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Stability of Transonic Characteristic Discontinuities in Two-Dimensional Steady Compressible Euler Flows

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For a two-dimensional steady supersonic Euler flow past a convex cornered wall with right angle, a characteristic discontinuity (vortex sheet and/or entropy wave) is generated, which separates the supersonic flow from the quiescent gas (hence subsonic). We proved that such a transonic characteristic discontinuity is structurally stable under small perturbations of the upstream supersonic flow in BV. The existence of a weak entropy solution and Lipschitz continuous free boundary (*i.e.* characteristic discontinuity) is established. To achieve this, the problem is formulated as a free boundary problem for a nonstrictly hyperbolic system of conservation laws; and the free boundary problem is then solved by analyzing nonlinear wave interactions and employing the front tracking method.

I. INTRODUCTION AND MAIN THEOREM

We are concerned with the structural stability of transonic characteristic discontinuities in two-dimensional steady full compressible Euler flows, which separate supersonic flows from the quiescent gases (that is, flows with zero-velocity, hence subsonic, *cf.* Fig. 1) under small perturbations of the upstream supersonic flow in the space of functions of bounded variation. In this paper, the term "characteristic discontinuity" means the discontinuity that is either a combination of vortex sheet/entropy wave or one of them in gas dynamics, see (3) below. We do not use the term "contact discontinuity" here to avoid confusion, since it also means specifically discontinuities for which the thermal pressure and the velocity are continuous, while only the mass density and thermal temperature change; this is the case indeed for the entropy waves, but not for the vortex sheets.

The flow is governed by the two-dimensional full Euler system, consisting of the conservation laws of mass, momentum, and energy:

$$\begin{cases} \partial_x(\rho u) + \partial_y(\rho v) = 0, \\ \partial_x(\rho u^2 + p) + \partial_y(\rho uv) = 0, \\ \partial_x(\rho uv) + \partial_y(\rho v^2 + p) = 0, \\ \partial_x(\rho u(E + \frac{p}{\rho})) + \partial_y(\rho v(E + \frac{p}{\rho})) = 0. \end{cases} \tag{1}$$

As usual, the unknowns $\mathbf{u} = (u, v)$, p, and ρ are respectively the velocity, the pressure, and the density of the flow, and

$$E = \frac{1}{2}(u^2 + v^2) + e(p, \rho)$$

is the total energy per unit mass with the internal energy $e(p, \rho)$. Let S be the entropy. For polytropic gas, the constitutive relations are

$$p = \kappa\rho^\gamma \exp\big(\frac{S}{c_\nu}\big), \qquad e = \frac{(\gamma-1)p}{\rho}$$

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for some positive constants κ, c_ν, and $\gamma > 1$. The sonic speed is given by $c = \sqrt{\gamma p/\rho}$. The flow is said to be *supersonic* (respectively *subsonic*) at a state point if $u^2 + v^2 > c^2$ (respectively $u^2 + v^2 < c^2$) there. It is well-known that the Euler system (1) is hyperbolic for supersonic flow, particularly hyperbolic in the positive x-direction if $u > c$; while it is of hyperbolic-elliptic composite-mixed type if the flow is subsonic. Hereafter, we use $U = (u, v, p, \rho)$ to represent the state of the flow under consideration.

An important physical case in which a characteristic discontinuity is generated is as follows: the characteristic discontinuity is a straight line emerging from a corner O (that is the positive x-axis); the gas flow above (*i.e.* in $\{x \in \mathbb{R},\ y > 0\}$) is a uniform supersonic flow with the velocity $(\underline{u}, 0)$, pressure $\underline{p}$, and density $\underline{\rho}^+$ such that $\underline{u} > \underline{c}^+$ for the sonic speed $\underline{c}^+ > 0$; below the characteristic discontinuity (*i.e.* in $\{x > 0, y < 0\}$), the gas is at rest with zero-velocity, pressure $\underline{p}$, and density $\underline{\rho}^-$. The question is whether such a transonic characteristic discontinuity is structurally stable under small perturbations of the upstream supersonic flow in the framework of two-dimensional steady full Euler equations, as shown in Fig. 1. Notice that the characteristic discontinuity is either a combination of a vortex sheet and an entropy wave or one of them.

For related cases, when the flows on both sides of the characteristic discontinuity are supersonic, it has been shown to be structurally stable by Chen-Zhang-Zhu in the framework of weak entropy solutions (Ref. 6), and the L^1–stability also holds as established by Chen-Kukreja (Ref. 4); when the flow is in an infinite duct and on both sides of the characteristic discontinuity the flows are subsonic, Bae proved that it is stable under small perturbations of the walls of the duct (Ref. 1). Characteristic discontinuities appear ubiquitously in Mach reflection and refraction/reflection of shock upon an interface. For such problems, Chen and Chen-Fang studied the stability of subsonic characteristic discontinuities (Refs. 7 and 8); Fang-Wang-Yuan showed the local stability of supersonic characteristic discontinuity in the framework of classical solutions (Ref. 10). Also see Ref. 16 for supersonic potential flows past a convex cornered bending wall and related geometry. As far as we know, there have been no results available so far concerning transonic characteristic discontinuities when the supersonic flows are not C^1 but only belong to the space of functions of bounded variation.

We remark that considerable progress has been made on the existence and stability of multidimensional transonic shocks in steady full Euler flows (see Refs. 3,7,13–15; also *cf.* Ref. 9). In these papers, the smooth supersonic flow is given, and the key point is to solve a one-phase elliptic free boundary problem. However, in order to solve the perturbed characteristic discontinuity in this paper, the key point is to solve a hyperbolic free boundary problem in the framework of weak entropy solutions.

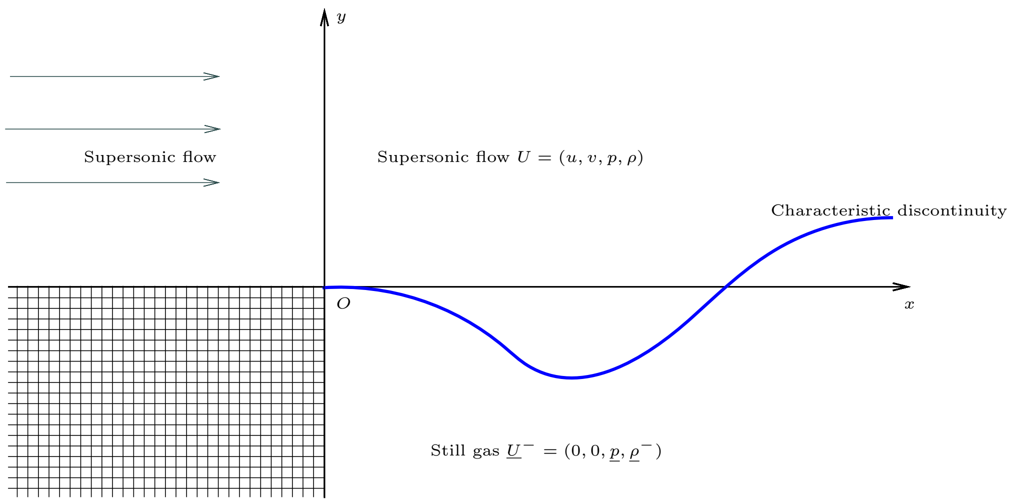

FIG. 1. A characteristic discontinuity emerged from the corner O that separates the static gas with zero-velocity below from the supersonic flow above

In the following, we first formulate the aforementioned stability problem for the characteristic discontinuity as a free boundary problem for the Euler equations. Then, in Sections II–IV, we establish the existence and stability of the free boundary, by a front tracking method (*cf.* Refs. 2,9,and 12).

To this end, we now introduce *characteristic discontinuities*, a kind of discontinuities that separate piecewise classical/weak solutions of (1). Suppose that Γ is a Lipschitz curve with normal $\mathbf{n} = (n_1, n_2)$ in the plane, and the flows $U = (u, v, p, \rho)$ on both sides of Γ satisfy the Euler equations (1) in the classical/weak sense. Then U is a weak solution to (1) provided it satisfies (1) on either side of Γ in the classical/weak sense, and the following Rankine-Hugoniot jump conditions hold along Γ:

$$\begin{cases} [\rho u]n_1 + [\rho v]n_2 = 0, \\ [\rho u^2 + p]n_1 + [\rho uv]n_2 = 0, \\ [\rho uv]n_1 + [\rho v^2 + p]n_2 = 0, \\ [\rho u(E + \frac{p}{\rho})]n_1 + [\rho v(E + \frac{p}{\rho})]n_2 = 0, \end{cases} \tag{2}$$

where $[\cdot]$ denotes the jump of the quantity across Γ. Such a discontinuity Γ is called a *characteristic discontinuity* if the mass flux $m = \rho\mathbf{u}\cdot\mathbf{n} = (\rho u)n_1 + (\rho v)n_2$ through Γ is zero. For a characteristic discontinuity, the first and fourth condition ($[\rho\mathbf{u}\cdot\mathbf{n}(E + \frac{p}{\rho})] = 0$) in (2) hold trivially, while the second ($[u\rho\mathbf{u}\cdot\mathbf{n}] + [p]n_1 = 0$) and the third ($[v\rho\mathbf{u}\cdot\mathbf{n}] + [p]n_2 = 0$) imply $[p] = 0$. Thus, we see that, for a characteristic discontinuity, the only jump conditions should be

$$[p] = 0 \qquad \text{and} \qquad \mathbf{u}\cdot\mathbf{n} = 0. \tag{3}$$

This implies that there might be jumps of the tangential velocity and the entropy (*i.e.* the density). Therefore, in general, a characteristic discontinuity in full Euler flow is either a vortex sheet or an entropy wave. We also note that (3) implies (2).

Consider the Cauchy problem of the hyperbolic-elliptic composite-mixed system (1):

$$\begin{cases} (1) \qquad \text{in } x \geq 0,\ y \in \mathbb{R}, \\ U = \begin{cases} U_0, & x = 0,\ y > 0, \\ \underline{U}^-, & x = 0,\ y < 0. \end{cases} \end{cases} \tag{4}$$

The discontinuous function

$$U = \begin{cases} \underline{U}^+ = (\underline{u}, 0, \underline{p}, \underline{\rho}^+), & x > 0,\ y > 0, \\ \underline{U}^- = (0, 0, \underline{p}, \underline{\rho}^-), & x > 0,\ y < 0, \end{cases}$$

with $\underline{u} > \underline{c}^+ = \sqrt{\gamma\underline{p}/\underline{\rho}^+}$ is a solution with a characteristic discontinuity of (1), when $U_0 = \underline{U}^+$, and $\underline{U}^-$ is the state of the static gas below $\{x > 0,\ y = 0\}$.

A weak entropy solution to problem (4) can be defined in the standard way (*cf.* Definition 1 below): In particular, it is defined as in (6)–(10), but the domain of integration Ω is replaced by $\{x \geq 0,\ y \in \mathbb{R}\}$, Σ is replaced by $\{x = 0,\ y \in \mathbb{R}\}$, and the right-hand sides of (7)–(8) are replaced by zero.

We note that the state of the static gas $\underline{U}^-$ should be unchanged under the perturbation of the supersonic flow. This is a merit of such a transonic characteristic discontinuity, which enables us to reduce the above problem to an initial-free boundary problem of the hyperbolic Euler equations.

Suppose that the characteristic discontinuity Γ is given by the equation:

$$y = g(x) \qquad \text{for } x \geq 0,$$

with $g(0) = 0$. Then

$$\mathbf{n} = \frac{(g'(x), -1)}{\sqrt{1 + (g'(x))^2}}.$$

The domain bounded by Γ and $\Sigma = \{(x, y) : x = 0,\ y > 0\}$ is written as Ω. We formulate the following free boundary problem of (1) in Ω:

$$\begin{cases} U = U_0 & \text{on } \Sigma, \\ p = \underline{p} & \text{on } \Gamma, \\ v = g'(x)u & \text{on } \Gamma, \end{cases} \tag{5}$$

where the first is the initial data and the last two conditions on Γ come from (3).

