbol{\theta} \in S^{n-3}$.

In fact, a more general theorem has been proved in [27] for elliptic operators with higher order. In applications, we usually need to verify that the kernel of $\mathcal{A}(\boldsymbol{\theta})$ is 0-dimensional, which is still a difficult problem. Fortunately, for our problem (A.16)–(A.17), we have a better theorem for $\ell = 0$ and $p = 2$, proved in Reisman [35, Lemma 3.1], by the energy method. In this theorem, a sufficient condition is presented which ensures that operator $\mathcal{A}(\boldsymbol{\theta})$ is an isomorphism.

Theorem A.4. *Suppose that*

$$-\frac{\Phi}{\omega_*} < \sigma < 0, \quad \text{or} \quad 0 < \sigma < -\frac{\Phi}{\omega_*}, \quad (\text{A.19})$$

and $\beta = 1 - \sigma$. Then the following holds:

(i). *If $U \in E_{2,\beta}^2(K)$ and satisfies problem (A.16)–(A.17), then*

$$\|U\|_{E_{2,\beta}^2(K)} \leq C \left(\|F\|_{E_{2,\beta}^0(K)} + \sum_{\pm} \|G^\pm\|_{E_{2,\beta}^{1-1/2}(\gamma^\pm)} \right); \quad (\text{A.20})$$

(ii). *For any $(F, G^+, G^-) \in E_{2,\beta}^0(K) \times \prod_{\pm} E_{2,\beta}^{1-1/2}(\gamma^\pm)$, there exists $U \in E_{2,\beta}^2(K)$ that solves problem (A.16)–(A.17).*

A.2.3. *Step 3: L^2 well-posedness for problem (4.2)–(4.3).* Now we go back to our problem (4.2)–(4.3). Applying the Fourier transform with respect to \mathbf{x}' , we have

$$\Delta_X \hat{u}(X; \boldsymbol{\eta}) - \boldsymbol{\eta}^2 \hat{u}(X; \boldsymbol{\eta}) = \hat{f}(X; \boldsymbol{\eta}) \quad \text{in } K, \text{ for } \boldsymbol{\eta} \in \mathbb{R}^{n-2}, \quad (\text{A.21})$$

$$\mathcal{P}^\pm(D_X; \boldsymbol{\eta})\hat{u} = \hat{g}^\pm(X; \boldsymbol{\eta}) \quad \text{on } \gamma^\pm, \text{ for } \boldsymbol{\eta} \in \mathbb{R}^{n-2}, \quad (\text{A.22})$$

where $X = (x_1, x_2)^\top$, and $\mathcal{P}^\pm(D_X; \boldsymbol{\eta}) := \partial_{\nu^\pm} + \alpha^\pm \partial_{\tau^\pm} + i\boldsymbol{\eta} \cdot \mathbf{c}^\pm$. We hope that, for all $\boldsymbol{\eta} \in \mathbb{R}^{n-2}$, problem (A.21)–(A.22) has a unique solution $\hat{u}(X; \boldsymbol{\eta})$, so that the inverse Fourier transform can be employed to obtain the solution for problem (4.2)–(4.3).

If $\boldsymbol{\eta} = 0$, by applying Theorem A.1, problem (A.21)–(A.22) is solvable in space $V_{2,\beta}^2(K)$ for $\beta = 1 - \sigma$ with σ satisfying (A.14).

If $\boldsymbol{\eta} \neq 0$, by introducing the coordinate transform:

$$(X; \boldsymbol{\eta}) \mapsto (Y, \boldsymbol{\theta}) := (|\boldsymbol{\eta}| X, |\boldsymbol{\eta}|^{-1} \boldsymbol{\eta}),$$

and defining

$$U(Y; \boldsymbol{\eta}) := |\boldsymbol{\eta}|^2 \hat{u}(|\boldsymbol{\eta}|^{-1} Y; \boldsymbol{\eta}),$$

we find that problem (A.21)–(A.22) becomes a boundary value problem with the form as problem (A.16)–(A.17) in Step 2:

$$\Delta_Y U(Y; \boldsymbol{\eta}) - U(Y; \boldsymbol{\eta}) = F(Y; \boldsymbol{\eta}) \quad \text{in } K, \text{ for } \boldsymbol{\eta} \in \mathbb{R}^{n-2}, \quad (\text{A.23})$$

$$\mathcal{P}^\pm(D_Y; \boldsymbol{\theta})U(Y; \boldsymbol{\eta}) = G^\pm(Y; \boldsymbol{\eta}) \quad \text{on } \gamma^\pm, \text{ for } \boldsymbol{\eta} \in \mathbb{R}^{n-2}, \quad (\text{A.24})$$

where $\mathcal{P}^\pm(D_Y; \boldsymbol{\theta}) := \partial_{\nu^\pm} + \alpha^\pm \partial_{\tau^\pm} + i\boldsymbol{\theta} \cdot \mathbf{c}^\pm$ with $\boldsymbol{\theta} \in S^{n-3}$, and

$$F(Y; \boldsymbol{\eta}) := \hat{f}(|\boldsymbol{\eta}|^{-1} Y; \boldsymbol{\eta}), \quad G^\pm(Y; \boldsymbol{\eta}) := |\boldsymbol{\eta}| \hat{g}^\pm(|\boldsymbol{\eta}|^{-1} Y; \boldsymbol{\eta}).$$

Thus, Theorem A.4, well-prepared in Step 2, can be employed to establish the existence and uniqueness, as well as the *a priori* estimates, of a solution to problem (A.23)–(A.24) for any parameter $\boldsymbol{\eta} \neq 0$. Then the inverse Fourier transform with respect to $\boldsymbol{\eta}$ leads to a solution u of problem (4.2)–(4.3). We eventually obtain the following L^2 well-posedness theorem for problem (4.2)–(4.3).

