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Existence and Stability of Global Solutions of Shock Diffraction by Wedges for Potential Flow

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Abstract We present our recent results on the mathematical analysis of shock diffraction by two-dimensional convex cornered wedges in compressible fluid flow governed by the potential flow equation. The shock diffraction problem can be formulated as an initial-boundary value problem, which is invariant under the self-similar scaling. Then, by employing its self-similar invariance, the problem is reduced to a boundary value problem for a first-order nonlinear system of partial differential equations of mixed elliptic-hyperbolic type in an unbounded domain. It is further reformulated as a free boundary problem for a nonlinear degenerate elliptic system of first-order in a bounded domain with a boundary corner whose angle is bigger than π. A first global theory of existence and regularity has been established for this shock diffraction problem for the potential flow equation.

Keywords: Compressible flow, potential flow equations, shock diffraction, mixed elliptic-hyperbolic type, free boundary.

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1 Introduction

We are concerned with shock diffraction by a two-dimensional convex cornered wedge, which is not only a longstanding open problem in fluid mechanics, but also fundamental in the mathematical theory of multidimensional conservation laws. When a vertical shock propagates to the right along the convex cornered wedge, the incident shock interacts with the wedge, and the shock diffraction occurs. The study of the shock diffraction problem can date back 1950's by the work of Bargman [3], Lighthill [21, 22], Fletcher-Weimer-Bleakney [13], and Fletcher-Taub-Bleakney [12] via asymptotic or experimental analysis. Also Courant-Friedrichs [10] and Whitham [24].

One of the main challenges of this problem is that the expected elliptic domain of the solution is concave, since the angle exterior to the wedge at the origin is bigger than π, besides the other mathematical difficulties including free boundary problems without unform oblique derivative conditions and optimal regularity estimates along the degenerate elliptic curves that meets the free boundary. In general, the expected regularity of solutions at the corner in this domain, even for Laplace's equation, is only C^{α} with $\alpha \in (0,1)$; however, the coefficients in (6) depend on the derivatives of ψ, due to the Bernoulli law (7). To overcome the difficulty, the physical boundary conditions must be exploited to force the finer regularity of solutions at the corner.

As far as we have known, some efforts have been made mathematically for the shock diffraction problem via simplified models. For one of these models, the nonlinear wave system, Kim [16] studied this problem for the right-angle wedge with an additional physical assumption that the transonic shock will not collide with the sonic circle of the right-state. Recently, in Chen-Deng-Xiang [6], this assumption was removed, and the existence and optimal regularity of shock diffraction configurations were established for *all* the angles of the convex wedge via a different approach, which has been further developed in Chen-Xiang [9] to deal with the problem for the potential flow equation.

The purpose of this paper is to present the recent results we have obtained in [9] on the mathematical analysis of this shock diffraction problem for the potential flow equation, which can be formulated as an initial-boundary value problem. By employing its self-similar invariance, this initial-boundary value problem is reduced to a boundary value problem for a first-order nonlinear system of partial differential equations of mixed elliptic-hyperbolic type in an unbounded domain. It is further reformulated as a free boundary problem for nonlinear degenerate elliptic systems of first-order in a bounded domain with a boundary corner whose angle is bigger than π. A first global theory of existence and regularity has been established for this shock diffraction problem for the potential flow equation. To achieve this, we develop several mathematical ideas and techniques, which are also useful for other related problems involving similar analytical difficulties.

The organisation of this paper is as follows. In Section 2, we first formulate the shock diffraction problem as an initial-boundary value problem for the potential flow equation, and then reduce it into the boundary value problem (Problem 1) for a first-order nonlinear system of partial differential equations of mixed elliptic-hyperbolic type, and finally present the main theorem (Theorem 2.1). In Section 3, we first introduce some notions of admissible solutions and weighted Hölder norms and then present some a priori estimates of admissible solutions in the Hölder norms. In Section 4, based on the *a priori* estimates in Section 2, we then prove the existence of the shock diffraction configuration by topological argument and establish Theorem 2.1.

Finally, we remark in passing that a closely related problem, shock reflection-diffraction by a concave cornered wedges for potential flow, has been analyzed in Chen-Feldman [7, 8] and Bae-Chen-Feldman [1], where the existence of regular shock reflection-diffraction configurations has been established up to the sonic wedge-angle. The Prandtl-Meyer reflection for supersonic potential flow impinging onto a solid wedge has also been analyzed first in Elling-Liu [11] and recently in Bae-Chen-Feldman [2]. For other related references, we refer the reader to Canic-Keyfitz-Kim [4, 5] for the unsteady transonic small disturbance equation and the nonlinear wave system, Zheng [25] for the pressure-gradient system, and the references cited therein.

2 The Potential Flow Equation and Shock Diffraction Problem

In this section, we first formulate the shock diffraction problem as an initial-boundary value problem for the potential flow equation, then reduce it into the boundary value problem (Problem 1) for a first-order nonlinear system of partial differential equations of mixed elliptic-hyperbolic type, and finally present the main theorem (Theorem 2.1).

2.1 The potential flow equation and the Rankine-Hugoniot conditions

The Euler equations for potential flow consist of the conservation law of mass and the Bernoulli law for the density ρ and velocity potential Φ with the velocity $(u,v) = \nabla_{\mathbf{x}}\Phi$:

$$\partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \nabla \Phi) = 0, \tag{1}$$

$$\partial_t \Phi + \frac{1}{2}|\nabla_{\mathbf{x}} \Phi|^2 + i(\rho) = B_0, \tag{2}$$

where $i(\rho) = \frac{\rho^{\gamma-1}-1}{\gamma-1}$ for $\gamma > 1$ and $i(\rho) = ln\rho$ for $\gamma = 1$, and B_0 is the Bernoulli constant determined by the incoming flow and/or boundary conditions.

The shock diffraction can be formulated as an initial-boundary value problem: Seek a solution of system (1)–(2) with the initial condition at $t = 0$:

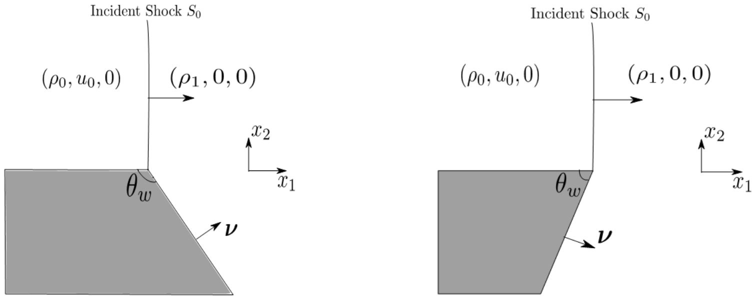

Fig. 1 Initial-Boundary Value Problem

$$(\rho, \Phi)|_{t=0} = \begin{cases} (\rho_0, u_0 x_1) & \text{in } \{x_1 < 0, x_2 > 0\}, \\ (\rho_1, 0) & \text{in } \{-\pi + \theta_w \le \arctan\left(\frac{x_2}{x_1}\right) \le \frac{\pi}{2}\}, \end{cases} \tag{3}$$

and the slip boundary condition along the wedge boundary $\partial\mathscr{W}$:

$$\nabla_{\mathbf{x}}\Phi \cdot \nu|_{\mathbb{R}_+\times\partial\mathscr{W}} = 0, \tag{4}$$

where ν is the exterior unit normal to $\partial\mathscr{W}$ (see Fig. 1).

Notice that the initial-boundary value problem is invariant under the self-similar scaling:

$$(\mathbf{x}, t) \to (\alpha\mathbf{x}, \alpha t), \quad (\rho, \Phi) \to (\rho, \frac{\Phi}{\alpha}) \qquad \text{for } \alpha \neq 0.$$

Thus we seek self-similar solutions with the form:

$$(\rho, \Phi)(\mathbf{x}, t) = (\rho(\xi, \eta), t(\psi(\xi, \eta) + \frac{1}{2}(\xi^2 + \eta^2)) \qquad \text{for } (\xi, \eta) = \frac{\mathbf{x}}{t}, \tag{5}$$

where ψ is the pseudo-velocity potential, that is,

$$(\psi_\xi, \psi_\eta) = (u - \xi, v - \eta) =: (U, V),$$

which is called a pseudo-velocity. Then the pseudo–potential function ψ is governed by the following Euler equations:

$$\operatorname{div}(\rho D\psi) + 2\rho = 0, \tag{6}$$

$$\frac{1}{2}|D\psi|^2 + \psi + \frac{\rho^{\gamma-1}}{\gamma-1} = 0, \tag{7}$$

where the divergence div and gradient D are with respect to the self–similar variables (ξ, η). Here we have replaced ψ by $\psi - \frac{\rho_1^{\gamma-1}}{\gamma-1}$ to make the right-hand side of (7) to be zero.

Then (6) and (7) can be deduced to the following potential flow equation of second-order for the potential function ψ:

$$\operatorname{div}\big(\rho(|D\psi|^2,\psi)D\psi\big)+2\rho(|D\psi|^2,\psi)=0, \tag{8}$$

with

$$\rho(|D\psi|^2,\psi)=\Big(-(\gamma-1)(\psi+\frac{1}{2}|D\psi|^2)\Big)^{\frac{1}{\gamma-1}}. \tag{9}$$

Equation (8) is a second-order equation of mixed hyperbolic-elliptic type: It is elliptic if and only if $|D\psi|<c(|D\psi|^2,\psi):=\sqrt{-(\gamma-1)(\psi+\frac{1}{2}|D\psi|^2)}$, which is equivalent to

$$|D\psi|<c_\star(\psi,\gamma):=\sqrt{-\frac{2(\gamma-1)}{\gamma+1}\psi}. \tag{10}$$

Since one of the corners on the boundary of the pseudo-elliptic domain, *i.e.* the origin, is bigger than π, it is not clear in general whether we could obtain the C^1-regularity of ψ to ensure the ellipticity of (8)–(9) near the point, in comparison with [7]. In fact, there is a counterexample even for Laplace's equation for the general case so that the solution is only in $C^\alpha, \alpha\in(0,1)$, at the corner. One of the key new ingredients here is to exploit the physical boundary conditions to ensure some additional regularity for the ellipticity. To achieve this, instead of studying (8)–(9) for ψ directly as in [7], we consider the corresponding system for $(\rho,U,V)=(\rho,u-\xi,v-\eta)$ to obtain the C^α–estimates of the solutions by exploiting the boundary conditions:

$$\begin{cases}\big(\rho(U,V,\psi)U\big)_\xi+\big(\rho(U,V,\psi)V\big)_\eta+2\rho(U,V,\psi)=0,\\ U_\eta=V_\xi,\\ \frac{\rho^{\gamma-1}(U,V,\psi)}{\gamma-1}+\frac{U^2+V^2}{2}=-\psi,\\ (\psi_\xi,\psi_\eta)=(U,V).\end{cases} \tag{11}$$

Since our global solutions involve shock waves in the problem, the solutions of (11) have to be considered as weak solutions in the distributional sense.