Definition 1. A pair (g, U) with $y = g(x) \in \mathrm{Lip}([0, \infty); \mathbb{R})$ and $U = (u, v, p, \rho) \in L^\infty(\Omega; \mathbb{R}^4)$ is called a *weak entropy solution* to problem (5) provided the following hold:

◇ U is a weak solution to (1) in Ω and satisfies the initial-boundary conditions in the trace sense: For any $\phi \in C_0^\infty(\mathbb{R}^2)$,

$$\int_\Omega \big(\rho u \partial_x \phi + \rho v \partial_y \phi\big)\,\mathrm{d}x\mathrm{d}y + \int_\Sigma \rho u \phi \,\mathrm{d}y = 0, \tag{6}$$

$$\int_\Omega \big((\rho u^2 + p)\partial_x \phi + \rho u v \partial_y \phi\big)\,\mathrm{d}x\mathrm{d}y + \int_\Sigma (\rho u^2 + p)\phi \,\mathrm{d}y = \underline{p}\int_\Gamma \phi n_1 \,\mathrm{d}s, \tag{7}$$

$$\int_\Omega \big((\rho u v)\partial_x \phi + (\rho v^2 + p)\partial_y \phi\big)\,\mathrm{d}x\mathrm{d}y + \int_\Sigma (\rho u v)\phi \,\mathrm{d}y = \underline{p}\int_\Gamma \phi n_2 \,\mathrm{d}s, \tag{8}$$

$$\int_\Omega \big(\rho u(E + \frac{p}{\rho})\partial_x \phi + \rho v(E + \frac{p}{\rho})\partial_y \phi\big)\,\mathrm{d}x\mathrm{d}y + \int_\Sigma \rho u(E + \frac{p}{\rho})\phi \,\mathrm{d}y = 0; \tag{9}$$

◇ U satisfies the entropy inequality, *i.e.* the steady Clausius inequality:

$$\partial_x(\rho u S) + \partial_y(\rho v S) \geq 0$$

in the sense of distributions in Ω: For any $\phi \in C_0^\infty(\mathbb{R}^2)$ with $\phi \geq 0$,

$$\int_\Omega \big(\rho u S \partial_x \phi + \rho v S \partial_y \phi\big)\,\mathrm{d}x\mathrm{d}y + \int_\Sigma \rho u S \phi \,\mathrm{d}y \leq 0. \tag{10}$$

We remark that, if (g, U) is a weak entropy solution to problem (5), then

$$\tilde{U} = \begin{cases} U & \text{in } \{y > g(x),\ x \geq 0\}, \\ \underline{U}^- & \text{in } \{y < g(x),\ x \geq 0\} \end{cases}$$

is a weak entropy solution to problem (4). This can be checked by integration by parts in $\{x \geq 0,\ y < g(x)\}$; thus, we omit the details. From now on, we focus on the solution of problem (5). The main result of this paper is the following.

Theorem 1. *There exist positive constants ε and C depending only on $\underline{U}^\pm$ so that, if*

$$\|U_0 - \underline{U}^+\|_{\mathrm{BV}(\Sigma)} \leq \varepsilon,$$

then problem (5) has a weak entropy solution (g, U). Moreover, the solution satisfies

(i) $g \in \mathrm{Lip}([0,\infty);\mathbb{R})$ *with* $g(0) = 0$ *and* $\|g'\|_{L^\infty([0,\infty))} \leq C\varepsilon$;

(ii) *There exists $\underline{U}_0 \in \mathbb{R}^4$ so that*

$$U - \underline{U}_0 \in C([0,\infty); L^1(g(x),\infty)), \qquad \|(U - \underline{U}^+)(x,\cdot)\|_{\mathrm{BV}([g(x),\infty))} \leq C\varepsilon.$$

Remark 1. We note that $\|U_0 - \underline{U}^+\|_{\mathrm{BV}(\Sigma)} \leq \varepsilon$ implies that $\lim_{y\to\infty}(U_0 - \underline{U}^+)(y)$ exists. Then there exists $\underline{U}_0 \in \mathbb{R}^4$ as claimed in Theorem 1 so that

$$\lim_{y\to\infty} U_0(y) = \underline{U}_0,$$

and

$$|\underline{U}_0 - \underline{U}^+| \leq \varepsilon.$$

To prove Theorem 1, we establish the compactness and convergence of approximate free boundaries to the free boundary of the exact solution in supersonic-subsonic flows in the framework of front tracking method, while some other essential tools/notions of the front tracking method are extended, modified, and further clarified working in the presence of the free boundary such as a generation of fronts to control the finiteness of physical fronts and the errors from approximate Riemann solvers for the nonstrictly hyperbolic free boundary problem. For this, two new nonlinear Riemann problems are involved: One is the Riemann problem at the convex corner connected with the quiescent gas state (subsonic state); and the other is the Riemann problem determining the evolution of the free boundary, for which we establish the boundedness of the key reflection coefficient of the reflected wave into the supersonic region after the interaction of the incident wave with the free boundary. To achieve the compactness, we have to identify the right scales and global weights to control the Glimm functional to make it monotonically decrease in the flow direction, while preserving the overall structural stability of the characteristic boundary as the hyperbolic region evolves in complicated ways under any small BV perturbation yet the subsonic state remains stable beneath the free boundary.

We also remark in passing that, as an example of one-phase hyperbolic free boundary problems for nonstrictly hyperbolic systems, we deal with the problem in the physical space, with the Euler coordinates throughout this paper. This represents a first example of an approach to apply the front-tracking method to study the structural stability of interfaces between different media, one of which is subsonic. Our approach offers further opportunities to initiate the study of vortex sheets/entropy waves in the space of bounded variation in nozzles, jets, etc. for mixed-type flows, transonic flows. In a subsequent paper (Ref. 5), we will deal with problem (5) for (1) and related L^1-stability in a different approach.

The rest of this paper is devoted to establishing Theorem 1. We will mainly employ a version of the front tracking method introduced in Holden-Risebro (Ref. 12) for convenience to deal with the problem. Thus, in Section II, we review some facts concerning the solvability of various Riemann problems for the steady Euler equations, and present some essential interaction estimates. It manifests clearly in the simplest case how such a hyperbolic free boundary problem can be solved. Then, in Section III, we construct approximate solutions by the front tracking algorithm. The key point is to show such an approximate solution can be established for $x \in [0, \infty)$ by constructing a Glimm functional. Then, in Section IV, with the uniform BV estimate of approximate solutions obtained from the Glimm functional, we establish the compactness of the family of approximate solutions and show that the limit is actually an entropy solution. Finally, we discuss the far field asymptotic behavior of the weak entropy solutions as $x \to \infty$ in Section V.

II. RIEMANN PROBLEMS AND INTERACTION ESTIMATES

In this section we first review certain basic properties of the steady hyperbolic Euler equations (1) that are used later for self-containedness (*cf.* Ref. 6, pp. 1665-1670). Then we show the solvability of "free boundary" Riemann problem and interaction estimate between weak waves and the free boundary, which are the *new* ingredients in this paper.

A. Euler Equations

As in Ref. 6, we write the Euler equations (1) in the form

$$\partial_x W(U) + \partial_y H(U) = 0, \qquad U = (u,\ v,\ p,\ \rho), \tag{11}$$

where

$$W(U) = (\rho u,\ \rho u^2 + p,\ \rho uv,\ \rho u(\frac{\gamma p}{(\gamma-1)\rho} + \frac{u^2+v^2}{2}))^\top$$

and

$$H(U) = (\rho v,\ \rho uv,\ \rho v^2 + p,\ \rho v(\frac{\gamma p}{(\gamma-1)\rho} + \frac{u^2+v^2}{2}))^\top.$$

The eigenvalues λ of this system are determined by $\det(\lambda\nabla_U W(U) - \nabla_U H(U)) = 0$, or explicitly,

$$(v - \lambda u)^2\big((v-\lambda u)^2 - c^2(1+\lambda^2)\big) = 0.$$

Thus, if $u > c$, we have four real eigenvalues:

$$\lambda_j = \frac{uv + (-1)^j c\sqrt{u^2+v^2-c^2}}{u^2 - c^2}, \quad j = 1, 4; \qquad \lambda_k = \frac{v}{u}, \quad k = 2, 3. \tag{12}$$

The associated linearly independent right-eigenvectors are

$$\mathbf{r}_j = \kappa_j(-\lambda_j,\ 1,\ \rho(\lambda_j u - v),\ \frac{\rho(\lambda_j u - v)}{c^2})^\top, \qquad j = 1, 4; \tag{13}$$

$$\mathbf{r}_2 = (u,\ v,\ 0,\ 0)^\top, \qquad \mathbf{r}_3 = (0,\ 0,\ 0,\ \rho)^\top, \tag{14}$$

where κ_j are renormalized factors so that $\mathbf{r}_j \cdot \nabla_U \lambda_j(U) \equiv 1$ since the j-th characteristic fields are genuinely nonlinear, $j = 1,\ 4$. Note that the second and third characteristic fields are linearly degenerate: $\mathbf{r}_j \cdot \nabla_U \lambda_j(U) \equiv 0,\ j = 2,\ 3$. Although the steady Euler system is not strictly hyperbolic, we can still employ the general ideas presented in Refs. 9 and 12 to treat related Riemann and Cauchy problems. The only difference is that, although the characteristic discontinuity has only one front in physical space (since two of the four characteristic eigenvalues coincide), we need two independent parameters (one corresponds to λ_2 for the vortex sheet, and the other to λ_3 for the entropy wave) to represent its strength.

At the unperturbed reference state $\underline{U}^+ = (\underline{u},\ 0,\ \underline{p},\ \underline{\rho}^+)$, we easily see that

$$\lambda_1(\underline{U}^+) < \lambda_2(\underline{U}^+) = 0 = \lambda_3(\underline{U}^+) < \lambda_4(\underline{U}^+) = -\lambda_1(\underline{U}^+).$$

Also, Lemma 2.3 in Ref. 6 indicates that the re-normalization factors $\kappa_j(U),\ j = 1,\ 4$, are positive in a small neighborhood of $\underline{U}^+$.

B. Wave Curves in the Phase Space

As shown in Ref. 6, at each state $U_0 = (u_0,\ v_0,\ p_0,\ \rho_0)$ with $u_0 > c_0$ in the phase space, there are four curves in a neighborhood of U_0:

- ♢ Vortex sheet curve $C_2(U_0) : U = (u_0 e^{\alpha_2},\ v_0 e^{\alpha_2},\ p_0,\ \rho_0)$.
  These are the states U that can be connected to U_0 by a vortex sheet with slope v_0/u_0 and strength $\alpha_2 \in \mathbb{R}$;
- ♢ Entropy wave curve $C_3(U_0) : U = (u_0,\ v_0,\ p_0,\ \rho_0 e^{\alpha_3})$.
  These are the states U that can be connected to U_0 by an entropy wave with slope v_0/u_0 and strength $\alpha_3 \in \mathbb{R}$.
- ♢ Rarefaction wave curve $R_j(U_0)$:

$$\mathrm{d}p = c^2 \mathrm{d}\rho,\ \mathrm{d}u = -\lambda_j \mathrm{d}v,\ \rho(\lambda_j u - v)\mathrm{d}v = \mathrm{d}p \qquad \text{for } \rho < \rho_0,\ u > c, \quad j = 1,\ 4.$$

  These are the states U that can be connected to U_0 from below (respectively, above) by a rarefaction wave of the first family (respectively, fourth family);
- ♢ Shock wave curve $S_j(U_0)$:

$$[p] = \frac{c_0^2}{b}[\rho],\ [u] = -s_j[v],\ \rho_0(s_j u_0 - v_0)[v] = [p] \qquad \text{for } \rho > \rho_0,\ u > c, \quad j = 1,\ 4.$$

  These are the states U that can be connected to U_0 from below (respectively, above) by a shock wave of the fourth family (respectively, first family), with the slope of the discontinuity to be

$$s_j = \frac{u_0 v_0 + (-1)^j \bar{c}\sqrt{u_0^2 + v_0^2 - \bar{c}^2}}{u_0^2 - \bar{c}^2}, \qquad j = 1,\ 4,$$

  where $\bar{c} = \frac{\rho c_0^2}{\rho_0 b}$ and $b = \frac{\gamma+1}{2} - \frac{\gamma-1}{2}\frac{\rho}{\rho_0}$.

