Report no. OxPDE-13/04

Oxford Centre for Nonlinear PDE

Well-Posedness of Transonic Characteristic Discontinuities in Two-Dimensional Steady Compressible Euler Flows

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March 2013

WELL-POSEDNESS OF TRANSONIC CHARACTERISTIC DISCONTINUITIES IN TWO-DIMENSIONAL STEADY COMPRESSIBLE EULER FLOWS

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ABSTRACT. In our previous work, we have established the existence of transonic characteristic discontinuities separating supersonic flows from a static gas in two-dimensional steady compressible Euler flows under a perturbation with small total variation of the incoming supersonic flow over a solid right-wedge. It is a free boundary problem in Eulerian coordinates and, across the free boundary (characteristic discontinuity), the Euler equations are of elliptic-hyperbolic composite-mixed type. In this paper, we further prove that such a transonic characteristic discontinuity solution is unique and L^1–stable with respect to the small perturbation of the incoming supersonic flow in Lagrangian coordinates.

1. INTRODUCTION

We are concerned with the well-posedness of transonic characteristic discontinuities that separate supersonic flows from a static gas in two-dimensional steady compressible Euler flows. The governing equations are the following Euler system that consists of conservation laws of mass, momentum, and energy (*cf.* [5, 7, 10]):

$$\partial_x(\rho u) + \partial_y(\rho v) = 0, \tag{1.1}$$

$$\partial_x(\rho u^2 + p) + \partial_y(\rho uv) = 0, \tag{1.2}$$

$$\partial_x(\rho uv) + \partial_y(\rho v^2 + p) = 0, \tag{1.3}$$

$$\partial_x(\rho uE + pu) + \partial_y(\rho vE + pv) = 0, \tag{1.4}$$

where $E = \frac{1}{2}(u^2+v^2)+e$. The unknowns ρ, p, e, and (u, v) represent the density, pressure, internal energy, and velocity of the fluid, respectively. Specifically, for a polytropic gas, the constitutive relations are

$$p = \rho^\gamma \exp\big(\frac{S}{c_\nu}\big), \qquad e = \frac{1}{\gamma-1}\frac{p}{\rho}, \tag{1.5}$$

where S is the entropy, c_ν is a positive constant, and $\gamma > 1$ is the adiabatic exponent. The sonic speed c is determined by

$$c = \sqrt{\frac{\partial p(\rho, S)}{\partial \rho}} = \sqrt{\frac{\gamma p(\rho, S)}{\rho}}. \tag{1.6}$$

Date: February 17, 2013.

2000 *Mathematics Subject Classification.* 35L50, 35L65, 35Q31, 35B35, 76H05, 76N10.

Key words and phrases. L^1 stability, uniqueness, transonic, characteristic discontinuity, vortex sheet, entropy wave, steady flow, Euler system, Lagrange coordinates, front tracking.

In our previous work [5], we have established the existence of a weak entropy solution to the following initial–boundary value problem:

$$\begin{cases} (1.1)-(1.4) & \text{in } \; x>0, y>g(x), \\ U=U_0 & \text{on } \; x=0, y>0, \\ p=\overline{p}, \quad \frac{v}{u}=g'(x) & \text{on } \; x>0, y=g(x), \end{cases} \tag{1.7}$$

provided that the incoming supersonic flow U_0 is close in BV to a reference state $\overline{U}_+$, where $U=(u,v,p,\rho)$, while $\overline{U}_+=(\overline{u},0,\overline{p},\overline{\rho}_+)$ is a uniform supersonic flow; that is, it is a constant vector and $\overline{u}>\overline{c}_+=\sqrt{\frac{\gamma\overline{p}}{\overline{\rho}_+}}$. The unknowns in problem (1.7) are the supersonic flow $U=U(x,y)$ and the free boundary $\Gamma_U=\{y=g(x), x\geq 0\}$, which is a Lipschitz curve passing through the origin. The free boundary is actually a transonic characteristic discontinuity (vortex sheet and/or entropy wave) separating the supersonic flow U from the unperturbed static gas $\overline{U}_-=(0,0,\overline{p},\overline{\rho}_-)$ that is subsonic (*cf.* Figure 1). Thus, problem (1.7) is a free boundary problem (in Eulerian coordinates) and, across the free boundary, the Euler equations are of elliptic-hyperbolic composite-mixed type.