Definition 2.1 *The vector function* (U,V) *is called a weak solution of* (11) *if there exists a function* $\psi\in W^{1,1}_{loc}(\Omega)$ *in a self-similar domain* Ω *such that*

(i) $\psi_\xi=U,\ \psi_\eta=V$ *a.e. in* Ω*;*
(ii) $-\psi-\frac{1}{2}(U^2+V^2)\ge 0$ *a.e. in* Ω*;*
(iii) $(\rho(U,V,\psi),\rho(U,V,\psi)U,\rho(U,V,\psi)V)\in(L^1_{loc}(\Omega))^3$*;*
(iv) *For every* $\zeta\in C_c^\infty(\Omega)$*,*

$$\int_\Omega\rho(U,V,\psi)\big((U,V)\cdot D\zeta-2\zeta\big)\,\mathrm{d}\xi\mathrm{d}\eta=0,$$

and

$$\int_\Omega (V,-U)\cdot D\zeta \,\mathrm{d}\xi\mathrm{d}\eta = 0.$$

For a piecewise smooth solution separated by a shock wave, it satisfies the conditions in Definition 2.1 if and only if it is a classical solution of (11) in each smooth subregion and satisfies the following Rankine-Hugoniot conditions across the shock wave:

$$\big[(\rho(U,V,\psi)U,\rho(U,V,\psi)V)\cdot\nu\big]_S = 0, \tag{12}$$

$$[\psi]_S = 0. \tag{13}$$

Condition (12) is from the conservation of mass, while (13) is from irrotationality.

2.2 *The shock diffraction problem*

If the initial left-state $(\rho_0,u_0,0)$ is subsonic, *i.e.* $u_0 < c_0 := c(\rho_0)$, the degenerate boundary is the sonic circle centered at $(u_0,0)$ with radius c_0, and the center rarefaction wave does not occur near the origin. In this paper, our focus is on this case to consider system (11) in the pseudo-subsonic region.

A discontinuity of $D\psi$ satisfying the Rankine-Hugoniot conditions (12)–(13) is called a shock if it satisfies the following physical entropy condition: *The density ρ increase across a shock in the pseudo–flow direction*. From (12), the entropy condition indicates that the normal derivative function ψ_ν on a shock always decreases across the shock in the pseudo–flow direction, which implies that $\rho_0 > \rho_1$ and $u_0 > u_1 = 0$.

On the other hand, (13) equals to

$$[v-\eta]\mathrm{d}\eta = -[u-\xi]\mathrm{d}\xi. \tag{14}$$

Then, as a direct corollary of (14), the Rankine-Hugoniot conditions are equivalent to:

$$u\big(\rho(u-\xi)+\rho_1\xi\big)+v\big(\rho(v-\eta)+\rho_1\eta\big) = 0, \tag{15}$$

and

$$\psi = \psi_1 \tag{16}$$

along the incident shock. Let $\xi = \xi_1$ is the location of the incident shock. By a straightforward calculation, the incident shock position is

$$\xi_1 = \sqrt{\frac{2\rho_0^2(c_0^2-c_1^2)}{(\gamma-1)(\rho_0^2-\rho_1^2)}} > 0 \tag{17}$$

with the property:

$$0 < u_0 = \frac{\rho_0-\rho_1}{\rho_0}\xi_1 < \xi_1. \tag{18}$$

Furthermore, we can show that the incident shock hits the sonic circle of the left-state, *i.e.* state (0), by the following relation:

$$0 < \xi_1 - u_0 < c_0. \tag{19}$$

In the self-similar plane, the domain outside the wedge is

$$\Lambda := \{(\xi, \eta) : -\pi + \theta_w \leq \arctan\big(\frac{\eta}{\xi}\big) \leq \pi\}.$$

Then the shock diffraction problem can be formulated as the following boundary value problem in the self-similar coordinates (ξ, η).

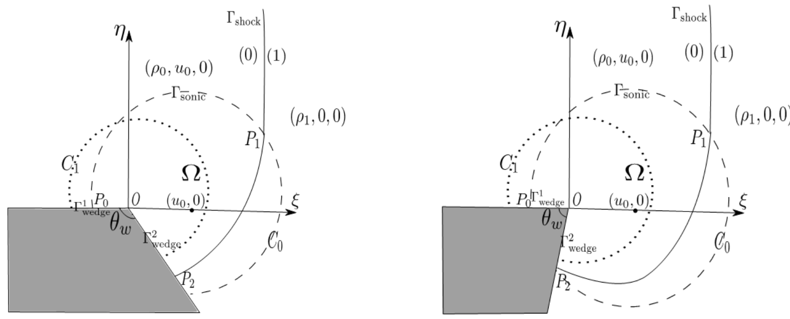

Fig. 2 Boundary Value Problem

Problem 1 (Boundary Value Problem) (See Fig. 2). Seek a solution (U,V) of equations (11) in the self–similar domain Λ with the slip boundary condition on the wedge boundary $\partial\Lambda$:

$$(U,V)\cdot\nu|_{\partial\Lambda} = 0$$

and the asymptotic boundary condition at infinity:

$$(\rho, U, V) \to (\bar{\rho}, \bar{U}, \bar{V}) \qquad \text{when } \xi^2 + \eta^2 \to \infty,$$

in the sense that

$$\lim_{R\to\infty} \|(\rho, U, V) - (\bar{\rho}, \bar{U}, \bar{V})\|_{C(\Lambda \setminus B_R(0))} = 0,$$

where

$$(\bar{\rho}, \bar{U}, \bar{V}) \begin{cases} (\rho_0, u_0 - \xi, -\eta) & \text{for } \{\xi < \xi_1, \eta > 0\}, \\ (\rho_1, -\xi, -\eta) & \text{for } \{\xi > \xi_1, \eta > 0\} \cup \{-\pi + \theta_w \leq \arctan\big(\frac{\eta}{\xi}\big) \leq 0\}. \end{cases}$$

Since $(u_0 - \xi, -\eta)$ does not satisfy the slip boundary condition for $\xi \geq 0$, the solution must differ from state (0) in $\{\xi < \xi_1\} \cap \Lambda$ near the wedge corner, which forces the shock to be diffracted by the wedge. In the domain Ω bounded by the pseudo-sonic circle of the left-state, *i.e.* state (0) with center $(u_0, 0)$ and radius $c_0 > 0$, and the shock wave, the solution is expected to be pseudo-subsonic and smooth, to satisfy the slip boundary condition along the wedge, and to be at least continuous across the pseudo-sonic circle to be pseudo-supersonic. The main theorem we have established is

Theorem 2.1 (Main Theorem) *Let θ_c be the critical angle of the given data such that the corresponding wedge boundary Γ^2_{wedge} passes the intersection point of the two sonic circles of the given Riemann data. Then there exists $\alpha = \alpha(\rho_0, \rho_1, u_0, \gamma) \in (0, \frac{1}{2})$ such that, when $\theta_w \in (\theta_c, \pi)$, there exists a pair of global self–similar solutions:*

$$\rho(\mathbf{x},t) = \Big(-(\gamma-1)(\partial_t \Phi(\mathbf{x},t) + \frac{1}{2}|\nabla_{\mathbf{x}}\Phi(\mathbf{x},t)|^2) \Big)^{\frac{1}{\gamma-1}},$$
$$(u,v)(\mathbf{x},t) = \nabla_{\mathbf{x}}\Phi(\mathbf{x},t) \qquad \text{for } \frac{\mathbf{x}}{t} \in \Lambda,\ t>0$$

for the shock diffraction by the wedge, with $\psi(\mathbf{x},t)$ defined by (5) *which satisfies*

$$\Phi(\mathbf{x},t) = t\psi(\frac{\mathbf{x}}{t}) + \frac{|\mathbf{x}|^2}{t} \qquad \text{for } \frac{\mathbf{x}}{t} \in \Lambda,\ t>0;$$

Equivalently, (U,V) with the pseudo-potential velocity ψ solving Problem 1 *satisfies that, for $(\xi,\eta) = \frac{\mathbf{x}}{t}$,*

$$(U,V) \in C^\infty(\Omega) \cap C^\alpha(\overline{\Omega}),$$

$$(\rho,U,V) = \begin{cases} (\rho_0, u_0-\xi, -\eta) & \text{for } \xi<\xi_1 \text{ and above the sonic circle } \widehat{P_1P_2}, \\ (\rho_1, -\xi, -\eta) & \text{on the right of or below the diffracted shock.} \end{cases} \tag{20}$$

In addition, the corresponding pseudo–potential velocity ψ is $C^{1,1}$ across the part $\widehat{P_0P_1}$ including the endpoints P_0 and P_1, the $C^{1,1}$–regularity is optimal, and the limit of $D^2\psi$ at P_1 does not exist; The transonic shock $\widehat{P_1P_2}$ is C^2 at P_1 and C^∞ except P_1. Furthermore, the solution pair (U,V) is stable with respect to the wedge-angle θ_w, i.e. ψ, as well as (U,V), converges to the unique incident plane shock solution at $\xi = \xi_1$ as $\theta_w \to \pi$.

We remark here that, when the wedge-angle $\theta_w \leq \frac{\pi}{2}$, it needs a transformation and other technical arguments in order to prove the existence of the solutions. To illustrate the key ideas more directly, we will restrict our sketch of the proof to the case that $\theta_w > \frac{\pi}{2}$, for which such a transformation is not needed.

3 Admissible Solutions and A Priori Estimates

In this section, we introduce some notions of admissible solutions and weighted Hölder norms, and present some *a priori* estimates of admissible solutions in the Hölder norms.