One can also parameterize $R_j(U_0)$ and $S_j(U_0)$ $(j = 1, 4)$ so that there is a curve given by a C^2 map $\alpha_j \mapsto \Phi_j(\alpha_j; U_0)$ in a neighborhood of U_0, with $\alpha_j \geq 0$ being the part of $R_j(U_0)$, and $\alpha_j < 0$ the part of $S_j(U_0)$, and

$$\Phi_j(0; U_0) = U_0, \qquad \partial_{\alpha_j}\Phi_j(0; U_0) = \mathbf{r}_j(U_0). \tag{15}$$

We can also write the curve $C_j(U_0)$ $(j = 2, 3)$ as $\alpha_j \mapsto \Phi_j(\alpha_j; U_0)$ which is still C^2 so that (15) hold for $j = 2, 3$. Since $\{\mathbf{r}_j(U_0)\}_{j=1}^4$ are linearly independent, such curves comprise locally a (curved) coordinate system in a neighborhood of U_0. This guarantees the solvability of the Riemann problems stated below.

For simplicity, we set

$$\Phi(\alpha_4, \alpha_3, \alpha_2, \alpha_1; U_0) = \Phi_4(\alpha_4; \Phi_3(\alpha_3; \Phi_2(\alpha_2; \Phi_1(\alpha_1; U_0)))). \tag{16}$$

Then

$$\Phi(0, 0, 0, 0; U_0) = U_0, \quad \partial_{\alpha_j}\Phi(0, 0, 0, 0; U_0) = \mathbf{r}_j(U_0), \qquad j = 1, 2, 3, 4. \tag{17}$$

C. Standard Riemann Problem

We now consider the standard Riemann problem, that is, system (1) with the piecewise constant (supersonic) initial data

$$U|_{x=x_0} = \begin{cases} U^+, & y > y_0, \\ U^-, & y < y_0, \end{cases} \tag{18}$$

where U^+ and U^- are the constant states which are regarded as the *above* state and *below* state with respect to the line $y = y_0$, respectively.

Lemma 1 (Lemma 2.2 in Ref. 6)**.** *There exists $\epsilon > 0$ such that, for any states U^- and U^+ lie in the ball $O_\epsilon(U_0) \subset \mathbb{R}^4$ with radius ϵ and center U_0, the Riemann problem* (18) *admits a unique admissible solution consisting of four elementary waves. In addition, the state U^+ can be represented by*

$$U^+ = \Phi(\alpha_4, \alpha_3, \alpha_2, \alpha_1; U^-). \tag{19}$$

It is noted (*cf.* Lemma 4.1 in Ref. 6) that one can use the parameters $\alpha_j, j = 1, \ldots, 4$, to bound $|U^+ - U^-|$: There is a constant B depending continuously on U_0 and ϵ so that, for $U^\pm$ connected by (19),

$$\frac{1}{B}\sum_{j=1}^{4} |\alpha_j| \le |U^+ - U^-| \le B\sum_{j=1}^{4} |\alpha_j|.$$

For later applications, it is also important to express the Riemann solver from the above state U^+ to the below state U^-, rather than the usual way given above. For $U^+ = \Phi_j(\alpha_j; U^-)$, we may have a C^2–map $U^- = \Psi_j(\alpha_j; U^+)$ with $\Psi_j(0; U) = U$ and $\partial_{\alpha_j}\Psi_j(0; U) = -\mathbf{r}_j(U)$. Thus, for $U^+ = \Phi(\alpha_4, \alpha_3, \alpha_2, \alpha_1; U^-)$, we may express U^- in terms of U^+ by

$$U^- = \Psi(\alpha_1, \alpha_2, \alpha_3, \alpha_4; U^+) = \Psi_1(\alpha_1; \Psi_2(\alpha_2; \Psi_3(\alpha_3; \Psi_4(\alpha_4; U^+)))).$$

Then $\Psi(0, 0, 0, 0; U) = U$ and $\partial_{\alpha_j}\Psi(0, 0, 0, 0; U) = -\mathbf{r}_j(U)$.

D. Free Boundary Riemann Problem

We now consider the following Riemann problem of (1) involving a free boundary — a characteristic discontinuity. The initial data is a constant state $U = U^+$ given on the positive y-axis, and the free boundary is a straight line $y = kx$ with $k \in \mathbb{R}$ to be solved. The boundary conditions on the free boundary are $p = \underline{p}$ and $k = \frac{v}{u}$. Since the free boundary — characteristic discontinuity — is of the second/third characteristic family, the Riemann solver should contain only one 4-wave with parameter α_4 and a middle constant state U^*; see Fig. 2 below.

Lemma 2. *There exists $\epsilon > 0$ so that, for $U^+ \in O_\epsilon(\underline{U}^+)$, there is only one admissible solution consisting of a 4-wave that solves the free boundary Riemann problem. The middle state U^* can be represented by $U^* = \Psi_4(\alpha_4; U^+)$, and the free boundary is determined by $k = \frac{v^*}{u^*}$. There also holds*

$$\alpha_4 = K_1(p^+ - \underline{p}) + M_1|U^+ - \underline{U}^+|^2, \qquad |k| \le K_1'|U^+ - \underline{U}^+|, \tag{20}$$

with the constants $K_1, K_1' > 0$ and a bounded quantity M_1 depending continuously on $\underline{U}^+$ and ϵ.

Proof. 1. We write $U^{(k)}$ to denote the k-th argument of the vector U, $k = 1, \ldots, 4$. Consider the function:

$$L(\alpha, U^+) = (\Psi_4(\alpha; U^+))^{(3)} - \underline{p} = (\Psi_4(\alpha; U^+) - \Psi_4(0; \underline{U}^+))^{(3)},$$

for which $L(0; \underline{U}^+) = 0$. Then

$$\partial_\alpha L(0; \underline{U}^+) = -(\mathbf{r}_4(\underline{U}^+))^{(3)} = -(\kappa_4 \rho u \lambda_4)|_{\underline{U}^+} < 0.$$

From the implicit function theorem, we infer that α can be viewed as a function of $U^+ \in O_\epsilon(\underline{U}^+)$ for suitably small $\epsilon > 0$. In particular, $\alpha(\underline{U}^+) = 0$. This completes the existence proof.

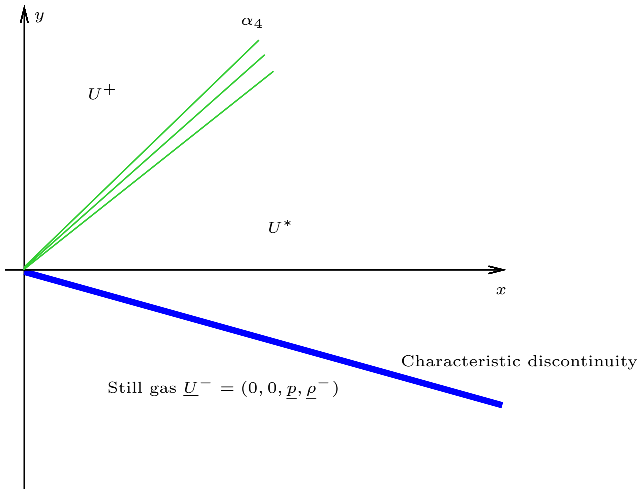

FIG. 2. A Riemann problem with a free boundary that is a characteristic discontinuity

2. Since $\nabla_U \Psi_4(0; U) = I_4$, $\partial_U L(0; \underline{U}^+) = (0, 0, 1, 0)$. Then

$$\nabla_U \alpha(\underline{U}^+) = \frac{(0, 0, 1, 0)}{(\kappa_4 \rho u \lambda_4)|_{\underline{U}^+}}.$$

Thus, by the Taylor expansion, we conclude

$$\alpha = K_1(p^+ - \underline{p}) + M_1|U^+ - \underline{U}^+|^2,$$

where $K_1 = \frac{1}{(\kappa_4 \rho u \lambda_4)|_{\underline{U}^+}} > 0$, and M_1 is a constant depending continuously and only on $\underline{U}^+$ and ϵ.

3. From the above, we have

$$|U^* - U^+| \le B|\alpha| \le B'|U^+ - \underline{U}^+|.$$

Then we have

$$|U^* - \underline{U}^+| \le B''|U^+ - \underline{U}^+|$$

for some constant $B'' > 0$. Hence, regarding v/u as a function of U and by the mean value theorem, we have

$$\left|\frac{v^*}{u^*}\right| \le C|U^* - \underline{U}^+| \le K_1'|U^+ - \underline{U}^+|$$

as desired. □

E. Approximate Riemann Solver

The front tracking method involves approximating the rarefaction waves appeared in the Riemann problems or (free) boundary Riemann problems by several artificial discontinuities separating piecewise constant states.

Suppose that $U^+ = \Phi(\alpha_4, \alpha_3, \alpha_2, \alpha_1; U^-)$ gives the solution to the standard Riemann problem (18), with middle states $U^1 = \Phi_1(\alpha_1; U^-)$ and $U^2 = \Psi_4(\alpha_4; U^+)$. For any $\delta > 0$, we define a δ-approximate solution U^δ to the Riemann problem as follows:

- If $\alpha_1 > 0$, then the 1-wave is a rarefaction wave that requires modification as follows. Set ν be the closest integer to α_1/δ (that is, $\nu \in \mathbb{Z}$ and $\frac{\alpha_1}{\delta} - \frac{1}{2} \le \nu < \frac{\alpha_1}{\delta} + \frac{1}{2}$), as well as $U_{1,0} = U^-$, $U_{1,\nu} = U^1$, and $U_{1,k} = \Phi_1(\frac{1}{\nu}\alpha_1; U_{1,k-1})$ for $k \in \{1, \dots, \nu - 1\}$. Then, in the wedge $\{(x, y) : x > 0,\ y < \lambda_* x\}$, we define

$$U^\delta = \begin{cases} U^-, & y < \lambda_1(U^-)x, \\ U_{1,k}, & \lambda_1(U_{1,k-1})x < y < \lambda_1(U_{1,k})x, \quad k = 1, \dots, \nu - 1, \\ U^1, & \lambda_1(U_{1,\nu-1})x < y < \lambda_* x. \end{cases} \tag{21}$$

Here λ_* is a constant chosen so that $\sup_{U\in O_\epsilon(\underline{U}^+)}\lambda_1(U)<\lambda_*<\inf_{U\in O_\epsilon(\underline{U}^+)}\lambda_2(U)$, which exists when ϵ is small. Then the rarefaction wave is replaced by "step" functions with width (strength) α_1/ν, and the discontinuity between two steps moves with the characteristic speed of the below state.

- If $\alpha_1<0$, then the 1-wave is a shock, and no change is necessary. In the wedge $\{(x,y): x>0,\ y<\lambda_* x\}$, we define
$$U^\delta=\begin{cases}U^-, & y<s_1x,\\ U^1, & s_1x<y<\lambda_*x,\end{cases}$$
where s_1 is the speed of the shock front.
- For α_2,α_3, there is always no change.
- Similar to the case of the 1-wave, we can define U^δ in $\{x>0,\ y>-\lambda_*x\}$ by considering whether the 4-wave is a rarefaction wave (with modification) or a shock (without modification).

F. Interaction of Weak Waves

The following weak wave interaction estimate is classical; see Lemma 3.2 in Ref. 6, p. 1670. We remark that in this paper "weak waves" always refer to waves (discontinuities) whose strengths are small compared to the free boundary.