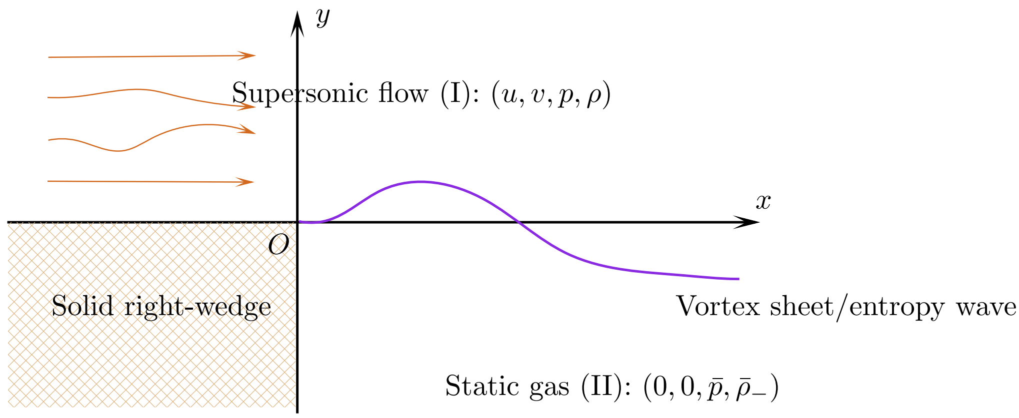

FIGURE 1.1. For supersonic flow passing the corner O with a static gas $\overline{U}_-$ on the right of the solid right wedge (*i.e.* with velocity zero), a combined vortex sheet/entropy wave is generated to separate the static gas below from the supersonic flow above.

To further study the L^1–stability, we run into a difficulty to define the L^1–distance between two solutions, since two solutions U and V describing the supersonic flows are generically defined in different domains depending on the location of their respective free boundaries Γ_U and Γ_V: For some given "time" $x^*>0$ and at some point $y^*\in\mathbb{R}$, it may happen that, at (x^*,y^*), it is a static state for one solution, but is a supersonic flow for the other. To this end, we employ the special structure of the Euler equations of the two-dimensional steady flows, and introduce the Lagrangian coordinates (ξ,η) associated to a solution, as done in the study of transonic shocks [8, 11, 16], to transform the free boundary to the positive ξ-axis, so that the solutions U and V are then defined in the same domain $\{\xi>0,\eta>0\}$ in Lagrangian coordinates. Then the initial data functions $U_0(0,y)$ and $V_0(0,y)$ are transformed to the corresponding initial data functions $U_0(0,\eta)$ and $V_0(0,\eta)$ in their respective Lagrange coordinates. Similar to the analysis in [5] for the construction of approximate solutions, we construct the approximate front tracking solutions $U(\xi,\eta)$ and $V(\xi,\eta)$ in the domain $\{\xi>0,\eta>0\}$ with the initial data $U_0(0,\eta)$ and $V_0(0,\eta)$ on $\xi=0$, and the boundary data $p=\overline{p}$ on $\eta=0$, respectively, and study the L^1–stability property of these solutions for the initial-boundary value problem (2.17) in Lagrangian coordinates, as proposed accurately below. With this merit of Lagrangian coordinates, we can show the characteristic discontinuity solutions are both unique for the given incoming supersonic flow in Eulerian coordinates and L^1–stable with respect to the small perturbations of the incoming supersonic flows in Lagrangian coordinates.

In Section 2, we present the Lagrangian coordinates associated to a solution to problem (1.7) and formulate this problem in Lagrangian coordinates as problem (2.17). Then we briefly review how the existence of a weak entropy solution is shown by using the front tracking method. Section 3 is devoted to establishing the L^1–stability of problem (2.17). Then, in Section 4, we show the existence of a semigroup associated with problem (2.17). Finally, in Section 5, we show the uniqueness of solutions to problem (2.17) in the larger class of viscosity solutions. By Wagner's theorem [15, Theorem 2], there is a one-to-one correspondence between a bounded measurable weak solution of a hyperbolic system of conservation laws and a bounded measurable weak solution of the corresponding equations in Lagrangian coordinates. Thus, the uniqueness results established for problem (2.17) imply the uniqueness of transonic characteristic discontinuities in Eulerian coordinates. This is clarified in Section 6. The well-posedness results are summarized in Theorem 6.2. These results on the existence, uniqueness, and L^1–stability in Lagrangian coordinates, together with the existence theorem established in [5], may be considered as a complete mathematical well-posedness theory of such transonic characteristic discontinuity solutions.