3.1 Weighted Hölder spaces and norms

Let $\mathscr{P}$ denote the corner points of $\partial\Omega$, and $B_\delta(\mathscr{P})$ be the union of the balls of radius δ centered at the corner points in $\mathscr{P}$. We then define the following weighted Hölder norms and Hölder spaces:

$$
\begin{aligned}
&[u]_{k,\Omega}^{(-\sigma),\mathscr{P}} = [u]_{k,0,\Omega}^{(-\sigma),\mathscr{P}} = \sup_{\delta>0} \sup_{\substack{x\in\Omega\setminus B_\delta(\mathscr{P})\\ |\beta|=k}} \left(\delta^{k-\sigma}|D^\beta u(x)|\right);\\
&[u]_{k+\alpha,\Omega}^{(-\sigma),\mathscr{P}} = \sup_{\delta>0} \sup_{\substack{x,y\in\Omega\setminus B_\delta(\mathscr{P})\\ |\beta|=k}} \left(\delta^{k+\alpha-\sigma}\frac{|D^\beta u(x)-D^\beta u(y)|}{|x-y|^\alpha}\right);\\
&\|u\|_{k,\Omega}^{(-\sigma),\mathscr{P}} = \textstyle\sum_{j=0}^k [u]_{j,\Omega}^{(-\sigma),\mathscr{P}};\\
&\|u\|_{k+\alpha,\Omega}^{(-\sigma),\mathscr{P}} = \|u\|_{k,\Omega}^{(-\sigma),\mathscr{P}} + [u]_{k+\alpha,\Omega}^{(-\sigma),\mathscr{P}};\\
&C_{k+\alpha,\Omega}^{(-\sigma),\mathscr{P}} := \{u \,:\, u\in C^{k,\alpha}(\Omega) \text{ and } \|u\|_{k+\alpha,\Omega}^{(-\sigma),\mathscr{P}} < \infty\},
\end{aligned}
\tag{21}
$$

where k is integer and $0<\alpha<1$. We remark that the weight near the wedge corner O will be separately dealt with from the others since the angle is bigger than π here. It is easy to verify that

$$
\|fg\|_{0+\alpha,\Omega}^{(\tau_1+\tau_2),\mathscr{P}} \le \|f\|_{0+\alpha,\Omega}^{(\tau_1),\mathscr{P}} \|g\|_{0+\alpha,\Omega}^{(\tau_2),\mathscr{P}} \qquad \text{for } \tau_1+\tau_2\ge 0. \tag{22}
$$

As in [17], there are two important properties of these norms:

(A) $\|u\|_{\alpha,\Omega}^{(-\sigma),\mathscr{P}} \le C\|u\|_{\sigma,\Omega}^{(-\sigma),\mathscr{P}} = C\|u\|_{\sigma,\Omega}$ for $0<\alpha\le\sigma$, where $\|u\|_{\sigma,\Omega}$ is the non-weighted Hölder norms for u.

(B) If $a\ge b>0$ and if $\{u_m\}$ is a bounded sequence in $C_a^{(-b),\mathscr{P}}$, then there is a subsequence $\{u_{m_j}\}$ which converges in any $C_{a'}^{(-b'),\mathscr{P}}$, with $0<b'<b$, $0<a'<a$, and $a'\ge b'$.

Before introducing the parabolic norm near the sonic circle, first we define the Ω' and Ω'' for any domain Ω as

$$
\begin{aligned}
\Omega' &:= \Omega\cap\{(\xi,\eta) \,:\, \text{dist}\{(\xi,\eta),\Gamma_{\text{sonic}}\} < 2\varepsilon_0\},\\
\Omega'' &:= \Omega\cap\{(\xi,\eta) \,:\, \text{dist}\{(\xi,\eta),\Gamma_{\text{sonic}})\} > \varepsilon_0\}
\end{aligned}
\tag{23}
$$

with the small constant $\varepsilon_0 > 0$. Obviously, $\Omega = \Omega' \cup \Omega''$, and it will be seen later that the equation studied is uniformly elliptic in Ω'' and elliptic in Ω', in fact degenerate on $\Gamma_{sonic} := \Omega \cap \{(\xi,\eta) : \sqrt{(\xi-u_0)^2+\eta^2} = c_0\}$.

In Ω', the equation is degenerate elliptic, for which the Hölder norms with parabolic scaling are natural. We define the norm $||\psi||^{\rm par}_{2,\alpha,\Omega'}$ as follows: First introduce new coordinate (x,y) in Ω' as

$$x = c_0 - \sqrt{(\xi-u_0)^2+\eta^2}, \qquad y = \arctan\Big(\frac{\eta}{\xi-u_0}\Big).$$

Denoting $z=(x,y)$ and $\bar{z}=(\bar{x},\bar{y})$ with $x,\bar{x}\in(0,2\varepsilon_0)$ and

$$\delta^{\rm par}_{\alpha}(z,\bar{z}) := \big(|x-\bar{x}|^2+\min\{x,\bar{x}\}|y-\bar{y}|^2\big)^{\frac{\alpha}{2}},$$

then, for $\psi\in C^2(\Omega')$ in the (x,y)–coordinates, we define

$$\begin{aligned}
&\|\psi\|^{\rm par}_{2,0,\Omega'} := \textstyle\sum_{0\le m+l\le 2}\sup_{z\in\Omega'}\big(x^{m+\frac{l}{2}-2}|\partial_x^m\partial_y^l u(z)|\big)\\
&[\psi]^{\rm par}_{2,\alpha,\Omega'} := \textstyle\sum_{m+l=2}\sup_{z,\bar{z}\in\Omega', z\neq\bar{z}}\Big(\big(\min\{x,\bar{x}\}\big)^{\alpha-\frac{l}{2}}\frac{|\partial_x^m\partial_y^l u(z)-\partial_x^m\partial_y^l u(\bar{z})|}{\delta^{\rm par}_{\alpha}(z,\bar{z})}\Big)\\
&\|\psi\|^{\rm par}_{2,\alpha,\Omega'} := \|u\|^{\rm par}_{2,0,\Omega'}+[u]^{\rm par}_{2,\alpha,\Omega'}
\end{aligned} \tag{24}$$

We refer [7] for more details for the motivation of this definition.

3.2 Notion of admissible solutions

The proof of Theorem 2.1 is based on the local existence and the uniform *a priori* estimates of admissible solutions. More precisely, we define the set

$$I \subset [0,\pi] \tag{25}$$

satisfies that, for any $\theta_w \in I$, there exists an admissible solution $(\rho^{(\theta_w)}, U^{(\theta_w)}, V^{(\theta_w)})$ for the shock diffraction problem. Here, the admissible solutions are defined as follows:

Definition 3.1 *Let $\gamma > 1$, $\rho_0 > \rho_1 > 0$, and $u_0 < c_0$, and let (ρ_0,ρ_1,u_0) satisfy* (17) *and* (18). *For any wedge-angle $\theta_w\in(\theta_c,\pi)$ and function $W=(U,V)\in(C^{\alpha}(\Lambda))^2$, $\theta_w\in I$ if and only if*

(i) *The function W is a weak solution to the shock diffraction problem, i.e. W satisfies Definition* 2.1 *and the Rankine-Hugoniot conditions* (12)–(13).
(ii) *The free boundary Γ_{shock}, with endpoints $P_1=(\xi_1,\eta_1)$ and $P_2=(\xi_2,\eta_2)$, lies between the two sonic circles of state* (0) *and state* (1), *i.e.,* $(\rho_0,u_0,0)$ *and* $(\rho_1,0,0)$ *respectively, and meets the wedge at P_2 perpendicularly. In addition, Γ_{shock} is C^∞ everywhere except the point P_1.*

(iii) (U,V) *satisfies* (20) *outside of* Ω, *and*

$$(U,V) \in \left(C^{\alpha}(\overline{\Omega}) \cap C^{1}(\overline{\Omega}\backslash\overline{OP_0P_1}) \cap C^{\infty}(\overline{\Omega}\backslash\overline{\Gamma_{sonic}\cup\{O\}})\right)^2,$$

where $\alpha \in (0,1)$ *depends only on* θ_w *and the given data.*

(iv) *Equation* (8) *is strictly elliptic in* $\overline{\Omega}\backslash\overline{\Gamma_{sonic}}$, *that is,*

$$|\nabla\psi| < c_\star(\psi,\gamma) := \sqrt{-\frac{2(\gamma-1)}{\gamma+1}\psi}.$$

(v) $u > 0$ *and* $v < 0$ *in* Ω.

In fact, admissible solutions have the following additional properties. Some of them need some technical proofs, which can be found in Chen-Xiang [9].

Remark 3.1 (Extension of the background solutions to a smaller wedge-angle) *The property that* Γ_{shock} *meets the wedge at* P_2 *perpendicularly in* (ii) *of Definition* 3.1 *and the slip boundary condition yield that, for any* $\theta_w \in I$ *and any* $\Theta_w < \theta_w$, *there are functions* $\tilde{W} = (\tilde{U},\tilde{V})$ *such that they satisfy equations in* (11) *in* $\Omega^{(\Theta_w)}$ *and* $\tilde{W} = W$ *in* $\Omega^{(\theta_w)}$, *where* $\Omega^{(\theta_w)}$ *is the domain corresponding to the wedge-angle* θ_w. *We call* $\tilde{W}$ *is the extension of the admissible solution* W, *which will be used as a background solution in our proof of Theorem* 1.1.

Remark 3.2 (Existence of the shock up to the wedge) *The property that* $v < 0$ *in* (v) *of Definition* 3.1 *and the fact that* $v = 0$ *on the right-hand side mean that* Γ_{shock} *exists up to the wedge boundary due to the jump of the velocity* v.

Remark 3.3 (Positivity of the horizontal speed u along $\Gamma_{\mathbf{shock}}$) *Properties* (v)–(vi) *of Definition* 3.1 *can deduce that, along* Γ_{shock}, *the horizontal velocity* u *is positive.*

Remark 3.4 (Uniform estimates of the size of domain Ω) *The property that the shock lies between two sonic circles in* (ii) *and the fact that* Γ_{shock} *exists up to the wedge boundary mean that the size of domain* Ω *is bounded.*

Remark 3.5 (The entropy condition) *Properties* (i) *and* (iv)–(v) *of Definition 3.1 deduce that*

$$\partial_\nu\varphi_1 > \partial_\nu\varphi > 0 \qquad on\ \Gamma_{shock},$$

where ν *is the unit normal to* Γ_{shock} *interior to* Ω.