Lemma 3. *Suppose that U^+, U^m, and U^- are three states in a small neighborhood of U_0 with $U^+=\Phi(\alpha_4,\alpha_3,\alpha_2,\alpha_1;U^m)$, $U^m=\Phi(\beta_4,\beta_3,\beta_2,\beta_1;U^-)$, and $U^+=\Phi(\gamma_4,\gamma_3,\gamma_2,\gamma_1;U^-)$. Then*
$$\gamma_j=\alpha_j+\beta_j+O(1)\triangle(\beta,\alpha),\tag{22}$$
where $\triangle(\beta,\alpha)=|\beta_4|(|\alpha_1|+|\alpha_2|+|\alpha_3|)+(|\beta_2|+|\beta_3|)|\alpha_1|+\sum_{j=1,4}\triangle_j(\beta,\alpha)$, with
$$\triangle_j(\beta,\alpha)=\begin{cases}0, & \alpha_j\ge 0,\ \beta_j\ge 0,\\ |\alpha_j||\beta_j|, & \text{otherwise.}\end{cases}$$

G. Interaction of Weak Wave and Free Boundary

We now consider the change of strength when a weak wave interacts with the free boundary (see Fig. 3). It is only possible that a weak 1-wave α_1 impinges on the characteristic discontinuity S, resulting in a reflected 4-wave with parameter α_4, and the characteristic discontinuity itself is also deflected to a new direction, denoted to be S^*. We note that both U^* and S^* can be solved by the free boundary Riemann problem with initial data U^+.

Lemma 4. *Suppose that U^-,U^+ are two states in $O_\epsilon(\underline{U}^+)$ for sufficiently small ϵ, and $U^+=\Phi_1(\alpha_1;U^-)$. Then, for another state $U^*\in O_\epsilon(\underline{U}^+)$ so that $U^+=\Phi_4(\alpha_4;U^*)$, there holds*
$$\alpha_4=-K_2\alpha_1+M_2|\alpha_1|^2,\tag{23}$$
with the constant $K_2>0$ and the quantity M_2 bounded in $O_\epsilon(\underline{U}^+)$. Furthermore, for $U^-=(u^-,v^-,p^-,\rho^-)$, we have $|K_2|>1, |K_2|<1$, and $|K_2|=1$ when $v^-<0$, $v^->0$, and $v^-=0$, respectively.

Proof. 1. We have $U^+=\Phi_1(\alpha;U^-)$ and $U^*=\Psi_4(\beta;U^+)$. Consider the following function:
$$L(\beta,\alpha):=(\Psi_4(\beta;\Phi_1(\alpha;U^-))-U^-)^{(3)}.$$
Then $L(0,0)=0$, and $\partial_\beta L(0,0)=-(\mathbf{r}_4(U^-))^{(3)}<0$. By the implicit function theorem, there exists a function $\beta=\beta(\alpha)$ so that $L(\beta(\alpha),\alpha)=0$ for small α. We see $\beta(0)=0$.

2. We calculate $\partial_\alpha L(0,0)=(\mathbf{r}_1(U^-))^{(3)}<0$. Thus, $\frac{\mathrm{d}\beta(0)}{\mathrm{d}\alpha}=-K_2:=\frac{(\mathbf{r}_1(U^-))^{(3)}}{(\mathbf{r}_4(U^-))^{(3)}}<0$. Therefore, the equality in (23) follows from the Taylor expansion.

3. The coefficient
$$K_2:=-\frac{(\mathbf{r}_1(U^-))^{(3)}}{(\mathbf{r}_4(U^-))^{(3)}}=\frac{\frac{v^-}{u^-}-\lambda_1(U^-)}{-\frac{v^-}{u^-}+\lambda_4(U^-)}>0$$
and, for any state $U=(u,\ v,\ p,\ \rho)\in O_\epsilon(\underline{U}^+)$, there holds $\lambda_1(U)<\lambda_{2,3}(U)=v/u<\lambda_4(U)$. Using these two facts with the expressions for $\lambda_1(U)$ and $\lambda_4(U)$ given in (12), it follows that $|K_2|<1,\ |K_2|>1,$ and $|K_2|=1$ when $v_l>0,\ v_l<0,$ and $v_l=0$, respectively. ∎

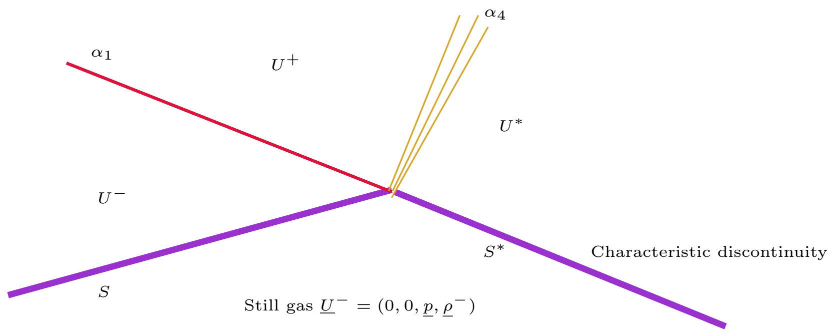

FIG. 3. A 1-wave α_1 is reflected by the characteristic discontinuity S, resulting in a reflected 4-wave α_4 and deflected characteristic discontinuity S^*

III. CONSTRUCTION OF APPROXIMATE SOLUTIONS AND UNIFORM ESTIMATES

In this section we adopt the front tracking method in Ref. 12 to construct a family of approximate solutions $\{(g^\delta, U^\delta)\}_{\delta>0}$ of problem (5) and present some uniform estimates independent of δ, which is necessary for a compactness argument in Section IV below to show the existence of a weak entropy solution to (5).

A. Construction of Approximate Solutions

For any given $\delta > 0$, we now describe the construction of an approximate solution (g^δ, U^δ) to the free boundary problem (5).

We first approximate the initial data $U_0(y)$ by a piecewise constant function $U_0^\delta(y)$ as done in the study of the Cauchy problem. We require that

$$\lim_{\delta\to 0} \left\| U_0 - U_0^\delta \right\|_{L^1([0,\infty))} = 0. \tag{24}$$

By Remark 1, we may also assume that, for each $\delta > 0$, there holds $U_0^\delta(y) = \underline{U}_0$ for large y.

We solve the Riemann problems with initial data on $\{x = 0,\ y > 0\}$ and a free boundary Riemann problem at the corner $(0, 0)$, and then approximate rarefaction waves as carried out in Section II E with parameter δ to obtain new discontinuities. Note the resulting (approximate) solution is piecewise constant.

Then we need do nothing until x increases to some value $x = \tau$, where

(i) two fronts (discontinuities) interact;

(ii) or there is a weak 1-wave that interacts with the free boundary (it is obtained by solving the free boundary Riemann problem before) from above.

As noted in Ref. 2, by adjusting the slopes of the discontinuities, we can assume that, at each $\{x = \tau\}$, only one of the two cases happens. This is harmless since the error can be made to be arbitrarily small.

For case (i), as mentioned above, by adjusting the slopes of these discontinuities (with arbitrarily small error), we may assume that only two discontinuities collide. Suppose that the discontinuity below is of r-family and has a parameter α with the below (constant) state U^l and above (constant) state U^m, while the discontinuity above is of s-family and has a parameter β with the below (constant) state U^m and above (constant) state U^r, and they collide at the point (τ, η). Then, as before, we solve a Riemann problem at (τ, η) with the below state U^l and above state U^r, by applying the approximate Riemann solver to obtain new discontinuities.

For case (ii), we may still assume only one discontinuity collides with the free boundary (transonic characteristic discontinuity). Then we solve a wave reflection-deflection problem with a 1-wave reflected by the free boundary, obtaining a reflected 4-wave and a deflected characteristic discontinuity (see Fig. 2). If the reflected 4-wave is a rarefaction wave, by approximating the rarefaction wave, we obtain again the approximate solver containing new discontinuities.

Continuing this procedure and, in some cases, removing certain quite weak fronts (see Section III D 2 below for details), we obtain an approximate solution (g^δ, U^δ).

Remark 2. To ensure that the above procedure works to construct an approximate solution for all $x \in [0,\infty)$, we need to show that, for any $0 < x < \infty$,

- The total variation is small: $\text{T.V.}(U^\delta(x,\cdot)) \leq C\varepsilon$,
- An L^∞–bound: The solution still lies in a small neighborhood of $\underline{U}^+$,
- Given any finite $T > 0$, there happen only a finite number of collisions/reflections for $\{0 < x < T\}$,

where C is a universal constant independent of ε, δ and $x > 0$. The first two are necessary so that we can actually solve the standard or free boundary Riemann problems. The third one guarantees that the global approximate solutions defined up to any $x > 0$ can be actually obtained.

In Secs. III B–III D, we deal with these three issues.

B. Bounds of Total Variation

We now establish the bounds of total variation of the approximate solutions $U^\delta(x,y)$.

1. Glimm Functional

We introduce the following version of Glimm functional

$$G(x) = V(x) + \kappa Q(x), \tag{25}$$

where $\kappa > 0$ is a large constant to be chosen. The terms V and Q are explained below. By the properties of the approximate Riemann solver, $\text{T.V.}(U^\delta(x,\cdot))$ is equivalent to $V(x)$. Then it suffices to prove

$$V(x) \leq C_0\varepsilon \tag{26}$$

for a constant C_0 depending only on $\underline{U}^+$. Recall here $\varepsilon = \left\|U_0 - \underline{U}^+\right\|_{\text{BV}([0,\infty))}$ measures the strength of the perturbation of initial data.

For a weak wave/discontinuity α of i_α-family, we define its weighted strength as

$$b_\alpha = \begin{cases} k_+\alpha & \text{if } \alpha \in \Upsilon_t \text{ and } i_\alpha = 1, \\ \alpha & \text{if } \alpha \in \Upsilon_t \text{ and } i_\alpha = 2,3,4, \end{cases} \tag{27}$$

where $k_+ > |K_2|$ for the coefficient K_2 appeared in Lemma 4, and we use Υ_t to denote the set of weak waves (not including the free boundary) that cross the line $\{x = t\}$.

• *The weighted strength term* $V(t)$*.* We define the total (weighted) strengths of weak waves at $x = t$ as

$$V(t) = \sum_{\alpha\in\Upsilon_t} |b_\alpha|. \tag{28}$$

• *The interaction potential term* $Q(t)$*.* The interaction potential term we use here is the same as the one introduced by Glimm (Ref. 11), that is,

$$Q(t) = \sum_{(b_\alpha,b_\beta)\in\mathcal{A}(t)} |b_\alpha b_\beta|, \tag{29}$$

where $\mathcal{A}(t)$ is the *approaching set* defined by pairs (b_α, b_β) so that, for $x = t$, the waves/discontinuity with strength b_α lies below the waves/discontinuity with strength b_β, and b_α is of family i_α and b_β of family i_β, where $i_\alpha > i_\beta$, or both are of the same family but at least one of them is a shock. Note we never consider the free boundary as a wave/discontinuity in this paper.

As shown by Lemma 6.2 in Ref. 12, at $x = \tau$, if two discontinuities of strengths b_α and b_β collide, we have

$$Q(\tau+) - Q(\tau-) = -\frac{1}{2}|b_\alpha b_\beta|, \tag{30}$$

provided that

$$V(\tau-) \leq \mu := \frac{1}{2}O(1). \tag{31}$$

It is here one needs Lemma 3. If no discontinuities collide at $x = \tau$, then $Q(\tau+) = Q(\tau-)$.

2. Non-increasing of the Glimm Functional

We now show the bounds of total variation by proving that the Glimm functional $G(x)$ is non-increasing for x. There are the following three cases.

(i) *Collision of discontinuities.* For $x = \tau$ where two discontinuities b_α and b_β collide, there are no other wave interactions and reflections upon the free boundary as we assumed. Therefore, the decreasing of $G(\tau)$ is classical. By Lemma 3, we have

$$\begin{aligned} G(\tau+) - G(\tau-) &= (V(\tau+) - V(\tau-)) + \kappa(Q(\tau+) - Q(\tau-)) \\ &\leq M|b_\alpha b_\beta| + \kappa(-\frac{1}{2}|b_\alpha b_\beta|) \leq 0, \end{aligned}$$

if we choose $\kappa \geq 2M$ sufficiently large. Note that $O(1)$ is independent of the approximation parameter δ.