2. Formulation of the Problem in Lagrangian Coordinates

In this section we reformulate the problem in Eulerian coordinates to the problem in Lagrangian coordinates and show the existence of entropy solutions for the problem in Lagrangian coordinates.

2.1. Lagrangian coordinates.

For a piecewise C^1–smooth flow, we may introduce the Lagrangian transformation as used in [9, 16] to formulate problem (1.7) in Lagrangian coordinates, which enables us to straighten the streamlines and hence treat a strict hyperbolic system derived from (1.1)–(1.4).

Let $U\in L^\infty(\mathbb{R}^+\times\mathbb{R}^+)$ and $g\in \mathrm{Lip}\,(\mathbb{R}^+)$ be a *piecewise* C^1 weak entropy solution to problem (1.7). Define

$$\eta=\eta(x,y;x_0,y_0)=\int_{(x_0,y_0)}^{(x,y)}(\rho u)(s,t)\,\mathrm{d}t-(\rho v)(s,t)\,\mathrm{d}s, \tag{2.1}$$

where (x_0,y_0) is a fixed point on the transonic characteristic discontinuity Γ_U, and the integration is on any smooth curve Γ connecting (x_0,y_0) with (x,y) and lies in the upper side of Γ_U. Since the domain $\{x>0,y>g(x)\}$ is simply connected, by the conservation of mass, as well as the Rankine-Hugoniot (R-H) conditions for crossing the shock-front, η is a well-defined function of (x,y) with $y\geq g(x)$, independent of the choice of Γ. Clearly, we have

$$\frac{\partial\eta}{\partial x}=-\rho v,\quad \frac{\partial\eta}{\partial y}=\rho u. \tag{2.2}$$

We also note that, by the last condition in (1.7), η is independent of (x_0, y_0) on $\mathcal{D}_U$. In fact, for (x_0, y_0) and (x_0', y_0') on the characteristic discontinuity, we have

$$\begin{aligned}
\eta(x, y; x_0, y_0) - \eta(x, y; x_0', y_0') &= \int_{(x_0,y_0)}^{(x_0',y_0')} (\rho u)(s,t)\,\mathrm{d}t - (\rho v)(s,t)\,\mathrm{d}s \\
&= \int_{x_0}^{x_0'} \big((\rho u)(s, g(s)) \cdot g'(s) - (\rho v)(s, g(s))\big)\,\mathrm{d}s \\
&= 0. \qquad (2.3)
\end{aligned}$$

Since the characteristic discontinuity $y = g(x)$ is in $\mathrm{Lip}\,(\mathbb{R}^+, \mathbb{R})$, it is differentiable almost everywhere, so that the integrand is discontinuous only at these points in a set of Lebesgue measure zero, which is harmless for the above calculation. Hence, in the following, we may write $\eta = \eta(x, y)$ with $\eta = 0$ whenever $y = g(x)$.

If u, v, and ρ belong to L^∞, and (1.1) is satisfied in the sense of distributions, we can also find a unique Lipschitz continuous function η so that $\eta(x_0, y_0) = 0$ and (2.2) holds. As a matter of fact, using standard mollification, we may approximate ρu and ρv in the weak convergence of distributions by C^∞–functions $(\rho u)^\epsilon$ and $(\rho v)^\epsilon$, for which the equality $\partial_x(\rho u)^\epsilon + \partial_y(\rho v)^\epsilon = 0$ still holds. Furthermore, by the Young inequality, we have $\|(\rho u)^\epsilon\|_{L^\infty} \le C\,\|\rho u\|_{L^\infty}$ and $\|(\rho v)^\epsilon\|_{L^\infty} \le C\,\|\rho v\|_{L^\infty}$. Then, as above, we can solve $\eta^\epsilon \in C^1$ so that $\eta^\epsilon(x_0, y_0) = 0$ and hence, in any bounded domain, $\{\eta^\epsilon\}$ is a family of C^1–functions that are uniformly bounded and equicontinuous. Thus, there is a subsequence $\{\eta^k\}$ that converges uniformly to some η; by taking a diagonal subsequence, we can find η that is defined in the whole domain and η^k converges uniformly to η in any compact subregion, hence $\eta^k \to \eta$ in $\mathscr{D}'$. It is obvious that $\eta(x_0, y_0) = 0$, and (2.2) holds by uniqueness of limits in the sense of distributions. This then ensures that η is Lipschitz continuous. The uniqueness follows from the well-known fact that a distribution with zero derivatives must be a constant. Differentiating η along a streamline (which is Lipschitz continuous) and using (2.2) yield that it is constant along the streamline.