Remark 3.6 (Monotonicity of $\Gamma_{\mathbf{shock}}$) *Properties* (i) *and* (v)–(vi) *deduce that, if* $\Gamma_{shock} = \{(\xi,\eta) : \xi = \xi(\eta)\} = \{(r,\theta) : r = r(\theta)\}$, *then*

$$\xi'(\eta) \geq 0, \quad r'(\theta) \geq 0.$$

Remark 3.7 (I is non-empty) *Based on the proof of the existence of the solutions to the wedge-angle near* π, *we have* $\theta_w \in I$ *when* $\pi - \theta_w$ *small. Thus,* $I \neq \emptyset$. *Then Theorem* 2.1 *is established if we can prove that the subset* I *is both open and closed.*

3.3 A second-order equation for v and the boundary conditions

It is important to deduce first an equation for v from the potential flow equation for our study. To do so, we first introduce an elliptic cut-off function which will be given in detail later, take the derivative on the equation of the conservation of mass with respect to η, and then use the irrotationality to obtain a second-order equation for v in Ω as

$$\begin{aligned} Q(v;u) &= \bar{a}_{11}v_{\xi\xi} + 2\bar{a}_{12}v_{\xi\eta} + \bar{a}_{22}v_{\eta\eta} + b_{11}v_\xi^2 + b_{12}v_\xi v_\eta + b_{22}v_\eta^2 + c_1 v_\xi + c_2 v_\eta \\ &= 0, \end{aligned} \tag{26}$$

where

$$|b_{11}| + |b_{12}| + |b_{22}| \leq \frac{C}{a_{11}}$$

with C depending on the C^1–bounds of $\hat{\psi}$ and the cut–off functions ζ_i and ζ_M, while

$$d_O^\alpha(|c_1| + |c_2|) \leq \frac{C}{a_{11}}$$

with C depending on the $||\hat{\psi}||^{(-1-\alpha),\{O,P_0\}}_{2,\alpha,\Omega''}$, where $d_O(X) = \mathrm{dist}\{X, O\}$.

Modify the Rankine-Hugoniot condition $F(u,v,\varphi,\eta) = 0$ to be

$$G := \zeta_s F + (1-\zeta_s)(L_1(u-\hat{u}) + L_2(v-\hat{v})),$$

where ζ_s is a special cut-off function such that $(\zeta_s)_u(u-\hat{u}) + (\zeta_s)_v(v-\hat{v})$ is a small term, and L_2 is chosen to be close to $F_v(\hat{u},\hat{v},\hat{\varphi},\hat{\eta})$ and L_1 is appropriately determined by $F_v(\hat{u},\hat{v},\hat{\varphi},\hat{\eta})$ and $\frac{F_u(\hat{u},\hat{v},\hat{\varphi},\hat{\eta})}{F_v(\hat{u},\hat{v},\hat{\varphi},\hat{\eta})}$. Then, differentiating it along the shock, we have the following boundary condition on Γ_{shock}:

$$M^{(2)}v := \beta_1^s v_\xi + \beta_2^s v_\eta = \bar{a}_{11}A_{s,1}v + g_s(u,v,\varphi) \qquad \text{on } \Gamma_{\mathrm{shock}}. \tag{27}$$

One of the points in designing ζ_s, L_1, and L_2 is to make sure that $\bar{a}_{11}A_{s,1} \geq 0$, and $\|g_s\|_\infty \leq C$, independent of s.

On the other hand, taking the derivative on the slip boundary condition along the boundary, we have the following boundary condition on $\Gamma^2_{\mathrm{wedge}}$:

$$M^{(1)}v = \beta_1^{(1)}v_\xi + \beta_2^{(1)}v_\eta = 0 \qquad \text{on } \Gamma^2_{\mathrm{wedge}}. \tag{28}$$

Moreover, v satisfies the Dirichlet boundary condition:

$$v = 0 \qquad \text{on } \Gamma_{\mathrm{sonic}} \cup \Gamma^1_{\mathrm{wedge}}, \tag{29}$$

and the one point boundary condition:

$$v = -g(\xi_w, \theta_w)\tan(\pi - \theta_w) \tag{30}$$

to guarantee the equivalence of the deduced equations and the original equations. The one point boundary condition is obtained from the slip boundary condition and the Rankine-Hugoniot condition.

3.4 Uniform estimates of the obliqueness along Γ_{shock}

The crucial proof to guarantee the obliqueness of the operator $M^{(2)}$ is that, if $(\hat{u},\hat{v},\hat{\varphi})$ is the solution in the sense of Definition 3.1, then

$$F_u(\hat{u},\hat{v},\hat{\varphi},\eta) > 0 \qquad \text{along } \Gamma_{\text{shock}}.$$

With this result, after carefully calculation, we can prove that the operators $M^{(i)}$ are oblique along Γ^2_{wedge} and Γ_{shock} respectively. Here the fact that $\hat{u} > 0$ and $\hat{v} < 0$ along Γ_{shock} plays a fundamental role. At the same time, $-\bar{a}_{11}A_{s,1} \leq 0$ is important for the maximum principle.

3.5 Uniform estimates of the approximate solutions near the origin

Consider the approximate solutions v^ε governed by

$$Q(v^\varepsilon;u^\varepsilon) + \varepsilon \triangle v^\varepsilon = 0,$$

and the boundary conditions (27)–(30), where Q is defined in (26). We prove that there exist $\sigma^* > 0$ and $\alpha_0 > 0$ such that, for each $\sigma \leq \sigma^*$ and $\alpha \leq \alpha_0$, and for any approximate solution v^ε, near the wedge corner O, we have

$$\|v^\varepsilon\|^{(-\sigma)}_{2+\alpha,\Omega} \leq C(\lambda,\theta_w,\Lambda,\Omega)\big(\|g(\xi_w,\theta_w)\| + \|g_s\|_\infty\big). \tag{31}$$

Furthermore, if the solution $(u^\varepsilon, v^\varepsilon, \varphi^\varepsilon)$ is sufficiently close to the background solution $(\hat{u},\hat{v},\hat{\varphi})$, then the boundary condition on Γ_{shock} will not be involved with inhomogeneous term:

$$M^{(2)}v^\varepsilon = \beta_1^{(2)} v^\varepsilon_\xi + \beta_2^{(2)} v^\varepsilon_\eta = 0;$$

thus the solution v^ε has a better estimate:

$$-g(\xi_w,\theta_w)\tan(\pi-\theta_w) \leq v^\varepsilon \leq 0. \tag{32}$$

3.6 Impossibility of Γ_{shock} meeting the sonic circle of state (1) and the sonic circle of state (0) except P_1

We prove that $r'(\theta) \geq 0$ along Γ_{shock}, which means that Γ_{shock} will not meet the sonic circle of state (0) again away from P_1. Next, we prove that there exists a constant $C > 0$ such that

$$\text{dist}\{\Gamma_{\text{shock}}, B_{c_1}(O)\} > \frac{1}{C},$$

for any solution in the sense of Definition 3.1, where c_1 is the sonic speed of state (1), *i.e.* the right-state. These estimates are crucial to guarantee the ellipticity in the domain Ω.

3.7 Uniform Hölder estimates of $(u^\varepsilon, v^\varepsilon)$ near Γ_{sonic}, and uniform upper and lower estimates of density ρ^ε

In order to pass the limit $\varepsilon \to 0$, we need uniform estimates of the approximate solutions near Γ_{sonic}, where the ellipticity may degenerate. In fact, we prove the uniform estimates near Γ_{sonic} by scaling,

$$|v^\varepsilon| \leq A(c_0 - r)^{1/4}, \tag{33}$$

$$|u^\varepsilon - u_0| + |\rho^\varepsilon - \rho_0| \leq A(c_0 - r)^{\frac{1}{6}} \qquad \text{for } 0 \leq c_0 - r \leq m. \tag{34}$$

As in Section 3.5, if the solution $(u^\varepsilon, v^\varepsilon, \varphi^\varepsilon)$ is sufficiently close to the background solution $(\hat{u}, \hat{v}, \hat{\varphi})$, we have

$$-A(c_0 - r)^{\frac{1}{4}} \leq v^\varepsilon \leq 0 \qquad \text{for } 0 \leq c_0 - r \leq m. \tag{35}$$

From the uniformly estimates away from Γ_{sonic}, $\|u^\varepsilon\|_0$ and $\|v^\varepsilon\|_0$ are uniformly bounded, then $\|\varphi^\varepsilon\|_{C^{0,1}}$ is also uniformly bounded, and

$$\Big(\frac{2}{\gamma+1}\Big)^{\frac{1}{\gamma-1}} \rho_1 \leq \rho^\varepsilon \leq C \qquad \text{in } \Omega.$$

3.8 Monotonicity of the solution v along Γ_{shock}

From now on, we consider the solutions without the viscosity term $\varepsilon \triangle v$, *i.e.* after passing the limit $\varepsilon \to 0$. What actually we can prove for the monotonicity of v is that, if the solution (u, v, φ) is sufficiently close to the background solution $(\hat{u}, \hat{v}, \hat{\varphi})$, then the solution v is monotonically increasing along Γ_{shock}.

3.9 Uniform estimates of the ellipticity in Ω up to Γ_{shock}

Note that the Mach number

$$M^2 = \frac{(u-\xi)^2+(v-\eta)^2}{c^2} \in C^\alpha(\overline{\Omega}) \cap C^\infty(\overline{\Omega}\backslash(\overline{\Gamma_{\text{sonic}} \cup \{P_1\}})).$$

Then we can show that there exists a constant $\mu > 0$ such that, for any $\theta_w \in (\theta_c, \pi)$, we have

$$M^2(\xi,\eta) \le 1-\mu d \qquad \text{for all } (\xi,\eta)\in\Omega,$$

where $d = \text{dist}\{(\xi,\eta),\Gamma_{\text{sonic}}\}$. It means that, for all $\xi = (\xi_1,\xi_2) \in \mathbb{R}^2$, we have

$$C^{-1}d|\xi|^2 \le \sum_{i,j=1}^{2} a_{ij}\xi_i\xi_j \le C|\xi|^2.$$

3.10 Regularity away and near Γ_{sonic}

For the regularity away from Γ_{sonic}, we employ the weighted Hölder norms and a transformation to control the behaviour of the quadratic nonlinear terms to estimate the solution v near the corners and other points. Next we use the irrotationality to obtain the regularity of u and then the regularity of ρ.