(ii) *Weak 1-wave interacts with the free boundary.* For $x = \tau$, a weak wave α_1 of 1-family interacts with the free boundary from above, resulting in a reflected 4-wave α_4. By Lemma 4, we have

$$\begin{aligned} G(\tau+) - G(\tau-) &= (V(\tau+) - V(\tau-)) + \kappa(Q(\tau+) - Q(\tau-)) \\ &\leq |b_{\alpha_4}| - |b_{\alpha_1}| + \kappa\mu|b_{\alpha_4}| \\ &\leq ((\kappa\mu + 1)(-K_2 + M_2\mu) - k_+)|\alpha_1| \leq 0, \end{aligned}$$

if we choose k_+ sufficiently large (independent of δ).

(iii) *Other situation.* If, for $x = \tau$, no collision or reflection upon the free boundary happens, then we still have $G(\tau+) = G(\tau-)$.

In the above, we have determined κ and k_+ independent of δ, and proved that, for any $x = \tau > 0$, there holds $G(\tau+) \leq G(\tau-)$, provided (31) holds.

3. Boundedness of Total Variation

The bound $V(\tau) \leq C_0\varepsilon$ then follows from an induction argument as shown in Ref. 12, p.217, for the proof of Lemma 6.3 there, provided that ε is small.

We first set $0 < \tau_1 < \ldots < \tau_k < \ldots$ to be the sequence so that, for $x = \tau_k$, either collision or reflection upon the free boundary occurs, and set $V_k := V(\tau_k-)$ and $G_k := G(\tau_k-)$ respectively.

We know that there exists a constant C_1 independent of $\delta > 0$ so that $V(\tau) \leq C_1 \mathrm{T.V.}(\mathrm{U}^\delta(\tau, \cdot))$ for all $x \geq 0$. Note here that the choice of weight k_+ is in essence only determined by $\underline{U}^+$. Define

$$C_0 = C_1 + \kappa C_1^2.$$

We choose positive $\varepsilon < 1$ small so that

$$C_1\varepsilon + \kappa(C_1\varepsilon)^2 \leq \mu, \qquad C_2C_0\varepsilon \leq \epsilon.$$

Here ϵ is the value so that the Riemann problems or the free boundary Riemann problems can be solved when the Riemann data are in $O_\epsilon(\underline{U}^+)$, and C_2 is the constant depending only on $\underline{U}^+$ so that $\mathrm{T.V.}(U^\delta(x, \cdot)) \leq C_2V(x)$ for any $x > 0$.

By assumption on the initial data, we have $\mathrm{T.V.}(U_0^\delta) \le \varepsilon$. Thus, by a property of the Riemann problem, we may have

$$V_1 \le C_1\varepsilon \le \min\{C_0\varepsilon, \mu\},$$

and furthermore,

$$G_1 \le V_1 + \kappa V_1^2 \le C_1\varepsilon + \kappa(C_1\varepsilon)^2 \le \min\{C_0\varepsilon, \mu\}.$$

Suppose that, for $n \le k$, we have proved

$$V_n \le \min\{C_0\varepsilon, \mu\}.$$

Then, by the decreasing of the Glimm functional, there holds

$$V_{k+1} \le G_{k+1} \le G_k \le \ldots \le G_1.$$

This shows

$$V_n \le \min\{C_0\varepsilon, \mu\} \qquad \text{for all } n.$$

If we further choose ε small so that $C_0\varepsilon \le \mu$, we obtain the bound $V(\tau) \le C_0\varepsilon$ as desired. This again implies the uniform estimate:

$$\mathrm{T.V.}(U^\delta(x,\cdot)) \le C_2C_0\varepsilon. \tag{32}$$

C. L^∞–Estimate of $\{U^\delta\}$ and Lipschitz Estimate of $\{g^\delta\}$

The fact that $\{U^\delta\}_{\delta>0}$ is uniformly bounded follows directly. For each x, the solution $U^\delta(x,y)$ is just the constant state $\underline{U}_0$ for sufficiently large y, by the finiteness of propagation speed and the fact that the initial data $U_0^\delta(y) \to \underline{U}_0$ as $y \to \infty$. Since we have proved $\mathrm{T.V.}(U^\delta(t,\cdot)) \le C_2C_0\varepsilon$ for any $t > 0$, then, by definition of the total variation, we conclude

$$\left\|U^\delta(x,\cdot) - \underline{U}^+\right\|_{L^\infty} \le C_2C_0\varepsilon \tag{33}$$

for some new constant C_2.

Estimate (33) implies the following uniform estimate on the free boundary that is given by the equation $y = g^\delta(x)$:

$$\left\|(g^\delta)'\right\|_{L^\infty} \le C_3\varepsilon. \tag{34}$$

with a constant C_3 depending only on $\underline{U}^+$. In particular, by construction, for fixed $\delta > 0$, g^δ is a piecewise linear (affine) function, and except for countable points $\{\tau_k\}$, it is differentiable, with $(g^\delta)'(x) = \frac{v^\delta(x,g^\delta(x))}{u^\delta(x,g^\delta(x))}$. Thus, by the mean value theorem,

$$|(g^\delta)'(x)| \le C'|U^\delta(x,g^\delta(x)) - \underline{U}^+| \le C'\left\|U^\delta - \underline{U}^+\right\|_{L^\infty} \le C'C_2C_0\varepsilon,$$

where the constant C' depends only on $\underline{U}^+$.

D. Finiteness of Collisions and Reflections

To show that the numbers of fronts/discontinuities and collisions/reflections do not approach infinity in $\{0 < x < \tau\}$ for any finite $\tau > 0$, the basic idea presented in Ref. 12 for the Cauchy problem works well, but we have to consider additional issues such as the reflections off the free boundary and the fact that the Euler system is not strictly hyperbolic in the argument. For completeness, we give the proof below, which closely follows that in Ref. 12.

1. Generation of Fronts and Modified Construction of Approximate Solutions

Firstly, we define the notion of *generation* of a front. We set that each initial front starting at $x = 0$ belongs to the first generation. Take two first-generation fronts of families d and h, respectively, that collide. The resulting fronts of families d and h belong to the first generation, while all the remaining fronts resulting from the collision are called second-generation fronts. Generally, if a front of family d and generation m interacts with a front of family h and generation n, the resulting front of families d and h are still of generation m and n, respectively, while the remaining fronts resulting from this collision are given generation $n+m$. The fronts of 4-family resulting from reflection of a front α of 1-family off the free boundary has the same generation of the front α. The point of this notion is that the fronts of high generation are quite weak.

Given the approximation parameter $\delta > 0$, we remove all fronts with generation higher than

$$N := \big[\ln_{4KT}(\delta)\big] \tag{35}$$

in our construction of approximate solution (g^δ, U^δ). Here $[z]$ denotes the integer larger than but closest to z and, following the notations in Ref. 12, p.218, we set

$$T = T(x) = \sum_{\alpha\in\Upsilon_x} |\alpha| \le V(x), \qquad K = \frac{1}{4C_0\varepsilon_0},$$

with ε_0 sufficiently small and fixed, and taking later $\varepsilon < \varepsilon_0$, so that $T < \frac{1}{4K}$.

More precisely, if two fronts of generation n and m collide, at most two waves will retain their generation. If $n+m > N$, then the remaining waves will be removed; however, if $n+m \le N$, we use the original (approximate) solution. When we remove the fronts, we let the function U^δ be equal to the value that has to be below the removed fronts, provided that the removed fronts are not the upmost fronts in the solution of the Riemann problem. If the upmost fronts are removed, then U^δ is set equal to the value immediately to the above of the removed fronts.

We remark that this process of removing (very) weak waves in approximate Riemann solver in our construction of approximate solutions will not influence the uniform estimates we obtained in Sections III B–III C. In particular, we still have $T < \frac{1}{4K}$.

2. Finiteness of Fronts and Collisions

We will show that there exist only a finite number of fronts of generation less than or equal to N and that, for a fixed δ, there is only a finite number of collisions/reflections.

For this, as we have known that $T < \frac{1}{4K}$, then the strength of each individual front is bounded by $\frac{1}{4K}$. For later reference, we also note that, by (35),

$$(4KT)^{N+1} \le \delta. \tag{36}$$

First we consider the number of fronts of first generation. This number can increase when the first-generation rarefaction waves split into several rarefaction fronts. By the term *rarefaction front* we mean a front approximating a rarefaction wave. Note that, by the construction of the approximate Riemann problem, the strength of each split rarefaction front is at least $\frac{3}{4}\delta$. Given that T is uniformly bounded, we find

$$(\text{Number of first generation fronts}) \le (\text{Number of initial fronts}) + \frac{4T}{3\delta}. \tag{37}$$

Thus, the number of first-generation fronts is finite. This also means that there will be only a finite number of collisions/reflections between first-generation fronts and free boundary. To see this, note first that strict hyperbolicity would have implied that each wave family will have speeds that are distinct. However, we see that, although the Euler system is not strictly hyperbolic, the multiplicity of the eigenvalues are constant for the states U near the background state $\underline{U}^+$. That is, $\lambda_1(U) < \lambda_2(U) = \lambda_3(U) < \lambda_4(U)$ for any state $U \in O_\epsilon(\underline{U}^+)$, and hence the eigenvalues are separable in the same way for any state U.

Hence, we can still conclude that each first-generation front will remain in a wedge in the (x, y)–plane determined by the slowest and fastest speeds of that family. Eventually, all first-generation fronts will have interacted at most finite times, and we can also conclude that there can be only a finite number of collisions between first-generation fronts and free boundary globally, since once a front is reflected, it will never meet the free boundary again.

Assume now that, for some $m \geq 1$, there will be only a finite number of fronts of generation i, for all $i < m$, and that there will only be a finite number of interactions between the fronts and fronts reflection off free boundary of generation less than m. Then, in analogy to (37), we find

$$\begin{aligned}&\text{Number of } m\text{-th generation fronts}\\ &\leq 2 \times (\text{Number of } j\text{-th and } i\text{-th-generation fronts; } i+j=m) + \frac{4T}{3\delta} < \infty. \qquad (38)\end{aligned}$$

Consequently, the number of fronts of generation less than or equal to m is finite. We can now repeat the arguments above showing that there is only a finite number of collisions between the first-generation fronts (and reflections off free boundary), just replacing "first generation" by "of generation less than or equal to m" and show that there is only a finite number of collisions producing the fronts of generation of $m+1$. Thus, we can conclude that there is only a finite number of fronts of generation less than $N+1$, and that these interact (reflect off free boundary) only a finite number of times.

IV. CONVERGENCE AND EXISTENCE OF WEAK ENTROPY SOLUTIONS

In this section we show the strong convergence of a subsequence of the approximate solutions to a weak entropy solution of problem (5).

A. Compactness

We first prove that there exists a subsequence of approximate solutions $\{(g^\delta, U^\delta)\}_{\delta>0}$ that converges to some (g, U) almost everywhere. In Section IV B, we will show that (g, U) is actually a weak entropy solution to problem (5).

1. Compactness of $\{g^\delta\}$

We now demonstrate the compactness of the approximate free boundaries $\{g^\delta\}_{\delta>0}$. More explicitly, we have

Lemma 5. *Let $g^\delta(x)$ be the free boundary for the approximate solution $U^\delta(x,y)$. Then there is a subsequence $\delta_j \to 0$ so that $g^{\delta_j}(x) \to g(x)$ uniformly in any compact set. Furthermore, the limit $g(x)$ is Lipschitz continuous: $|g(x_1) - g(x_2)| \leq C_3\varepsilon|x_1 - x_2|$ for some constant C_3.*

Proof. By (34), that is, $\|(g^\delta)'\|_{L^\infty([0,\infty))} \leq C_3\varepsilon$ and $g^\delta(0) = 0$, we see that, for fixed $T > 0$, the family $\{g^\delta\}$ is uniformly bounded and equicontinuous on $[0,T]$. Then, by the Arzela-Ascoli compactness criterion, there is a subsequence $\delta_j \to 0$ so that $g^{\delta_j} \rightrightarrows g$ uniformly for some g in $[0,T]$, and one easily proves that $|g(x_1) - g(x_2)| \leq C_3\varepsilon|x_1 - x_2|$ for $x_1, x_2 \in [0,T]$. By taking a diagonal subsequence for $2T, 3T, \ldots$, we can prove that g is defined for $x \in [0,\infty)$ and $g^{\delta_j} \to g$ uniformly in any compact subset of $[0,\infty)$, and $|g(x_1) - g(x_2)| \leq C_3\varepsilon|x_1 - x_2|$ for any finite x_1 and x_2. □

2. Compactness of $\{U^\delta\}$

We use the following compactness lemma, which is a modification of Theorem A.8 in Ref. 12.