Theorem A.5. *Suppose that σ satisfies condition (A.19) and $\beta = 1 - \sigma$. Then the operator of the boundary value problem (4.2)–(4.3) induces an isomorphism*

$$V_{2,\beta}^2(\mathcal{D}) \approx V_{2,\beta}^0(\mathcal{D}) \times \prod_{\pm} V_{2,\beta}^{1/2}(\Gamma^\pm).$$

Proof. It suffices to prove the unique solvability of problem (4.2)–(4.3) and to obtain the following estimate for homogeneous boundary conditions:

$$\|u\|_{V_{2,\beta}^2(\mathcal{D})} \leq C \|f\|_{V_{2,\beta}^0(\mathcal{D})}. \quad (\text{A.25})$$

Under the assumption that σ satisfies condition (A.19), by applying Theorem A.4, there exists a unique solution $U(Y; \boldsymbol{\eta}) \in E_{2,\beta}^2(K)$ for any $\boldsymbol{\eta} \neq 0$. Then

$$u(\mathbf{x}) = u(X, \mathbf{x}') = \mathcal{F}_{\boldsymbol{\eta} \rightarrow \mathbf{x}'}^{-1} (|\boldsymbol{\eta}|^{-2} U(Y; \boldsymbol{\eta}))$$

is the solution to problem (4.2)–(4.3). Noting that

$$\begin{aligned} \|u\|_{V_{2,\beta}^2(\mathcal{D})} &= \int_{\mathbb{R}^{n-2}} |\boldsymbol{\eta}|^{-2(\beta+1)} \|U\|_{E_{2,\beta}^2(K)}^2 \, d\boldsymbol{\eta}, \\ \|f\|_{V_{2,\beta}^0(\mathcal{D})} &= \int_{\mathbb{R}^{n-2}} |\boldsymbol{\eta}|^{-2(\beta+1)} \|F\|_{E_{2,\beta}^0(K)}^2 \, d\boldsymbol{\eta}, \end{aligned}$$

we obtain estimate (A.25), which completes the proof. \square

A.2.4. Step 4: Schauder estimates for problem (4.2)–(4.3). In this step, we present the weighted Hölder estimates, which have been established in [29, 30]. The theorem below in this step is a direct consequence of these theorems for oblique derivative boundary value problems of the Poisson equation.

With the L^2 well-posedness established in Step 3, the *a priori* Schauder estimates for solution u have also been established in [29, 30], by employing Green’s function and its delicate estimates. The Schauder estimates imply the well-posedness of problem (4.2)–(4.3) in the weighted Hölder spaces (*cf.* [29, 30] for detail calculations). As a direct consequence by using condition (A.19) in Theorem A.4, we have the following theorem:

Theorem A.6. *Let $\alpha \in (0, 1)$, $\underline{\sigma} = \min(0, -\frac{\Phi}{\omega_*})$, and $\bar{\sigma} = \max(0, -\frac{\Phi}{\omega_*})$. Suppose that $\sigma_* \in (\underline{\sigma}, \bar{\sigma})$, $\kappa = 1 - \sigma_*$, $\sigma_j \in (\underline{\sigma}, \bar{\sigma})$ and $\beta_j = 2 + \alpha - \sigma_j$, $j = 1, 2$. Then, for any $\ell = 0, 1, \dots$, if*

$$f \in \mathcal{C}_{\ell+\beta_1, \ell+\beta_2}^{\ell, \alpha}(\mathcal{D}), \quad g^\pm \in \mathcal{C}_{\ell+\beta_1, \ell+\beta_2}^{\ell+1, \alpha}(\Gamma^\pm),$$

solution $u \in V_{2, \kappa}^2(\mathcal{D})$ of problem (4.2)–(4.3) is also in $\mathcal{C}_{\ell+\beta_1, \ell+\beta_2}^{\ell+2, \alpha}(\mathcal{D})$ and satisfies the following estimate:

$$\|u\|_{(\ell+2, \alpha; \mathcal{D})}^{(\ell+\beta_1, \ell+\beta_2)} \leq C \left(\|f\|_{(\ell, \alpha; \mathcal{D})}^{(\ell+\beta_1, \ell+\beta_2)} + \sum_{\pm} \|g^\pm\|_{(\ell+1, \alpha; \Gamma_j)}^{(\ell+\beta_1, \ell+\beta_2)} \right).$$

Theorem 4.4 is a special case of Theorem A.6 with $\ell = 0$.

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