Now we introduce the following Lagrangian transformation $(x, y) \mapsto (\xi, \eta)$:

$$(\xi, \eta) = (x, \eta(x, y)). \qquad (2.4)$$

Then we have

$$\frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{pmatrix} 1 & 0 \\ -\rho v & \rho u \end{pmatrix}, \qquad (2.5)$$

thus

$$\partial_x = \partial_\xi - \rho v \partial_\eta, \quad \partial_y = \rho u \partial_\eta. \qquad (2.6)$$

This transformation is Lipschitz continuous and one-to-one provided $\rho u > 0$. A simple computation shows that equations (1.1)–(1.3) may be written in divergence form:

$$\begin{cases}
\partial_\xi(\frac{1}{\rho u}) - \partial_\eta(\frac{v}{u}) = 0, \\
\partial_\xi(u + \frac{p}{\rho u}) - \partial_\eta(\frac{pv}{u}) = 0, \\
\partial_\xi v + \partial_\eta p = 0,
\end{cases} \qquad (2.7)$$

or, as a symmetric system for $U = (u, v, p)^\top$,

$$A\partial_\xi U + B\partial_\eta U = 0, \tag{2.8}$$

with

$$A = \begin{pmatrix} u & 0 & \frac{1}{\rho} \\ 0 & u & 0 \\ \frac{1}{\rho} & 0 & \frac{u}{\rho^2 c^2} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & -v \\ 0 & 0 & u \\ -v & u & 0 \end{pmatrix}. \tag{2.9}$$

For $\rho u \neq 0$, the conservation law of energy becomes $\partial_\xi(\frac{u^2+v^2}{2} + \frac{c^2}{\gamma-1}) = 0$, that is,

$$\frac{u^2+v^2}{2} + \frac{c^2}{\gamma-1} = b(\eta). \tag{2.10}$$

This is the well-known Bernoulli law. As $b(\eta)$ is given by the initial data, from now on, we focus on system (2.7) with ρ determined by $U = (u, v, p)$ through (2.10).

The eigenvalues λ of (2.8) (*i.e.* $|\lambda A - B| = 0$) are

$$\lambda_1 = \frac{\rho c^2 u}{u^2 - c^2}\left(\frac{v}{u} - \sqrt{M^2-1}\right), \tag{2.11}$$

$$\lambda_2 = 0, \tag{2.12}$$

$$\lambda_3 = \frac{\rho c^2 u}{u^2 - c^2}\left(\frac{v}{u} + \sqrt{M^2-1}\right), \tag{2.13}$$

where $M = \frac{\sqrt{u^2+v^2}}{c}$ is the Mach number of the flow. Then, for $u > c$, system (2.8) is strictly hyperbolic. The associated right-eigenvectors are

$$r_1 = \kappa_1(\frac{\lambda_1}{\rho} + v, -u, -\lambda_1 u)^\top, \tag{2.14}$$

$$r_2 = (u, v, 0)^\top, \tag{2.15}$$

$$r_3 = \kappa_3(\frac{\lambda_3}{\rho} + v, -u, -\lambda_3 u)^\top, \tag{2.16}$$

where κ_j can be chosen that $r_j \cdot \nabla\lambda_j \equiv 1$, since the jth-characteristics fields, $j = 1, 3$, are genuinely nonlinear. Note that the second characteristic field is always linearly degenerate: $r_2 \cdot \nabla\lambda_2 = 0$.

2.2. **Formulation of the problem in Lagrangian coordinates.** As we noted above that the transonic characteristic discontinuity becomes the positive ξ-axis in Lagrangian coordinates, we formulate the problem in Eulerian coordinates into the following initial-boundary value problem for equations (2.7):

$$\begin{cases} (2.7) & \text{in } \ \xi > 0, \eta > 0, \\ U(0,\eta) = U_0(\eta) & \text{on } \ \xi = 0, \eta > 0, \\ p = \overline{p} & \text{on } \ \xi > 0, \eta = 0. \end{cases} \tag{2.17}$$

Once we solve U from this problem, we then obtain the free boundary in Eulerian coordinates:

$$g(x) = \int_0^x \frac{v(\xi,0)}{u(\xi,0)}\,\mathrm{d}\xi. \tag{2.18}$$

These are the corresponding forms in Lagrangian coordinates for problem (1.7).