Lemma 6. *Let $\{u_\eta : [0,\infty) \times [0,\infty) \to \mathbb{R}^4\}_\eta$ be a family of functions such that, for each positive T,*

(a) *$|u_\eta(x,\theta)| \leq C_T$ for $(x,\theta) \in [0,T] \times [0,\infty)$ with a constant C_T independent of η;*

(b) *For all $t \in [0,T]$, there holds*

$$\sup_{|\xi|\leq\rho} \int_B |u_\eta(x,\theta+\xi) - u_\eta(x,\theta)|\,\mathrm{d}\theta \leq \nu_{B,T}(|\rho|),$$

for a modulus of continuity ν and all compact $B \subset [0,\infty)$ (here $u_\eta(x,t)$ is extended to be zero for $x \notin [0,\infty)$);

(c) *Furthermore, for any $R>0$, for s and t in $[0,T]$, there holds*

$$\int_0^R |u_\eta(t,\theta)-u_\eta(s,\theta)|\,\mathrm{d}\theta \le \omega_T(|t-s|) \qquad \text{as } \eta\to 0,$$

for some modulus of continuity ω_T.

Then there exists a sequence $\eta_j \to 0$ such that, for each $x\in[0,T]$, the function $u_{\eta_j}(x)$ converges to a function $u(x)$ in $L^1([0,\infty))$. The convergence is in the topology of $C([0,T];L^1[0,\infty))$.

For any $T>0$, note that $U^\delta(x,y)$ is defined for $0<x<T$ and $g^\delta(x)<y<\infty$. By introducing $\theta=y-g^\delta(x)$, we may regard U^δ as a function of $\theta\in[0,\infty)$ and $x\in[0,T]$ by defining

$$\breve{U}^\delta(x,\theta)=U^\delta(x,\theta+g^\delta(x))$$

to apply Lemma 6. Obviously, $\left\|\breve{U}^\delta\right\|_{L^\infty}=\left\|U^\delta\right\|_{L^\infty}$ and $\mathrm{T.V.}(\breve{U}^\delta)(x,\cdot)=\mathrm{T.V.}(U^\delta)(x,\cdot)$. Then, by (33), we see immediately that (a) is valid for $\{\breve{U}^\delta\}_{\delta>0}$.

Using the boundedness of L^∞ norm and total variation of $\breve{U}^\delta$ (*cf.* (32)), the verification of (b) is elementary. Without loss of generality, we assume $\xi>0$. Then, by monotone convergence theorem,

$$\begin{aligned}
\int_{\mathbb{R}^+} |\breve{U}^\delta(x,\theta+\xi)-\breve{U}^\delta(x,\theta)|\,\mathrm{d}\theta &= \sum_{k=0}^{\infty}\int_{k\xi}^{(k+1)\xi} |\breve{U}^\delta(x,\theta+\xi)-\breve{U}^\delta(x,\theta)|\,\mathrm{d}\theta\\
&= \int_0^\xi \sum_{k=0}^{\infty}\left|\breve{U}^\delta(x,z+(k+1)\xi)-\breve{U}^\delta(x,z+k\xi)\right|\mathrm{d}z\\
&\le (\mathrm{T.V.}\breve{U}^\delta(x,\cdot))|\xi|\\
&\le (C_2C_0\varepsilon)|\xi|.
\end{aligned}$$

The verification of (c) is also not difficult. For $0<s<t<T$, we will prove that

$$\int_0^R |\breve{U}^\delta(t,\theta)-\breve{U}^\delta(s,\theta)|\,\mathrm{d}\theta \le C(t-s), \tag{39}$$

for any $R>0$ and a constant C independent of δ, t, and s.

To this end, for given approximate solution U^δ, suppose the "collision times" are

$$0<\tau_1<\ldots<\tau_k<\ldots.$$

Then, for $x\in(\tau_i,\tau_{i+1})$, nothing happens on the (approximate) free boundary, and then we may ignore the free boundary and write $U^\delta(x,y)$ in the form

$$U^\delta(x,y)=\sum_{k=1}^{N_i}(U^i_{k+1}-U^i_k)H(y-y^i_k(x))+U^i_1, \tag{40}$$

with $H(\cdot)$ the Heaviside step function (whose value is 0 for the negative argument and 1 for the positive argument). Here we have assumed that, for $x\in(\tau_i,\tau_{i+1})$, there are N_i discontinuities with equation $y=y^i_k(x)$ (from below to above as $k=1,\ldots,N_i$), and the state below $\{y=y^i_k(x)\}$ is U^i_k. From Section III D 2, we know that $N_i<\infty$.

With the above expression, for $\tau_i < s < t < \tau_{i+1}$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^+} \left|\breve{U}^\delta(t,\theta) - \breve{U}^\delta(s,\theta)\right| \mathrm{d}\theta \\
&= \int_{\mathbb{R}^+} \left| \int_s^t \frac{\mathrm{d}}{\mathrm{d}\tau} \breve{U}^\delta(\tau,\theta)\,\mathrm{d}\tau \right| \mathrm{d}\theta \\
&\le \int_{\mathbb{R}^+} \int_s^t \sum_{k=1}^{N_i} \left|U_{k+1}^i - U_k^i\right| \left|H'((g^\delta(\tau)+\theta) - y_k^i(\tau))\right| \left(\left|\frac{\mathrm{d}g^\delta(\tau)}{\mathrm{d}\tau}\right| + \left|\frac{\mathrm{d}y_k^i(\tau)}{\mathrm{d}\tau}\right| \right) \mathrm{d}\tau\,\mathrm{d}\theta \\
&\le (L + C_3\varepsilon) \int_s^t \sum_{k=1}^{N_i} \left|U_{k+1}^i - U_k^i\right| \int_{\mathbb{R}^+} \left|H'((g^\delta(\tau)+\theta) - y_k^i(\tau))\right| \mathrm{d}\theta\,\mathrm{d}\tau \\
&= (L + C_3\varepsilon) \int_s^t \sum_{k=1}^{N_i} \left|U_{k+1}^i - U_k^i\right| \mathrm{d}\tau \\
&\le (L + C_3\varepsilon) \mathrm{T.V.}\big(U^\delta(\tau_i+,\cdot)\big)(t-s) \\
&\le \big(L + C_3\varepsilon\big) C_2 C_0 \varepsilon (t-s).
\end{aligned} \tag{41}$$

Here we have set

$$L = \sup_{U \in O_\epsilon(\underline{U}^+)} \left(|\lambda_1(U)|, |\lambda_{2,3}(U)|, |\lambda_4(U)|\right) \tag{42}$$

to be the maximal characteristic speed, and used the fact that $\left|\frac{\mathrm{d}y_k^i(x)}{\mathrm{d}x}\right| \le L$. Estimate (34) is also used to control $\left|\frac{\mathrm{d}g^\delta}{\mathrm{d}\tau}\right|$.

We note (41) also holds for $s = \tau_i$ and/or $t = \tau_{i+1}$. Then, for $s \in (\tau_i, \tau_{i+1})$ and $t \in (\tau_j, \tau_{j+1})$ with $i < j$, using (41) repeatedly in the intervals $(s, \tau_{i+1}), (\tau_{i+1}, \tau_{i+2}), \ldots, (\tau_{j-1}, \tau_j)$, and (τ_j, t), we obtain (39) with $C = (L + C_3\varepsilon) C_2 C_0 \varepsilon$.

Therefore, by Lemma 6, we can find a subsequence $\{\breve{U}^{\delta_j}\}$ that converges to some $\breve{U}$ under the metric of $C([0,T]; L^1([0,\infty)))$. In addition, upon at most a further subsequence, $g^{\delta_j} \to g$. Now set $U(x,y) = \breve{U}(x, y - g(x))$, which is defined in the domain $\Omega = \{x > 0, y > g(x)\}$, with $D = \{y = g(x)\}$ being the lateral (free) boundary. In Section IV B below, we show that (g, U) is actually a weak entropy solution of problem (5). In the following, for simplification, we also write δ_j as δ.

B. Existence of a Weak Entropy Solution

For $0 \le s \le t \le T_0$, define $\Omega_{s,t} := \Omega \cap \{x \in [s,t]\}$, $\Sigma_s := \Omega \cap \{x = s\}$, and $\Gamma_{s,t} := D \cap \{s \le x \le t\}$. By the definition of weak entropy solutions (Definition 1), a pair of bounded measurable functions $(g, U) = (g(x), U(x,y))$ is a weak entropy solution of problem (5) provided that

- For any $\psi \in C_0^\infty(\mathbb{R}^2)$,

$$F_s^t(U) := \int_{\Omega_{s,t}} \big(\rho u \partial_x \psi + \rho v \partial_y \psi\big)\,\mathrm{d}y\,\mathrm{d}x + \int_{\Sigma_s} \rho u \psi\,\mathrm{d}y - \int_{\Sigma_t} \rho u \psi\,\mathrm{d}y = 0; \tag{43}$$

- For any $\psi \in C_0^\infty(\mathbb{R}^2)$,

$$\begin{aligned}
G_s^t(U) := &\int_{\Omega_{s,t}} \big((\rho u^2 + p)\partial_x \psi + \rho u v \partial_y \psi\big)\,\mathrm{d}y\,\mathrm{d}x + \int_{\Sigma_s} (\rho u^2 + p)\psi\,\mathrm{d}y \\
&- \int_{\Sigma_t} (\rho u^2 + p)\psi\,\mathrm{d}y - \underline{p} \int_{\Gamma_{s,t}} \psi n_1\,\mathrm{d}s = 0;
\end{aligned} \tag{44}$$

- For any $\psi \in C_0^\infty(\mathbb{R}^2)$,

$$I_s^t(U) := \int_{\Omega_{s,t}} \big(\rho u v \partial_x \psi + (\rho v^2 + p)\partial_y \psi\big)\,\mathrm{d}y\,\mathrm{d}x + \int_{\Sigma_s} \rho u v \psi\,\mathrm{d}y - \int_{\Sigma_t} \rho u v \psi\,\mathrm{d}y - \underline{p} \int_{\Gamma_{s,t}} \psi n_2\,\mathrm{d}s = 0; \tag{45}$$

- For any $\psi \in C_0^\infty(\mathbb{R}^2)$,

$$J_s^t(U) := \int_{\Omega_{s,t}} \big(\rho u(E+\frac{p}{\rho})\partial_x\psi + \rho v(E+\frac{p}{\rho})\partial_y\psi\big)\,\mathrm{d}y\,\mathrm{d}x + \int_{\Sigma_s} \rho u(E+\frac{p}{\rho})\psi\,\mathrm{d}y - \int_{\Sigma_t} \rho u(E+\frac{p}{\rho})\psi\,\mathrm{d}y = 0; \tag{46}$$

- For any $\psi \in C_0^\infty(\mathbb{R}^2)$ that is nonnegative,

$$E_s^t(U) := \int_{\Omega_{s,t}} \big(\rho u S\partial_x\psi + \rho v S\partial_y\psi\big)\,\mathrm{d}y\,\mathrm{d}x + \int_{\Sigma_s} \rho u S\psi\,\mathrm{d}y - \int_{\Sigma_t} \rho u S\psi\,\mathrm{d}y \le 0. \tag{47}$$

1. Estimate on the Total Strength of the Removed Fronts

For any approximate solution (g^δ, U^δ), we set

$$\Omega^\delta := \{x>0,\ y>g^\delta\}$$

and

$$\Gamma^\delta := \{y = g^\delta(x)\}.$$

For $0 \le s \le t \le T_0$, define

$$\Omega^\delta_{s,t} := \Omega^\delta \cap \{x\in[s,t]\}, \qquad \Sigma^\delta_s := \Omega^\delta \cap \{x=s\}, \qquad \Gamma^\delta_{s,t} := \Gamma^\delta \cap \{x\in[s,t]\}.$$

We note by our construction of approximate solutions that U^δ may not be a weak entropy solution of the Euler equations (1) in Ω^δ, since there are possible errors introduced by approximating the rarefaction waves via several fronts and by removing weak fronts of higher generations. In the following we will estimate these errors and show that they actually vanish as $\delta \to 0$. The analysis is again quite similar to Ref. 12. We first list below Lemma 6.5 in Ref. 12 for later reference.