For the regularity near Γ_{sonic}, we use the parabolic norms and a scaling to make the equation non-degenerate. Introduce new coordinates

$$(x,y) = (c_0 - r, \theta - \theta_1)$$

to flatten Γ_{sonic}, where (r,θ) are the polar coordinates, c_0 is the sonic speed of state (0), and (r_1,θ_1) is P_1. Then, following the procedures in [1] exactly, we can derive the following property:

Theorem 3.1 (Optimal regularity) *Let ψ be a solution obtained as before. Then we have*

(i) *ψ cannot be C^2 across the pseudo-sonic circle Γ_{sonic};*
(ii) *$\varphi = \psi - \psi_0$ is $C^{2+\alpha}$ in Ω up to Γ_{sonic} away from the point P_1 for any $\alpha \in (0,1)$;*
(iii) *for any $(\xi_0,\eta_0) \in \Gamma_{sonic}\backslash\{P_1\}$,*

$$\lim_{\substack{(\xi,\eta)\to(\xi_0,\eta_0)\\(\xi,\eta)\in\Omega}} D_{rr}\varphi = \frac{1}{\gamma+1}, \qquad \lim_{\substack{(\xi,\eta)\to(\xi_0,\eta_0)\\(\xi,\eta)\in\Omega}} D_{\theta\theta}\varphi = 0, \qquad \lim_{\substack{(\xi,\eta)\to(\xi_0,\eta_0)\\(\xi,\eta)\in\Omega}} D_{r\theta}\varphi = 0;$$

(iv) *$D^2\varphi$ has a jump across Γ_{sonic}: for any $(\xi_0,\eta_0) \in \Gamma_{sonic}\backslash\{P_1\}$,*

$$\lim_{\substack{(\xi,\eta)\to(\xi_0,\eta_0)\\(\xi,\eta)\in\Omega}} D_{rr}\varphi - \lim_{\substack{(\xi,\eta)\to(\xi_0,\eta_0)\\(\xi,\eta)\in\Lambda\backslash\Omega}} D_{rr}\varphi = \frac{1}{\gamma+1};$$

(v) *the limit* $\lim_{\substack{(\xi,\eta)\to P_1\\(\xi,\eta)\in\Omega}} D^2\varphi$ *does not exist.*

4 Existence of the Shock Diffraction Configuration

Once the *a priori* estimates are proved, the existence of the shock diffraction configuration can be established by topological argument. Thanks to the uniform estimates in Section 3, the set I is obviously closed. Then the remaining task is to prove that the set I is open.

The main idea of the existence proof is that, instead of studying the potential flow equation of φ, we study a system for (ρ,u,v) directly. In order to do that, we first introduce the degenerate elliptic cut-off, the higher order cut-off near the pseudo-sonic circle, and the uniform elliptic cut-off away from the pseudo-sonic circle, and introduce the modified Rankine-Hugoniot condition along Γ_{shock}. Then differentiate them to obtain a second-order equation for v with the oblique boundary conditions on Γ_{shock} and Γ^2_{wedge}. Once the existence of v is obtained, we use the irrotational equation to recover u by v. Next, passing the limit to obtain a solution (u,v,φ) which is actually equivalent to the original potential flow equation of φ. Using this scalar equation, we can obtain a better regularity to remove the cut-off function introduced and prove that the solution we have obtained is actually sufficiently close to the background solution, if the wedge-angle is sufficiently close to the background wedge-angle.

For the main part, the existence of the modified free boundary problem for v, we in fact have the following theorem.

Theorem 4.1 (Modified free boundary problem) *Assume that $\Theta_w\in I$. Then there exist $\delta_0=\delta(\rho_0,\rho_1,u_0,\gamma,\Theta_w)>0$ small enough, $\sigma^*>0$, $\alpha_0>0$, and $\varepsilon^*>0$ such that, for each $\theta_w\in[\Theta_w-\delta_0,\Theta_w)$, $\sigma<\sigma^*$, $\alpha<\alpha_0$, and $\varepsilon\in(0,\varepsilon^*)$, there exists a solution $(u^\varepsilon,v^\varepsilon,\xi^\varepsilon(\eta))\in\big(C^{2+\alpha}_{(-\sigma)}(\Omega^\varepsilon)\big)^2\times C^{2+\alpha}$ to the regularized free boundary problem:*

$$\begin{cases} Q^\varepsilon(v;u):=Q(v;u)+\varepsilon\triangle v=0,\\ u_\eta=v_\xi, \end{cases} \tag{36}$$

with the free boundary position:

$$\xi'=-\zeta_s\frac{v}{u}-(1-\zeta_s)\frac{\hat v}{\hat u} \tag{37}$$

and the following boundary conditions:

$$(u,v,\psi) = (u_0, 0, \psi_0) \qquad \text{on } \Gamma_{sonic}, \tag{38}$$
$$v = 0 \qquad \text{on } \Gamma^1_{wedge}, \tag{39}$$
$$M^{(1)}v = 0 \qquad \text{on } \Gamma^2_{wedge}, \tag{40}$$
$$M^{(2)}v - \bar{a}_{11}A_{s,1}v = g_s(u,v,\varphi) \qquad \text{on } \Gamma_{shock}, \tag{41}$$
$$v = -g(\xi_w, \theta_w)\tan(\pi - \theta_w) \qquad \text{at } P_2. \tag{42}$$

In addition, the solution satisfies the following estimates:

$$|\xi(\eta) - \hat{\xi}(\eta)| \le \delta_1, \qquad 0 \le \xi'(\eta) \le K_2 \tag{43}$$
$$a^\varepsilon_{11}u^\varepsilon_\xi + 2a^\varepsilon_{12}v^\varepsilon_\xi + a^\varepsilon_{22}v^\varepsilon_\eta = C_1(\varepsilon) \to 0 \qquad \text{when } \varepsilon \to 0, \tag{44}$$
$$|v^\varepsilon| \le A(c_0 - r)^{\frac{1}{4}} \qquad \text{for } 0 \le c_0 - r \le m, \tag{45}$$
$$|u^\varepsilon - u_0| + |\rho^\varepsilon - \rho_0| \le A(c_0 - r)^{\frac{1}{6}} \qquad \text{for } 0 \le c_0 - r \le m, \tag{46}$$
$$\|(u^\varepsilon, v^\varepsilon)\|^{(-\sigma)}_{2+\alpha,\Omega} + \|(u^\varepsilon, v^\varepsilon)\|^{(-\sigma-1)}_{2+\alpha,\Omega\setminus B_{d_0}(O)} \le C_2(\varepsilon), \tag{47}$$
$$\|v^\varepsilon\|^{(-\sigma)}_{2+\alpha,\Omega\cap\{c_0-r\ge s\}} + \|v^\varepsilon\|^{(-\sigma-1)}_{2+\alpha,\Omega\cap\{c_0-r\ge s\}\setminus B_{d_0}(O)} \le C(s), \tag{48}$$

and

$$\|u^\varepsilon\|^{(-\sigma)}_{1+\alpha,\Omega\cap\{c_0-r\ge s\}} + \|u^\varepsilon\|^{(-\sigma-1)}_{1+\alpha,\Omega\cap\{c_0-r\ge s\}\setminus B_{d_0}(O)} \le C(s) \tag{49}$$

for some small positive constants δ_1 and K_2, while $C_1(\varepsilon)$, $C_2(\varepsilon)$, and $C(s)$ depend only on the data, the background solution, as well as ε and s respectively. Meanwhile, A and m are independent of θ, and ε_0 is chosen such that $\varepsilon_0 < m$.

The proof of this theorem is long and technical. Thus, instead of showing that here, we would like to illustrate the ideas by proving a simpler case that the wedge-angle is near π. In this case, the background solution is constant, namely, $(\hat{u}, \hat{v}) = (u_0, 0)$. Then the inhomogeneous terms vanish, and the uniform estimate of the smallness between the solution and the background solution can be easier obtained. In fact, the constants on the right-hand side of inequalities (47)–(49) are all multiplied with a small term $\pi - \theta_w$. We now illustrate its proof below.

4.1 The degenerate elliptic cut-off near the pseudo-sonic circle

First define the regions Ω' and Ω'' for any domain Ω as

$$\begin{aligned} \Omega' &:= \Omega \cap \{(\xi,\eta) \,:\, \text{dist}\{(\xi,\eta), \Gamma_{\text{sonic}}\} < 2\varepsilon_0\}, \\ \Omega'' &:= \Omega \cap \{(\xi,\eta) \,:\, \text{dist}\{(\xi,\eta), \Gamma_{\text{sonic}}\} > \varepsilon_0\} \end{aligned} \tag{50}$$

with a small constant $\varepsilon_0 > 0$. Obviously, $\Omega = \Omega' \cup \Omega''$. In this subsection, we will introduce a degenerate elliptic cut-off function ζ_1 and also a cut-off function ζ_M of higher order smallness in Ω'. Since the equation we study requires more precise estimates near Γ_{sonic}, the elliptic cut-off function introduced in this subsection is more accuracy in comparison with that in [7]. In addition, the elliptic cut–off function does not take its values simply on φ_x, but on $\varphi_x - a$ in order to remove the elliptic cut–off function, where a is some constant which is defined in the following statement.

The leading term of the second-order elliptic equation for v is of the following form:

$$\big(c^2-(u-\xi)^2\big)v_{\xi\xi}-2(u-\xi)(v-\eta)v_{\xi\eta}+\big(c^2-(v-\eta)^2\big)v_{\eta\eta}. \tag{51}$$

Thus, in the polar coordinates, introduce the cut–off function ζ_M for small quantities of higher order as

$$\zeta_M=\begin{cases} s & \text{for } |s|\le M,\\ M+1 & \text{for } |s|\ge M+2,\end{cases}$$

so that

$$\zeta_M(-s)=-\zeta_M(s),\quad 0\le \zeta_M'(s)\le 1 \qquad \text{on } \mathbb{R},$$

for some constant M to be determined later. Then rewrite the above form by plugging the cut–off function into the terms involving higher order small quantities as

$$\Big((c_0-r)c_0+(\gamma+1)\big((u-u_0)\cos\theta+v\sin\theta+\tfrac{c_0-r}{\gamma+1}\big)r+O_1\Big)v_{rr}$$
$$+\tfrac{2}{r}O_3v_{r\theta}+\tfrac{1}{r^2}(c_0^2+O_2)v_{\theta\theta}+\tfrac{1}{r}(c_0^2+O_2)v_r-\tfrac{2}{r^2}O_3v_\theta,$$

with

$$O_1=(c_0-r)^2\zeta_M\big(\tfrac{O_1^\varphi}{(c_0-r)^2}\big),$$
$$O_2=(c_0-r)\zeta_M\big(\tfrac{\hat{c}^2-c_0^2-(O_2^\varphi)^2}{(c_0-r)}\big),$$
$$O_3=(c_0-r)^{\frac{3}{2}}\zeta_M\big(\tfrac{-O_2^\varphi r+O_3^\varphi}{(c_0-r)^{3/2}}\big).$$

Therefore, the ellipticity of this form equals to

$$(c_0-r)c_0+(\gamma+1)\Big((u-u_0)\cos\theta+v\sin\theta+\frac{c_0-r}{\gamma+1}\Big)r>0 \quad\text{and}\quad c^2>0.$$