2.3. **Existence of entropy solutions.** The entropy solutions of problem (2.17) can be defined in the standard way via integration by parts. We note that the existence of entropy solutions to problem (2.17) can be constructed easily by using the front tracking method as carried out in [5] provided that $\|U_0 - \overline{U}_+\|_{\mathrm{BV}(\mathbb{R}^+)}$ is sufficiently small. In particular, the following lateral Riemann problem with the boundary data $p = \overline{p}$ on the characteristic boundary $\{\eta = 0\}$ is uniquely solvable.

Lemma 2.1. *Consider the following lateral Riemann problem:*

$$\begin{cases} (2.7) & \text{in } \ \xi > 0, \eta > 0, \\ U = U_+ & \text{on } \ \xi = 0, \eta > 0, \\ p = \overline{p} & \text{on } \ \xi > 0, \eta = 0. \end{cases}$$

There exists $\varepsilon > 0$ so that, if U_+ lies in the ball $O_\varepsilon(\overline{U}_+)$ with center $\overline{U}_+$ and radius ε, then there is a unique admissible solution that contains only a 3-wave.

Proof. 1. Note that system (2.7) is strictly hyperbolic for $u > c$. For each point U with $u > c$, in its small neighborhood, we can obtain C^2-wave curves $\Phi_j(\alpha; U), j = 1, 2, 3$, so that $\Phi_j(0; U) = U$ and $\frac{\mathrm{d}\Phi_j}{\mathrm{d}\alpha}|_{\alpha=0} = r_j(U)$: $\Phi_j(\alpha; U)$ is connected to U from the upper side by a simple wave of j-family with strength $|\alpha|$; for $\alpha > 0$ and $j = 1, 3$, this wave is a rarefaction wave, while, for $\alpha < 0$ and $j = 1, 3$, this wave is a shock. For $j = 2$, the wave is always a characteristic discontinuity.

For our purpose, we note that there is also a C^2-curve $\Psi_3(\beta; U)$ which consists of those states that can be connected to U from the lower side by a 3-wave of strength β. We have $\Psi_3(0; U) = U$ and $\frac{\mathrm{d}\Phi_j}{\mathrm{d}\beta}|_{\beta=0} = -r_3(U)$.

2. We set $U = (u, v, p)^\top$ in Lagrangian coordinates and use U_3 to represent the third argument of the vector U. Then, to solve the lateral Riemann problem, it suffices to show that there exists a unique β so that $(\Psi_3(\beta; U_+))_3 = \overline{p}$. Therefore, we consider the following function:

$$L(\beta; U_+) = (\Psi_3(\beta; U_+))_3 - (\Psi_3(0; \overline{U}_+))_3.$$

It is clear that $L(0; \overline{U}_+) = 0$, and

$$\frac{\partial L(0; \overline{U}_+)}{\partial \beta} = -(r_3(\overline{U}_+))_3 = \kappa_3 \bar{\lambda}_3 u \neq 0.$$

By the implicit function theorem, there exists $\varepsilon > 0$ such that, for $U_+ \in O_\varepsilon(\overline{U}_+)$, there is a function $\beta = \beta(U_+)$ so that $L(\beta(U_+); U_+) = 0$. Then, similar to Lemma 2.2 in [5], by using the Taylor expansion up to second order (recall that Ψ_j is C^2), we obtain the following estimate:

$$\beta = K|p_+ - \overline{p}| + O(1)|U_+ - \overline{U}_+|^2, \tag{2.19}$$

with a suitable constant K depending only on $\overline{U}_+$. □

For the reflection of weak 1-waves off the characteristic boundary $\{\eta = 0\}$, we have

Lemma 2.2. *Suppose that U^l, U^m, and U^r are three states in $O_\epsilon(\overline{U}^+)$ for sufficiently small ϵ, with $U^m = \Phi_1(\alpha_1; U^l) = \Phi_3(\alpha_3; U^r)$. Then*

$$\alpha_3 = -K_2\alpha_1 + M_2|\alpha_1|^2, \tag{2.20}$$

with the constant $K_2 > 0$ *and the quantity* M_2 *bounded in* $O_\epsilon(\underline{U}^+)$. *Furthermore, for* $U^l = (u_l, v_l, p_l)$, $|K_2| > 1, |K_2| < 1,$ *and* $|K_2| = 1$ *when* $v_l < 0$, $v_l > 0$, *and* $v_l = 0$, *respectively.*

The proof here follows the similar argument for Lemma 2.4 in [5].