Lemma 7. *Let $\mathcal{G}_m$ denote the set of all fronts of generation m, and let $\mathcal{T}_m$ denote the sum of the strengths of fronts of generation m: $\mathcal{T}_m = \sum_{\alpha_j\in\mathcal{G}_m}|\alpha_j|$. Then $T = \sum_{m=1}^N \mathcal{T}_m$, and*

$$\mathcal{T}_m \le C(4KT)^m$$

for some constant C. In particular, for $m = N+1$, we have $\mathcal{T}_{N+1} \le C\delta$ (cf. (36)).

2. Exact Riemann Solutions

For a given approximate solution (g^δ, U^δ), suppose as before that the collision/reflection "times" are $x = \tau_1 < \tau_2 < \cdots$. For a fixed interval $[\tau_j, \tau_{j+1}]$, set $s_1 = \tau_j$. We solve the following initial–free boundary problem with $i = 1$ (*cf.* (11)):

$$\begin{cases} \partial_x W(\tilde U) + \partial_y H(\tilde U) = 0, & x>s_i,\quad y>\tilde g(x),\\ \tilde U = U^\delta, & x = s_i,\quad y > \tilde g(s_i) := g^\delta(s_i),\\ \tilde p = \underline{p}, & x>s_i,\quad \text{on } \tilde\Gamma := \{y=\tilde g(x)\},\\ \tilde v = \tilde g'\tilde u, & x>s_i,\quad \text{on } \tilde\Gamma. \end{cases} \tag{48}$$

Since the "initial data" $U^\delta(s_1,\cdot)$ is piecewise constant, the solution $(\tilde g_1, \tilde U_1)$ is obtained by solving the Riemann problems. It can be solved up to $x = s_2$ when two-wave interaction or reflection off the free boundary occurs (if $s_2 > \tau_{j+1}$, we set $s_2 = \tau_{j+1}$). Then we solve $(\tilde g_2, \tilde U_2)$ from problem (48) with $i = 2$ (note that the initial data is $U^\delta(s_2,\cdot)$), up to some s_3. Repeat this process, we obtain

$$(\tilde g_i, \tilde U_i) \qquad \text{in } [s_i, s_{i+1})$$

with $\cup_{i=1}^\infty [s_i, s_{i+1}) = [\tau_j, \tau_{j+1})$. We can then define $(\tilde g, \tilde U)$ piecewise in $x \in [\tau_j, \tau_{j+1})$ by $(\tilde g, \tilde U) = (\tilde g_i, \tilde U_i)$ for $x \in [s_i, s_{i+1})$.

3. Error of Splitting Rarefaction Waves

Let $\tilde{U}^\delta$ be the *approximate* solution obtained from problem (48) in $[s_i, s_{i+1}]$, with the approximating parameter δ. This means that the rarefaction waves in $\tilde{U}$ are separated into many discontinuities; while there is no front to be removed since each front in $\tilde{U}$ is of generation one. Also, by our rule of splitting rarefaction waves, the lowermost state of $\tilde{U}^\delta$ is the same as $\tilde{U}$. This implies that the corresponding free boundaries are the same, and both $\tilde{U}$, $\tilde{U}^\delta$ are defined in the same domain. The analysis below is similar to Ref. 12. We present details here to show the ideas there still work for our free boundary problem.

Suppose that there is a rarefaction wave in $\tilde{U}$ with the below state $\tilde{U}_l$ and above state $\tilde{U}_r$. Then this rarefaction wave is replaced by a step function $\tilde{U}^\delta$. There also holds

$$|\tilde{U}^\delta(x,y) - \tilde{U}(x,y)| \le O(\delta)$$

by our splitting process (it is zero for the points not in the rarefaction wave fan). We also want to find the error in the L^1–space. To this end, we note that there are at most $\frac{|\tilde{U}_r - \tilde{U}_l|}{O(\delta)}$ steps, and the width of each step is at most $(x - s_i)\triangle\lambda$, with $\triangle\lambda$ the difference of characteristic speeds of two adjacent approximate fronts of each step — it is less than $O(\delta)$ (*cf.* (21)). Using the mean value theorem (since we know uniform L^∞ bounds of $\tilde{U}$ and $\tilde{U}^\delta$) and summing up for all rarefaction wave fans across x, we find

$$\begin{aligned}
\int_{y>\tilde{g}(x)} |W(\tilde{U}^\delta)(x,y) - W(\tilde{U})(x,y)|\,\mathrm{d}y &\le C\int_{y>\tilde{g}(x)} |\tilde{U}^\delta(x,y) - \tilde{U}(x,y)|\,\mathrm{d}y \\
&\le O(\delta)\sum_k |\tilde{U}_r^k - \tilde{U}_l^k||x - s_i| \\
&\le O(\delta)\mathrm{T.V.}(U^\delta)|x - s_i| \\
&= O(\delta)|x - s_i|. \qquad (49)
\end{aligned}$$

We note here that $\sum_k |\tilde{U}_r^k - \tilde{U}_l^k|$ is actually controlled by the total variation of the initial data by using the property of the Riemann solution. A similar inequality also holds when $W(U)$ is replaced by $H(U)$.

4. Error of the Removing Weak Fronts

We then compare $\tilde{U}^\delta$ and U^δ in $x \in [s_i, s_{i+1}]$. We note that both $\tilde{U}^\delta$ and U^δ satisfy the same initial data. The only difference between them is that some fronts in $\tilde{U}^\delta$ of generation $N+1$ are ignored to obtain U^δ. Note that, by removing fronts of generation $N+1$, we always keep the lowermost state the same as before. This means that the free boundary of U^δ is the same as $\tilde{U}^\delta$, hence still to be $y = \tilde{g}(x) = g^\delta(x)$. Consequently, $\tilde{U}^\delta$ is different from U^δ in $x \in (s_i, s_{i+1})$ only in a number of wedges emanating from the discontinuities in $U^\delta(s_i, \cdot)$, and in each wedge, the difference is bounded by the strength of the removing fronts α that are of generation $N+1$. We also note the width of each wedge is controlled by $O(x - s_i)$. By Lemma 7, we then find

$$\begin{aligned}
\int_{y>\tilde{g}(x)} \left|W(\tilde{U}^\delta)(x,y) - W(U^\delta)(x,y)\right|\mathrm{d}y &\le C\int_{y>\tilde{g}(x)} \left|\tilde{U}^\delta(x,y) - U^\delta(x,y)\right|\mathrm{d}y \\
&\le O(|x - s_i|)\sum_{\alpha\in\mathcal{G}_{N+1}} |\alpha| \\
&\le O(\delta)|x - s_i|. \qquad (50)
\end{aligned}$$

A similar inequality is also true for $H(U)$.

5. Total Error of Approximate Solutions

Since $\tilde{U}$ is obtained by the exact Riemann solvers for $x \in [s_i, s_{i+1}]$, the following must hold (with $\Omega_{s,t}$, Σ_s, and $\Gamma_{s,t}$ in the integrals replaced by $\Omega^\delta_{s,t}$, Σ^δ_s, and $\Gamma^\delta_{s,t}$ respectively, since we have shown that the free boundary of $\tilde{U}$ is the same as U^δ):

$$F_{s_i}^{s_{i+1}}(\tilde{U}) = 0, \quad G_{s_i}^{s_{i+1}}(\tilde{U}) = 0, \quad I_{s_i}^{s_{i+1}}(\tilde{U}) = 0, \quad J_{s_i}^{s_{i+1}}(\tilde{U}) = 0, \quad E_{s_i}^{s_{i+1}}(\tilde{U}) \le 0.$$

From (49) and (50), we also obtain that, for any $x \in [s_i, s_{i+1}]$,

$$\int_{\Sigma_x^\delta} \big|W(U^\delta)(x,y) - W(\tilde{U})(x,y)\big|\,\mathrm{d}y \le O(\delta)|x - s_i|, \tag{51}$$

$$\int_{\Sigma_x^\delta} \big|H(U^\delta)(x,y) - H(\tilde{U})(x,y)\big|\,\mathrm{d}y \le O(\delta)|x - s_i|. \tag{52}$$

Therefore, as an example, we find (note that the boundary terms involving the pressure $\underline{p}\int_\Gamma \psi n_1\,\mathrm{d}s$ are canceled because the boundary is the same):

$$\begin{aligned}
\big|G_{s_i}^{s_{i+1}}(U^\delta)\big| &= \big|G_{s_i}^{s_{i+1}}(U^\delta) - G_{s_i}^{s_{i+1}}(\tilde{U})\big| \\
&\le \int_{s_i}^{s_{i+1}} \int_{\Sigma_x^\delta} \big|\big(W_2(U^\delta) - W_2(\tilde{U})\big)\partial_x\phi + \big(H_2(U^\delta) - H_2(\tilde{U})\big)\partial_y\phi\big|\,\mathrm{d}y\mathrm{d}x \\
&\quad + \int_{\Sigma_{s_i}^\delta} \big|\big(W_2(U^\delta) - W_2(\tilde{U})\big)\phi\big|\,\mathrm{d}y + \int_{\Sigma_{s_{i+1}}^\delta} \big|(W_2(U^\delta) - W_2(\tilde{U}))\phi\big|\,\mathrm{d}y \\
&\le M\int_{s_i}^{s_{i+1}} \int_{\Sigma_x^\delta} \big|W_2(U^\delta) - W_2(\tilde{U})\big|\,\mathrm{d}y\,\mathrm{d}x + M\int_{s_i}^{s_{i+1}} \int_{\Sigma_x^\delta} \big|H_2(U^\delta) - H_2(\tilde{U})\big|\,\mathrm{d}y\,\mathrm{d}x \\
&\quad + M\int_{\Sigma_{s_i}^\delta} \big|W_2(U^\delta) - W_2(\tilde{U})\big|\,\mathrm{d}y + M\int_{\Sigma_{s_{i+1}}^\delta} \big|W_2(U^\delta) - W_2(\tilde{U})\big|\,\mathrm{d}y \\
&\le O(\delta)(s_{i+1} - s_i)^2 + O(\delta)(s_{i+1} - s_i),
\end{aligned}$$

where $M := \|\phi\|_{W^{1,\infty}}$. Then we find

$$\big|G_{\tau_j}^{\tau_{j+1}}(U^\delta)\big| \le \mathcal{O}(\delta)\sum_{i=1}^{\infty}\big((s_{i+1} - s_i)^2 + (s_{i+1} - s_i)\big) \le O(\delta)\big((\tau_{j+1} - \tau_j)^2 + (\tau_{j+1} - \tau_j)\big).$$