Next, for the degenerate elliptic cut–off, let $\zeta_1\in C^\infty(\mathbb{R})$ satisfy

$$\zeta_1(s)=\begin{cases} s & \text{if } -\frac{1}{3(\gamma+1)}<s<\frac{7}{6(\gamma+1)},\\ -\frac{2}{3(\gamma+1)} & \text{if } s<-\frac{1}{(\gamma+1)},\\ \frac{5}{4(\gamma+1)} & \text{if } s>\frac{4}{3(\gamma+1)},\end{cases} \tag{52}$$

so that

$$\zeta_1'(s) \geq 0 \qquad \text{on } \mathbb{R}, \tag{53}$$
$$\pm\zeta_1''(s) \geq 0 \qquad \text{on } \{\pm s \leq 0\}. \tag{54}$$

The value s that the cut–off functions ζ_1 takes on is

$$\frac{(u-u_0)\cos\theta + v\sin\theta}{c_0 - r} + \frac{1}{\gamma+1}.$$

Then (51) becomes the following modified form:

$$A_{11}v_{\xi\xi} + 2A_{12}v_{\xi\eta} + A_{22}v_{\eta\eta},$$

where

$$\begin{aligned}
A_{11} &= c_0^2 - (\xi-u_0)^2 + (\gamma-1)(c_0-r)r\Big(\zeta\big(\tfrac{(u-u_0)\cos\theta+v\sin\theta}{c_0-r} + \tfrac{1}{\gamma+1}\big) - \tfrac{1}{\gamma+1}\Big)\\
&\quad + \tfrac{2(c_0-r)(\xi-u_0)^2}{r}\Big(\zeta\big(\tfrac{(u-u_0)\cos\theta+v\sin\theta}{c_0-r} + \tfrac{1}{\gamma+1}\big) - \tfrac{1}{\gamma+1}\Big)\\
&\quad + \tfrac{1}{r^2}\big(O_1(\xi-u_0)^2 - 2O_3(\xi-u_0)\eta + O_2\eta^2\big),\\
A_{12} &= -(\xi-u_0)\eta + \tfrac{2(c_0-r)(\xi-u_0)\eta}{r}\Big(\zeta\big(\tfrac{(u-u_0)\cos\theta+v\sin\theta}{c_0-r} + \tfrac{1}{\gamma+1}\big) - \tfrac{1}{\gamma+1}\Big)\\
&\quad + \tfrac{1}{r^2}\big((O_1-O_2)(\xi-u_0)\eta + O_3(\xi-u_0)^2 - O_3\eta^2\big),\\
A_{22} &= c_0^2 - \eta^2 + (\gamma-1)(c_0-r)r\Big(\zeta\big(\tfrac{(u-u_0)\cos\theta+v\sin\theta}{c_0-r} + \tfrac{1}{\gamma+1}\big) - \tfrac{1}{\gamma+1}\Big)\\
&\quad + \tfrac{2(c_0-r)\eta^2}{r}\Big(\zeta\big(\tfrac{(u-u_0)\cos\theta+v\sin\theta}{c_0-r} + \tfrac{1}{\gamma+1}\big) - \tfrac{1}{\gamma+1}\Big)\\
&\quad + \tfrac{1}{r^2}\big(O_1\eta^2 + 2O_3(\xi-u_0)\eta + O_2(\xi-u_0)^2\big).
\end{aligned}$$

4.2 The uniform elliptic cut-off away from the pseudo-sonic circle

Let $\zeta_2 \in C^\infty$ be a smooth increasing function such that

$$\zeta_2(s) = \begin{cases} s & \text{if } s \geq \varepsilon_1,\\ \frac{1}{2}\varepsilon_1 & \text{if } s < 0, \end{cases} \tag{55}$$

and $|\zeta_2'(s)| \leq 1$. Let ζ_2 be evaluated at $c^2 - U^2 - V^2$. In Ω', consider the following modified system:

$$\begin{cases} \frac{U^2\zeta_2 + V^2c^2}{U^2+V^2}u_\xi + \frac{2UV}{U^2+V^2}(\zeta_2 - c^2)u_\eta + \frac{V^2\zeta_2+U^2c^2}{U^2+V^2}v_\eta = 0,\\ v_\xi = u_\eta,\\ c^2 = -\frac{\gamma-1}{2}(U^2+V^2) - (\gamma-1)\psi. \end{cases} \tag{56}$$

Finally, we combine the coefficients introduced above in $\mathscr{D}$ as follows. Let $\zeta_3 \in C^\infty(\mathbb{R})$ satisfy

$$\zeta_3(s) = \begin{cases} 0 & \text{if } s \leq 2\varepsilon_0, \\ 1 & \text{if } s \geq 4\varepsilon_0, \end{cases} \qquad 0 \leq \zeta_3'(s) \leq \frac{10}{\varepsilon_0} \quad \text{on } \mathbb{R}.$$

Then we define that, for $(\rho, u, v) \in \mathbb{R}^3$ and $(\xi, \eta) \in \mathscr{D}$,

$$\begin{aligned} \bar{a}_{11} &= \zeta_3(c_0 - r)\tfrac{U^2\zeta_2 + V^2c^2}{U^2+V^2} + \big(1 - \zeta_3(c_0 - r)\big)A_{11}, \\ \bar{a}_{12} &= \zeta_3(c_0 - r)\tfrac{UV}{U^2+V^2}(\zeta_2 - c^2) + \big(1 - \zeta_3(c_0 - r)\big)A_{12}, \\ \bar{a}_{22} &= \zeta_3(c_0 - r)\tfrac{V^2\zeta_2 + U^2c^2}{U^2+V^2} + \big(1 - \zeta_3(c_0 - r)\big)A_{22}. \end{aligned} \tag{57}$$

This leads to system (11) to be the following modified system:

$$\begin{cases} \bar{a}_{11}u_\xi + 2\bar{a}_{12}u_\eta + \bar{a}_{22}v_\eta = 0, \\ v_\xi = u_\eta, \\ D(\psi - \psi_0) = (u - u_0,\ v), \\ \rho = \big(-\frac{\gamma-1}{2}(U^2+V^2) - (\gamma-1)\psi\big)^{\frac{1}{\gamma-1}}. \end{cases} \tag{58}$$

4.3 A second-order equation for v

In order to study the existence of solutions to system (58), we introduce a second-order equation from this system for v, $Q(v;u)$, by taking the derivative on the first equation with respect to η and then using the other equations to replace the unknown terms. We have

$$\begin{aligned} Q(v;u) &:= \bar{a}_{11}v_{\xi\xi} + 2\bar{a}_{12}v_{\xi\eta} + \bar{a}_{22}v_{\eta\eta} + b_{11}v_\xi^2 + b_{12}v_\xi v_\eta + b_{22}v_\eta^2 + c_1v_\xi + c_2v_\eta \\ &= 0, \end{aligned} \tag{59}$$

where

$$|b_{11}| + |b_{12}| + |b_{22}| < \frac{C}{a_{11}}$$

with C depending on the C^1–bounds of $\hat{\psi}$ and the cut–off functions ζ_i and ζ_M, while

$$d_O^\alpha(|c_1| + |c_2|) < \frac{C}{a_{11}}$$

with C depending on $\|\hat{\psi}\|_{2,\alpha,\Omega''}^{(-1-\alpha),\{O,P_0\}}$ and $d_O(X) = \text{dist}\{X, O\}$.

Near Γ_{sonic}, in the (r,θ)–coordinates, this equation reads

$$\begin{aligned}&\big((c_0-r)\big(1+(\gamma+1)\zeta_1\big)\big)v_{rr}+\tfrac{1}{c_0}v_{\theta\theta}+b_rv_r\\&+\tfrac{(\gamma+1)c_0^2\sin\theta\zeta_1'}{a_{11}}\Big(v_r^2+\tfrac{\cos\theta}{c_0}(-(u-u_0)\sin\theta+v\cos\theta)v_r+\tfrac{(u-u_0)\cos^2\theta+v\sin\theta\cos\theta}{c_0-r}v_r\Big)\\&-\tfrac{(2c_0^2+(\gamma-1)r^2)\cos\theta\zeta_1'}{a_{11}r}v_rv_\theta+O_1v_r+O_2v_rv_\theta+O_3v_\theta=0,\end{aligned} \tag{60}$$

where

$$b_r:=\frac{1}{a_{11}}\Big(\big(\sin^2\theta+1\big)\big(a_{11}\cos^2\theta+2a_{12}\sin\theta\cos\theta+a_{22}\sin^2\theta\big)$$
$$-(\gamma+1)\big(c_0+O(1)(c_0-r)\big)r\sin^2\theta\zeta_1\Big). \tag{61}$$

Lemma 4.1 *If*

$$\zeta_1\geq-\frac{2}{3(\gamma+1)},$$

then there exists $\varepsilon_0>0$ *such that, for any* $0\leq c_0-r\leq\varepsilon_0$*, we have*

$$-\frac{9}{8}(\gamma+1)\max\{\zeta_1,0\}\leq b_r\leq C, \tag{62}$$

where C *is a uniform constant independent on* θ*,* u*, and* v*.*

This lemma is crucial for the proof of the uniform Hölder estimate of v near Γ_{sonic}.

Finally, equation (60) can be rewritten in the divergent form by scaling as follows:

$$\Big((c_0-r)\big(1+(\gamma+1)\zeta_1\big)v_r\Big)_r+(\frac{1}{c_0}v_\theta)_\theta+O_1v_r+O_2(c_0-r)(v_r)^2$$
$$+O_3(c_0-r)v_rv_\theta+O_4v_\theta=0. \tag{63}$$

with $|O_i|\leq C$, provided that $\sin\theta>0$.

On the other hand, away from Γ_{sonic}, we notice that the equation is strictly and uniformly elliptic with the bounded coefficients depending only on δ_0 and C.

4.4 *The different boundary conditions from those stated in Theorem* 4.1

The difference comes out at the free boundary. First, the condition for the free boundary position can simply be proposed as

$$\xi'=-\frac{v}{u}. \tag{64}$$

Then take the derivative on the Rankine-Hugoniot condition along $\Gamma_{\rm shock}$ and use (64) to yield the oblique boundary condition on $\Gamma_{\rm shock}$:

$$M^{(2)}v=\beta_1^{(2)}v_\xi+\beta_2^{(2)}v_\eta=0 \qquad \text{on } \Gamma_{\rm shock}, \tag{65}$$

with

$$\beta_1^{(2)}=(-\bar a_{11}+2\bar a_{12}\xi')F_u-\bar a_{11}F_v\xi',$$
$$\beta_2^{(2)}=-\bar a_{11}F_v+\bar a_{22}F_u\xi',$$

where, along $\Gamma_{\rm shock}$, $F(u,v,\varphi,\eta)=0$.