For our current problem, we have to modify slightly the Glimm functional as done in [5]. In particular, this will allow us to show later that the Lyapunov functional Φ (see Section 3) decreases when a weak wave of 1-family is reflected off the boundary. We now introduce the following version of the Glimm functional:

$$\mathcal{G}(\xi) = \mathcal{V}(\xi) + \kappa \mathcal{Q}(\xi),$$

where $\kappa > 0$ is a large constant to be chosen. The terms $\mathcal{V}$ and $\mathcal{Q}$ are explained below:

• *The weighted strength term* $\mathcal{V}(\xi)$. We define the total (weighted) strength of weak waves in $U^\delta(\xi, \cdot)$ as

$$\mathcal{V}(\xi) = \sum_\alpha |b_\alpha|. \tag{2.21}$$

Here, for a weak wave α of i_α-family, we define its weighted strength as

$$b_\alpha = \begin{cases} k_+\alpha & \text{if } i_\alpha = 1, \\ \alpha & \text{if } i_\alpha = 2, 3, \end{cases}$$

where $k_+ > K_2$ is a positive constant which has been fixed by considering the reflection coefficient K_2 in the strength α_3 of the reflected 3-wave in the interaction between the weak 1-wave and the characteristic boundary $\{\eta = 0\}$ as given in Lemma 2.2.

• *The interaction potential term* $\mathcal{Q}(\xi)$. The interaction potential term we used here is a modification to the one introduced by Glimm, that is,

$$\mathcal{Q}(\xi) = \sum_{(\alpha,\beta)\in\mathscr{A}} |b_\alpha b_\beta| + \sum_{\beta\in\mathscr{A}_b} |b_\beta| = \mathcal{Q}_{\mathcal{A}} + \mathcal{Q}_b, \tag{2.22}$$

where the set $\mathscr{A}(\xi)$ is of all the couples (b_α, b_β) of approaching wave-fronts. The term $\mathcal{Q}_b$ in our wave interaction potential is an additional term, in comparison with the Cauchy problem; and if a weak wave α of 1-family is approaching the boundary, we write $\alpha \in \mathscr{A}_b$.

Now, for given initial data U_0, we may construct an approximate solution U^δ by a front tracking algorithm introduced by [12], where δ is a small parameter measuring the accuracy of the solution, which controls the following four types of errors generated by the algorithm:

- Errors in the approximation of initial data;
- Errors in the speeds of shocks, characteristic discontinuities (vortex sheets and entropy waves), and rarefaction fronts;
- Errors by approximating the rarefaction waves by piecewise constant rarefaction fronts;
- Errors from removing all the fronts with generation higher than $N \in \mathbb{Z}_+$ (N depends on δ).

The construction of a Glimm functional as above provides the necessary uniform estimates that guarantee the existence of a subsequence of U^δ which converges to a bounded entropy solution of (2.17) in $C([0, T]; L^1(\mathbb{R}^+))$ for any $T > 0$.

The following sections are devoted to the L^1–stability and uniqueness properties for problem (2.17).

3. A Lyapunov functional and L^1–stability

This section is devoted to establishing the L^1–stability of two entropy solutions U and V of problem (2.17) with respective initial data U_0 and V_0 obtained by the front tracking method:

$$\|U(\xi,\cdot)-V(\xi,\cdot)\|_{L^1(\mathbb{R}^+)} \le C\,\|U_0-V_0\|_{L^1(\mathbb{R}^+)}\,, \tag{3.1}$$

with a constant C depending only on the equations and the reference state $\overline{U}_+$. To this end, following an idea in [4], we introduce a Lyapunov functional $\Phi(U^{\delta_1},V^{\delta_2})$ by incorporating additional new waves generated from the weak wave interactions with the characteristic boundary $\{\eta=0\}$, which is equivalent to the L^1–distance:

$$C_1^{-1}\,\|U(\xi,\cdot)-V(\xi,\cdot)\|_{L^1(\mathbb{R}^+)} \le \Phi(U,V) \le C_1\|U(\xi,\cdot)-V(\xi,\cdot)\|_{L^1(\mathbb{R}^+)} \tag{3.2}$$

and decreases as ξ increases:

$$\Phi(U^{\delta_1}(\xi_2,\cdot),V^{\delta_2}(\xi_2,\cdot)) - \Phi(U^{\delta_1}(\xi_1,\cdot),V^{\delta_2}(\xi_1,\cdot)) \le C_2\max(\delta_1,\delta_2)(\xi_2-\xi_1),\ \ \forall\ \xi_2>\xi_1>0, \tag{3.3}$$

for some constants C_i, $i=1$, 2, where U^{δ_1} and V^{δ_2} are two approximate solutions corresponding to the initial data U_0 and V_0 obtained by the wave-front tracking, with accuracy δ_1 and δ_2, respectively.