Thus it is clear that

$$\big|G_s^t(U^\delta)\big| \le O(\delta)(|t - s|^2 + |t - s|) \qquad \text{for any } 0 \le s < t < \infty,$$

and

$$\lim_{\delta\to 0} G_s^t(U^\delta) = 0.$$

We now need to prove that

$$\lim_{\delta\to 0} G_s^t(U^\delta) = G_s^t(U).$$

6. Verification of Weak Entropy Solutions

Set

$$\breve{\phi}^\delta(x,\theta) = \phi(x, \theta + g^\delta(x)), \quad \breve{W}_2^\delta(x,\theta) = W_2(U^\delta)(x, \theta + g^\delta(x)), \quad \breve{H}_2^\delta(x,\theta) = H_2(U^\delta)(x, \theta + g^\delta(x)),$$

where $W_2^\delta = W_2(U^\delta)$ and $H_2^\delta = H_2(U^\delta)$. Then we have

$$\begin{aligned}
G_s^t(U^\delta) &= \int_s^t \int_{y > g^\delta(x)} \big(W_2^\delta\partial_x\phi + H_2^\delta\partial_y\phi\big)\,\mathrm{d}y\mathrm{d}x \\
&\quad + \int_{y > g^\delta(s)} \big(W_2^\delta\phi\big)|_{x=s}\,\mathrm{d}y - \int_{y > g^\delta(t)} \big(W_2^\delta\phi\big)|_{x=t}\,\mathrm{d}y - \underline{p}\int_{\Gamma_{s,t}^\delta} \phi n_1\,\mathrm{d}s \\
&= \int_s^t \int_{\mathbb{R}^+} \big(\breve{W}_2^\delta(\partial_x\breve{\phi}^\delta - \partial_\theta\breve{\phi}^\delta(g^\delta)') + \breve{H}_2^\delta\partial_\theta\breve{\phi}^\delta\big)\,\mathrm{d}\theta\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^+} \big(\breve{W}_2^\delta\breve{\phi}^\delta\big)|_{x=s}\,\mathrm{d}\theta - \int_{\mathbb{R}^+} \big(\breve{W}_2^\delta\breve{\phi}^\delta\big)|_{x=t}\,\mathrm{d}\theta \\
&\quad - \underline{p}\int_s^t \phi(x, g^\delta(x))\frac{(g^\delta)'(x)}{\sqrt{1 + ((g^\delta)'(x))^2}}\sqrt{1 + ((g^\delta)'(x))^2}\,\mathrm{d}x.
\end{aligned}$$

By Lemma 5, we know that $g^\delta \to g$ uniformly for $x \in [s,t]$. Since $\{\breve{U}^\delta\}$ is uniformly bounded and converges to $\breve{U}$ under the metric of $C([s,t];L^1(\mathbb{R}^+))$, $\breve{W}^\delta$ and $\breve{H}^\delta$ are also uniformly bounded and converge to $\breve{W}$ and $\breve{H}$ respectively in the topology of $C([s,t];L^1(\mathbb{R}^+))$, as $\delta \to 0$. From these facts, one can easily use the Lebesgue dominant convergence theorem to show (with $\breve{\phi} = \phi(x, \theta + g(x))$) that, as $\delta \to 0$,

$$\int_{\mathbb{R}^+} (\breve{W}_2^\delta \breve{\phi}^\delta)|_{x=s}\,\mathrm{d}\theta - \int_{\mathbb{R}^+} (\breve{W}_2^\delta \breve{\phi}^\delta)|_{x=t}\,\mathrm{d}\theta \to \int_{\mathbb{R}^+} (\breve{W}_2 \breve{\phi})|_{x=s}\,\mathrm{d}\theta - \int_{\mathbb{R}^+} (\breve{W}_2 \breve{\phi})|_{x=t}\,\mathrm{d}\theta,$$

$$\int_s^t \int_{\mathbb{R}^+} \big(\breve{W}_2^\delta \partial_x \breve{\phi}^\delta + \breve{H}_2^\delta \partial_\theta \breve{\phi}^\delta\big)\,\mathrm{d}\theta\mathrm{d}x \to \int_s^t \int_{\mathbb{R}^+} \big(\breve{W}_2 \partial_x \breve{\phi} + \breve{H}_2 \partial_\theta \breve{\phi}\big)\,\mathrm{d}\theta\mathrm{d}x.$$

Since $\{(g^\delta)'\}$ is uniformly bounded, we may assume that $(g^\delta)' \rightharpoonup h$ in the weak* topology of $L^\infty(\mathbb{R}^+)$. Since $g^\delta \to g$ uniformly in $[s,t]$, we find that $(g^\delta)' \to g'$ in the sense of distributions. Thus, we must have $h = g'$. Therefore, as $\phi(x, g^\delta(x)) \to \phi(x, g(x))$ uniformly in $[s,t]$ and $(g^\delta)' \rightharpoonup g'$ in the weak* L^∞ sense, we have

$$\int_s^t \phi(x, g^\delta)(g^\delta)'(x)\,\mathrm{d}x \to \int_s^t \phi(x,g) g'(x)\,\mathrm{d}x.$$

We also find

$$\begin{aligned}
&\int_s^t \int_{\mathbb{R}^+} \big(\breve{W}_2^\delta \partial_\theta \breve{\phi}^\delta (g^\delta)' - \breve{W}_2 \partial_\theta \breve{\phi} g'\big)\,\mathrm{d}\theta\mathrm{d}x \\
&= \int_s^t \int_{\mathbb{R}^+} \breve{W}_2^\delta \big(\partial_\theta \breve{\phi}^\delta - \partial_\theta \breve{\phi}\big)(g^\delta)'\,\mathrm{d}\theta\mathrm{d}x + \int_s^t \int_{\mathbb{R}^+} \breve{W}_2^\delta \partial_\theta \breve{\phi}\big((g^\delta)' - g'\big)\,\mathrm{d}\theta\mathrm{d}x \\
&\quad + \int_s^t \int_{\mathbb{R}^+} (\breve{W}_2^\delta - \breve{W}_2) \partial_\theta \breve{\phi} g'(x)\,\mathrm{d}\theta\mathrm{d}x.
\end{aligned}$$

Using the boundedness of $\{\breve{W}_2^\delta\}$ and $\{(g^\delta)'\}$, as well as the uniform convergence $\partial_\theta \breve{\phi}^\delta \to \partial_\theta \breve{\phi}$, the first integral on the right-hand side goes to zero as $\delta \to 0$. The third one converges to zero follows directly from $\breve{W}_2^\delta \to \breve{W}_2$ in $C([s,t];L^1(\mathbb{R}^+))$. For the second integral, it can be written as

$$\int_s^t \int_{\mathbb{R}^+} \big(\breve{W}_2^\delta - \breve{W}_2\big) \partial_\theta \breve{\phi}\big((g^\delta)' - g'\big)\,\mathrm{d}\theta\mathrm{d}x + \int_s^t \int_{\mathbb{R}^+} \breve{W}_2 \partial_\theta \breve{\phi}\big((g^\delta)' - g'\big)\,\mathrm{d}\theta\mathrm{d}x.$$

By the boundedness of $(g^\delta)' - g'$, the first one then converges to zero; for the second one, using again $(g^\delta)' \rightharpoonup g'$ in the weak* topology of L^∞, then we have

$$\int_s^t \int_{\mathbb{R}^+} \breve{W}_2^\delta \partial_\theta \breve{\phi}^\delta (g^\delta)'\,\mathrm{d}\theta\mathrm{d}x \to \int_s^t \int_{\mathbb{R}^+} \breve{W}_2 \partial_\theta \breve{\phi} g'\,\mathrm{d}\theta\mathrm{d}x.$$

Hence, we have

$$\begin{aligned}
\lim_{\delta \to 0} G_s^t(U^\delta) &= \int_s^t \int_{\mathbb{R}^+} \big(\breve{W}_2(\partial_x \breve{\phi} - \partial_\theta \breve{\phi} g') + \breve{H}_2 \partial_\theta \breve{\phi}\big)\,\mathrm{d}\theta\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^+} (\breve{W}_2 \breve{\phi})|_{x=s}\,\mathrm{d}\theta - \int_{\mathbb{R}^+} (\breve{W}_2 \breve{\phi})|_{x=t}\,\mathrm{d}\theta - \underline{p} \int_s^t \phi(x, g(x)) g'(x)\,\mathrm{d}x \\
&= G_s^t(U)
\end{aligned}$$

by a change of variables $(x,y) = (x, \theta + g(x))$.

Therefore, we have proved $G_s^t(U) = 0$ as desired. Similarly, we can conclude that

$$E_s^t(U) \le 0, \qquad F_s^t(U) = 0, \qquad I_s^t(U) = 0, \qquad J_s^t(U) = 0.$$

Hence, the limit (g, U) obtained from the approximate solutions (g^δ, U^δ) is actually a weak entropy solution to problem (5).

It is clear that g should satisfy the estimate listed in Theorem 1, as guaranteed by Lemma 5. To show $\|(U - \underline{U}^+)(x,\cdot)\|_{\mathrm{BV}} \le C\varepsilon$, we note that we have proved

$$\|(U^\delta - \underline{U}^+)(x,\cdot)\|_{\mathrm{BV}} \le C\varepsilon.$$

Then, by Helly's theorem, without loss of generality, we may assume

$$(U^\delta - \underline{U}^+)(x,\cdot) \to (\tilde{U} - \underline{U}^+)(x,\cdot) \qquad \text{pointwise}$$

for some $(\tilde{U} - \underline{U}^+)(x,\cdot)$ so that

$$\left\|(\tilde{U} - \underline{U}^+)(x,\cdot)\right\|_{\mathrm{BV}} \le C\varepsilon.$$

However, by uniqueness of the pointwise limit, we must have $\tilde{U} = U$. This completes the proof of Theorem 1.

V. FAR FIELD BEHAVIOR OF WEAK ENTROPY SOLUTIONS

Finally we discuss the asymptotic behavior of the weak entropy solution (g, U) as $x \to \infty$.

For any given $\delta > 0$ and the corresponding approximate solution (g^δ, U^δ), we know that there are a finite number of fronts and collisions/reflections. Thus, there exists $x_\delta > 0$ so that, for $x > x_\delta$, there are no collisions and reflections. Suppose then that there are $m+1$ different states $\{U_j^\delta\}_{j=0}^m$ from the above to below. It is obvious that $U_0^\delta = \underline{U}_0$ (*cf.* Remark 1), and there is m_0 with $1 \le m_0 < m$ so that each pair $(U_{j-1}^\delta, U_j^\delta)$ $(j = 1, \ldots, m_0)$ is connected by a discontinuity of the first characteristic family, while each $(U_{j-1}^\delta, U_j^\delta)$, $j = m_0+1, \ldots, m$, is connected by a characteristic discontinuity (of the second and/or third characteristic family). Since no fronts interact, it is only possible that, for $m_0 > 1$, all the discontinuities of the first family must be rarefaction waves; for $m_0 = 1$, this discontinuity might be a shock or a rarefaction wave. For states U_j^δ $(j = m_0+1, \ldots, m)$, the pressure must be $\underline{p}$ by the boundary condition and the Rankine–Hugoniot jump conditions of characteristic discontinuities.

Now we solve the free boundary Riemann problem of the Euler equations (1) with the initial data $U = \underline{U}_0$ and the boundary condition $p = \underline{p}$ on the free boundary $y = kx$. Suppose that the solution is given by $U_\infty = \Psi_4(\alpha; \underline{U}_0)$. Then $k = \frac{v_\infty}{u_\infty}$, and the resulting 4-wave is a shock if $\underline{p} > \underline{p}_0$, and a rarefaction wave if $\underline{p} < \underline{p}_0$. It is clear that both $\frac{v_j^\delta}{u_j^\delta}$ $(j = m_0+1, \ldots, m)$ and $(g^\delta)'$ should be $\frac{v_\infty}{u_\infty}$ for all $x > x_\delta$.

We now define $p_\delta(\xi, \theta) = p^\delta(\xi + x_\delta + 1, \theta + g^\delta(\xi + x_\delta))$ for $\xi \ge 0$ and $\theta \ge 0$. It is easy to see that $|p_\delta(\xi,\cdot)|$ and $\mathrm{T.V.}(p_\delta)(\xi,\cdot) = |\underline{p}_0 - \underline{p}|$ are bounded for all $\delta > 0$ and given ξ. Thus, by Helly's theorem, there is a subsequence (still denoted as δ) so that $\lim_{\delta\to 0} p_\delta(\xi,\theta) = p(\theta) = \underline{p}$ for $\theta > 0$ pointwise. This should imply that, for a.e. $\theta \ge 0$ and the weak entropy solution $U = (u, v, p, \rho)$,

$$\lim_{x\to\infty} p(x, \theta + g(x)) = \underline{p}.$$

Similarly, we have

$$\lim_{x\to\infty} \frac{v}{u}(x, \theta + g(x)) = \frac{v_\infty}{u_\infty}.$$

It then follows from $g' = \frac{v}{u}$ that

$$\lim_{x\to\infty} g'(x) = \frac{v_\infty}{u_\infty}.$$

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