4.5 Existence of solutions for the linearized viscous fixed boundary problem for v

We now linearize the modified problem for v, and first show the local existence of solutions near the wedge corner O (where $\Gamma^1_{\rm wedge}$ and $\Gamma^2_{\rm wedge}$ meet) by the method of continuity. Next we show the local existence near P_2, where $\Gamma_{\rm shock}$ and $\Gamma^2_{\rm wedge}$ meet. With this local solvability, we focus on the proof of the global existence of solutions by the Perron method, as used in [15], [17], and [18].

Before proving the existence of solutions, we introduce some notations which are important in the Perron method. The linearized problem is called locally solvable if, for each $y\in\bar\Omega$, there is a neighborhood $N=O(y)\cap\Omega$ such that, for any $h\in C(\bar N)$, there is a solution $v\in C^2(N)\cap C(\bar N)$ of the problem:

$$\begin{cases} Lv=0 \quad \text{in } N,\\ N^{(1)}v\big|_{\bar N\cap\Gamma^2_{\rm wedge}}=0,\\ N^{(2)}v\big|_{\bar N\cap\Gamma_{\rm shock}}=0,\\ v\big|_{\partial N'}=h,\\ v\big|_{P_2}=-g(\xi_w,\theta_w)\tan(\pi-\theta_w),\end{cases}$$

where $\partial N'=\partial N\cap\Omega$. For brevity, as in [17], denote this function v by $(h)_y$ to emphasize its dependence on h and y. Denote S^- (S^+) the set of all subsolutions (supersolutions) of the problem. A subsolution or supersolution $w\in S^\pm$ of the linearized problem is a function $w\in C(\bar\Omega)$ satisfying

$$\pm\big(g(\xi_w,\theta_w)\tan(\pi-\theta_w)+w\big)\le 0 \qquad \text{at } P_2$$

and

$$\pm w\le 0 \qquad \text{on } \bar N\cap(\Gamma_{\rm sonic}\cup\Gamma^1_{\rm wedge}),$$

such that, for any $y\in\bar\Omega$, if $\pm(h-w)\ge 0$ on $\partial N'$, then

$$\pm\big((h)_y - w\big) \geq 0 \qquad \text{in } N(y).$$

Then we show properties (i)–(vii) listed below to prove the global existence for the linearized problem:

(i) If $u_1, u_2 \in S^-$, then $\max\{u_1, u_2\} \in S^-$.
(ii) If $u_1 \in S^-$ and $y \in \bar{\Omega}$, and if $\bar{u}_1$ is given by $\bar{u}_1 = u_1$ in $\bar{\Omega}\backslash N(y)$ and $\bar{u}_1 = (u_1)_y$ in $N(y)$, then $\bar{u}_1 \in S^-$.
(iii) If $w^\pm \in S^\pm$, then $w^+ \geq w^-$ in Ω.
(iv) If $w^\pm \in C^2(N) \cap C(\bar{N})$ satisfy $Lw^+ = Lw^-$ in $N \cap \Omega$, $\tilde{M}w^+ = \tilde{M}w^-$ on $N \cap \Gamma_{\text{wedge}}$, and $w^+ \geq w^-$ in $N \cap \Omega$, then either $w^+ = w^-$ in N or else $w^+ > w^-$ in N.
(v) $S^\pm$ are non-empty.
(vi) Let $\{u_m\}$ be a bounded sequence of $C^2(N) \cap C(\bar{N})$–solutions of $Lu_m = 0$ in $N \cap \Omega$ and $\tilde{M}u_m = 0$ on $N \cap \Gamma_{\text{wedge}}$. Then there is a convergent subsequence $\{u_m\}$ such that $u = \lim u_{m_i}$ is a $C^2(N)$–solution of $Lu = 0$ in $N \cap \Omega$ and $\tilde{M}u = 0$ on $N \cap \Gamma_{\text{wedge}}$.
(vii) For each $x_0 \in \Gamma_{\text{shock}} \cup \Gamma_{\text{sonic}}$, there are sequences $\{w_m^\pm\}$ of subsolutions and supersolutions such that $\lim w_m^\pm(x_0) = u(x_0)$.

4.6 Existence of solutions for the modified nonlinear fixed boundary problem for v

Once the linearized problem is solved, the existence for the modified nonlinear problem can be proved by the Leray–Schauder fixed point theorem (*cf.* Theorem 11.3 in [15]).

To achieve this, we first introduce the sets $\mathscr{H}^\varepsilon$ that is defined in a bounded domain Ω and $\mathscr{K}^\varepsilon$ in a bounded domain $(-\xi_1 \tan(\pi - \theta_w), \eta_1]$, depending on given values θ_w, ρ_0, ρ_1 and u_0, as follows:

Definition 4.1 *The elements of* $\mathscr{H}^\varepsilon \in C^{2+\alpha}_{(-\nu)}$, *satisfy*

(H1) $u = u_0$ *on* Γ_{sonic};
(H2) $|u - u_0| \leq A_0(c_0 - r)^{1/6}$ *when* $|c_0 - r|$ *small independent on* θ;
(H3) $\|u\|^{(-\nu)}_{2+\alpha} \leq A_1(\varepsilon)$;
(H4) $\|u\|^{(-\nu-1)}_{2+\alpha} \leq A_2(\varepsilon)$ *away from the wedge-angle O.*

Definition 4.2 *The elements of* $\mathscr{K}^\varepsilon \in C^{2+\alpha}$, *satisfy*

(K1) $\xi(\eta_1) = \xi_1$;
(K2) $\xi'(\eta_1) = 0$;
(K3) $|\xi(\eta) - \hat{\xi}(\eta)| \leq \delta_*$;
(K4) $0 \leq \xi'(\eta) \leq K_2$

The weighted Hölder space is defined in (21). The values of α, $\nu \in (0,1)$, as well as K_i, δ_1, and A_i, will be specified later. Obviously, $\mathscr{H}^\varepsilon$ and $\mathscr{K}^\varepsilon$ are closed, bounded and convex.

Then the crucial step to apply the fixed point theorem is to prove the following uniform estimates which are also stated in Section 3:

Lemma 4.2 *For given K_i, δ_1, and A_i for $\mathscr{K}^\varepsilon$ and $\mathscr{H}^\varepsilon$, there exist σ^*, $\alpha_0 \in (0,1)$, and $d_0 > 0$ such that any solution $v \in C^{2+\alpha}_{(-\sigma)}(\Omega) \cap C^{2+\alpha}_{(-\sigma-1)}(\Omega \backslash B_{d_0}(O))$ to the nonlinear problem $v = \sigma \mathbf{T} v$ with $\alpha \leq \alpha_0$, $\sigma \leq \sigma^*$, and $\sigma \in [0,1]$ satisfies*

$$\|v\|^{(-\sigma)}_{2+\alpha,\Omega} \leq C \tan(\pi - \theta_w), \tag{66}$$

$$\|v\|^{(-1-\sigma)}_{2+\alpha,\{\Omega \backslash B_{d_0}(P_0)\}} \leq C \tan(\pi - \theta_w), \tag{67}$$

$$\tag{68}$$

and

$$-g(\xi_w, \theta_w) \tan(\pi - \theta_w) \leq v \leq 0, \tag{69}$$

where C independent of v.

Finally, we can show that the solution obtained in this subsection is unique by the maximum principle, which will be used to demonstrate that the mapping introduced in Subsection 4.7 is well–defined.

4.7 Existence of solutions for the modified nonlinear fixed boundary problem for (ρ, u, v).

Thanks to the uniform estimates of v and then u near Γ_{sonic} stated in Section 3, we can prove the existence for the modified nonlinear fixed boundary problem (36) and (38)–(42) by the Leray–Schauder fixed point theorem.

From Subsection 4.6, for every $u \in \mathscr{H}^\varepsilon$, there exists a unique $v \in C^{(-\sigma)}_{2+\alpha}$ satisfying $\|v\|^{(-\sigma)}_{2+\alpha} < C \tan(\pi - \theta_w)$. Thus, we can define a mapping for u as

$$\mathbf{S}: \ u \ \to \bar{u},$$

in the following:

$$\bar{u}(\xi, \eta) = \mathbf{S} u := u_0 + \int_{\eta(\xi)}^{\eta} v_\xi(\xi, s) \mathrm{d}s, \tag{70}$$

where $(\xi, \eta(\xi))$ denotes the point on the sonic circle Γ_{sonic}. For the other quantities ρ and ψ, we can obtain them once the nonlinear problem for u and v established as follows:

$$\begin{aligned} &\psi_\xi = U = u - \xi, \quad \psi_\eta = V = v - \eta, \\ &\rho = \big(-(\gamma - 1)\psi - \tfrac{\gamma-1}{2}(U^2 + V^2) \big)^{\frac{1}{\gamma-1}}. \end{aligned} \tag{71}$$

Furthermore, for the solutions to the nonlinear equations, we can prove v and then φ is monotone along Γ_{shock} by contradiction argument.

4.8 The free boundary problem

We can now prove the existence of solutions to the free boundary value problem. As indicated above, for any given boundary $\xi = \xi(\eta) \in \mathscr{K}^{\varepsilon}$, which is a small perturbation of the background solution $\xi = \xi_1$, we solve the fixed boundary problem and then give an update boundary by

$$\frac{J(\xi)(\eta)}{d\eta} = -\frac{v^{\varepsilon}}{u^{\varepsilon}} \quad \text{with } J(\xi)(\eta_1) = \xi_1. \tag{72}$$

The fixed point theorem which will be used here is the standard Schauder theorem (*cf.* Corollary 11.2, [15]). Then Theorem 4.1 is proved.

4.9 The limiting solution and the equivalence to the original system

We now study the limiting solution, as the elliptic regularization parameter ε tends to zero, to obtain a solution to system (58) and then to the original system, *i.e.* the potential flow equation, which we will study next to remove the elliptic cut–off. In fact, we can establish the following existence result.

Proposition 4.1 *There exist constants $\sigma^* > 0$, $\alpha_0 > 0$, and $\delta_0 > 0$ small enough such that, for any $\sigma < \sigma^*$, $\alpha < \alpha_0$, and $\pi - \delta_0 \le \theta_w < \pi$, there exists a solution*

$$(u, v, \psi) \in \left(C^{\alpha}(\overline{\Omega}) \cap C^1(\overline{\Omega} \backslash (\overline{\Gamma_{sonic}} \cup O)) \cap C^2(\Omega)\right)^3$$

with $(u - \xi, v - \eta) = (\psi_\xi, \psi_\eta)$ to problem (58)*,* (38)*–*(40)*,* (42)*,* (64)*, and* (65)*, i.e.*

$$\begin{cases} \bar{a}_{11} u_\xi + 2\bar{a}_{12} u_\eta + \bar{a}_{22} v_\eta = 0, \\ v_\xi = u_\eta, \\ D(\psi - \psi_0) = (u - u_0, v), \end{cases} \tag{73}$$

so that the velocity potential ψ satisfies (8) *in Ω, i.e.,*

$$\operatorname{div}\left(\rho(|\nabla\psi|^2, \psi) D\psi\right) + 2\rho(|\nabla\psi|^2, \psi) = 0, \tag{74}$$

the slip boundary condition on Γ_{wedge} with $\varphi = \psi - \psi_0$:

$$\varphi_\nu = 0 \tag{75}$$

with ν the normal direction and the following boundary conditions on Γ_{shock}:

$$\varphi = \varphi_1, \tag{76}$$
$$F(\varphi_\xi, \varphi_\eta, \varphi, \eta) = 0, \tag{77}$$

where $F(\mathbf{p}, z, \eta) = 0$ comes from the Rankine-Hugoniot condition satisfying $F(\mathbf{0}, 0, \eta) = 0$, $D_{\mathbf{p}}F \cdot \nu \neq 0$, and $D_z F \neq 0$. Moreover, on Γ_{sonic}, the velocity potential ψ satisfies the Dirichlet boundary condition:

$$\psi = \psi_0. \tag{78}$$

4.10 Removal of the cut–off function ζ_M for the higher order smallness

It is convenient to study this problem in a new coordinate introduced by $(x,y) = (c_0 - r, \theta - \theta_1)$ near Γ_{sonic}. Then the equation reads

$$\begin{aligned}&\Big(c_0 x + (\gamma+1)c_0 x \zeta\big(\tfrac{1}{\gamma+1} - \tfrac{\varphi_x}{x}\big) + O_1\Big)\varphi_{xx} + O_2\varphi_{xy} + (1+O_3)\varphi_{yy}\\ &\quad -(c_0 + O_3)\varphi_x - O_2\varphi_y = 0,\end{aligned} \tag{79}$$

with

$$O_1 \le (M+1)|x|^2, \quad O_2 \le (M+1)|x|^{\frac{3}{2}}, \quad |O_3| \le (M+1)|x|,$$

due to the cut-off function ζ_M. By scaling argument, we have the following estimates to remove the cut-off function ζ_M for the higher order smallness:

$$0 \le \varphi \le \frac{3}{5(\gamma+1)} x^2 \qquad \text{in } \Omega \cap \{c_0 - r \le 2\varepsilon_0\} \tag{80}$$

and

$$\|\varphi\|^{(-1-\alpha)}_{2+\alpha, \Omega\cap\{c_0 - r \ge s\}} \le C(s)(\pi - \theta_w) \tag{81}$$

for all $s \in (0, 8\varepsilon_0)$ with $C(s)$ depending only on the data and s.

4.11 Removal of the degenerate elliptic cut-off

Now we remove the degenerate elliptic cut–off ζ_1 in the (x,y)–coordinates with

$$(x,y) = (c_0 - r, \theta - \theta_1) \qquad \text{in } \Omega \cap \{c_0 - r < 4\varepsilon_0\}.$$

In this subsection, we let $|\pi - \theta_w|$ sufficiently small, depending only on the data, so that φ is a solution of the shock diffraction problem. Since the elliptic cut–off introduced here is more precise and $\sin\theta$ may be 0 at P_2, in comparison with that in [7], which means that the proof could not be used directly. Thus, we need more careful argument to re-control it.

First we bound φ_x near P_1 by the following lemma:

Lemma 4.3 *For $|\pi-\theta_w|$ sufficiently small, we have*

$$-\frac{x}{6(\gamma+1)} \le \varphi_x \le \frac{4x}{3(\gamma+1)} \qquad \textit{in } \Omega\cap\{x\le 4\varepsilon_0\}\cap\{y\le 4\varepsilon_2\}. \tag{82}$$

Next, away from P_1, we bound φ_x with one additional assumption, by the following lemma:

Lemma 4.4 *Assume that*

$$|\varphi-\frac{x^2}{2(\gamma+1)}| \le C_1 x^{2+\alpha} \qquad \textit{in } \Omega\cap\{x\le 2\varepsilon_0\}\cap\{y\ge 2\varepsilon_2\}. \tag{83}$$

Then, for $|\pi-\theta_w|$ sufficiently small, we have

$$-\frac{x}{3(\gamma+1)} \le \varphi_x - \frac{x}{\gamma+1} \le \frac{x}{3(\gamma+1)} \qquad \textit{in } \Omega\cap\{x\le 4\varepsilon_0\}\cap\{x\ge 2\varepsilon_2\}. \tag{84}$$

For this lemma, we first prove that the cut–off function can be removed when x small enough, which may depend on y. Then, in this domain, rewrite this equation in a more convenient form and scale it to obtain a uniform estimate to guarantee that the removal can be extended to $x=2\varepsilon_0$ without respect to y. With this proposition in hand, the remaining task is to show that (83) holds for some $\alpha<\frac{1}{2}$, which is proved in the following lemma.

Lemma 4.5 *For $|\pi-\theta_w|$ sufficiently small, we have*

$$|\varphi-\frac{x^2}{2(\gamma+1)}| \le C_1 x^{2+\alpha} \qquad \textit{in } \Omega\cap\{x\le \varepsilon'\}\cap\{y\ge 2\varepsilon_2\}, \tag{85}$$

where C_1 and ε' only depends on the data.

This completes the proof of the existence theory of the shock diffraction configuration with the required properties stated in Definition 3.1 when $\pi-\theta_w$ small. If it is large, using the same idea but much more technically, we can obtain that, for any $\Theta_w\in I$, there exists a constant $\delta_0>0$ such that, for any $\Theta_w-\delta_0<\theta_w\le\Theta_w$, there is a solution $W^{(\theta_w)}=(U^{(\theta_w)},V^{(\theta_w)})$ close to $W^{(\Theta_w)}$. Then, from the estimates stated above, we obtain that $(\theta_w,W^{(\theta_w)})$ belongs to the solution set defined in Definition 3.1. This means that the set I is open. Thus, from the fact that I is close and nonempty, we then finally have $(\theta_c,\pi)\subset I$.

For further details, see Chen-Xiang [9].

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References

1. M. Bae, G.-Q. Chen, and M. Feldman. Regularity of solutions to regular shock reflection for potential flow. Invent. Math. **175**, 505–543 (2009).
2. M. Bae, G.-Q. Chen, and M. Feldman. Global solutions to the Prandtl-Meyer reflection for supersonic flow impinging onto a solid wedge. Preprint, December 2012.
3. V. Bargman. *On nearly glancing reflection of shocks*, Office Sci. Res. and Develop. Rep. No. **5117**, 1945.
4. S. Čanić, B. L. Keyfitz, and E. H. Kim. Free boundary problems for the unsteady transonic small disturbance equation: transonic regular reflection. Methods Appl. Anal. **7**, 313–336 (2000).
5. S. Čanić, B. L. Keyfitz, and E. H. Kim. *Free foundary problems for nonlinear wave systems: Mach stems for interacting shocks*, SIAM J. Math. Anal. **37** (2006), 1947–1977.
6. G.-Q. Chen, X. Deng, and W. Xiang. The global existence and optimal regularity of solutions for shock diffraction problem to the nonlinear wave systems. Arch. Ration. Mech. Anal. 2013 (to appear).
7. G.-Q. Chen and M. Feldman. Global solutions of shock reflection by large-angle wedges for potential flow. Ann. of Math. (2), **171**, 1067–1182 (2010).
8. G.-Q. Chen and M. Feldman. *Mathematics of Shock Reflection-Diffraction and von Neumanns Conjectures*. Research Monograph, Preprint, 2012.
9. G.-Q. Chen and W. Xiang. Global solutions of the shock diffraction problem by wedges for potential flow. *Preprint*, 2012.
10. R. Courant and K. O. Friedrichs. *Supersonic Flow and Shock Waves*, Reprinting of the 1948 original. Applied Mathematical Sciences, Vol. 21. Springer-Verlag: New York-Heidelberg, 1976.
11. V. Elling and T.-P. Liu. *Supersonic flow onto a solid wedge*, Comm. Pure Appl. Math. **61** (2008), 1347–1448.
12. C. H. Fletcher, A. H. Taub, and W. Bleakney. *The Mach reflection of shock waves at nearly glancing incidence*, Rev. Modern Phys. **23(3)** (1951), 271–286.
13. C. H. Fletcher, D. K. Weimer, and W. Bleakney. *Pressure behind a shcok wave diffracted through a small angle*, Phys. Rev. **78(5)** (1950), 634–635.
14. D. Gilbarg and L. Hörmander. Intermediate schauder estimates. Arch. Rational Mech. Anal. **74**, 297–318 (1980).
15. D. Gilbarg and N. Trudinger. *Elliptic Partial Differential Equations of Second Order, 3rd edn*. Springer, Berlin, 1998.
16. E. H. Kim. global sub-sonic solution to an interacting transonic shock of the self-similar nonlinear wave equation. J. Diff. Eqs. **248**, 2906–2930 (2010).
17. G. Lieberman. The perron process applied to oblique derivative problems. Adv. Math. **55**, 161–172 (1985).
18. G. M. Lieberman. Mixed boundary value problems for elliptic and parabolic differential equations of second order. J. Math. Anal. Appl. **113**, 422–440 (1986).
19. G. M. Lieberman. Optimal Hölder regularity for mixed boundary value problems. J. Math. Anal. Appl. **143**, 572–586 (1989).
20. G. M. Lieberman and N. S. Trudinger. Nonlinear oblique boundary vallue problems for fully nonlinear elliptic equations. Trans. Amer. Math. Soc. **295**, 509–546 (1986).
21. M. J. Lighthill, *The diffraction of blast I*, Proc. Roy. Soc. **198A** (1949), 454–470.

22. M. J. Lighthill, *The diffraction of blast II*, Proc. Roy. Soc. **200A** (1950), 554–565.
23. K. Miller. Barriers on cones for uniformly elliptic operators. Ann. Mat. Pura. Appl. **76(4)**, 93–105 (1967).
24. G. B. Whitham. *Linear and Nonlinear Waves*, Reprint of the 1974 original. Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. John Wiley & Sons, Inc.: New York, 1999.
25. Y. Zheng. *Two-dimensional regular shock reflection for the pressure gradient system of conservation laws*, Acta Math. Appl. Sinica (English Ser). **22** (2006), 177–210.