Estimate (3.3) has many important implications. Firstly, taking $U_0=V_0$ and $\xi_1=0$ in (3.3) and using (3.2), we obtain $\lim_{\delta_1,\delta_2\to 0}\left\|U^{\delta_1}(\xi,\cdot)-U^{\delta_2}(\xi,\cdot)\right\|_{L^1(\mathbb{R}^+)}=0$. Thus, the whole sequence of approximate solutions $\{U^\delta\}$ converges to the same limit. Secondly, taking directly $\delta_1,\delta_2\to 0$ yields (3.1). This again implies that the entropy solution of problem (2.17) obtained by the front tracking method is unique.

In the following, we first define specifically the Lyapunov functional and then verify (3.2) and (3.3).

3.1. **Definition of the Lyapunov functional and its equivalence to the L^1–distance.** Similar to [4, 13, 14], when ξ is fixed, for each $\eta\ge 0$, we solve a "Riemann problem" with the below-state $U(\xi,\eta)$, the upper-state $V(\xi,\eta)$, by moving along the Hugoniot curves S_1, C_2, and S_3 of system (2.7) in the phase space. We use $h_i(\xi,\eta)$ to denote the strength of the i-Hugoniot wave in this "Riemann solver", which is totally determined by $U(\xi,\eta)$ and $V(\xi,\eta)$. As in the standard Riemann solver, $\sum_{i=1}^3|h_i(\xi,\eta)|$ is equivalent to $|U(\xi,\eta)-V(\xi,\eta)|$, with modulo a constant depending only on $\overline{U}_+$.

Now we define the weighted L^1–strengths:

$$q_i(\xi,\eta)=c_i^a h_i(\xi,\eta), \tag{3.4}$$

where the constants c_i^a are to be determined later, depending on the estimates on the wave interactions and reflections off the characteristic boundary $\{\eta=0\}$ (as done in Section 2).

Then we define the Lyapunov functional:

$$\Phi(U(\xi,\cdot),V(\xi,\cdot))=\sum_{i=1}^3\int_0^\infty |q_i(\xi,\eta)|W_i(\xi,\eta)\,\mathrm{d}\eta, \tag{3.5}$$

with the weight

$$W_i(\xi,\eta) = 1 + \kappa_1 \mathcal{A}_i(\xi,\eta) + \kappa_2\big(\mathcal{Q}(U)(\xi) + \mathcal{Q}(V)(\xi)\big). \tag{3.6}$$

Here κ_1 and κ_2 are two constants to be specified later; $\mathcal{Q}(U)(\xi)$ and $\mathcal{Q}(V)(\xi)$ are the Glimm's total wave interaction potentials in U and V at "time" ξ, respectively; $\mathcal{A}_i(\xi,\eta)$ is the total strength of waves in U and V which approach the i-wave $q_i(\xi,\eta)$ at "time" ξ, defined by

$$\mathcal{A}_i(\xi,\eta) = \mathcal{B}_i(\xi,\eta) + \mathcal{C}_i(\xi,\eta) \tag{3.7}$$

and

$$\begin{aligned}
\mathcal{B}_i &= \Bigg(\sum_{\substack{\alpha\in\mathcal{J}(U)\cup\mathcal{J}(V)\\ \eta_\alpha<\eta,\, i<k_\alpha\le 3}} + \sum_{\substack{\alpha\in\mathcal{J}(U)\cup\mathcal{J}(V)\\ \eta_\alpha>\eta,\, 1\le k_\alpha< i}}\Bigg)|\alpha|,\\
\mathcal{C}_i &= \begin{cases} \big(\sum_{\alpha\in\mathcal{J}(U),\eta_\alpha<\eta,k_\alpha=i} + \sum_{\alpha\in\mathcal{J}(V),\eta_\alpha>\eta,k_\alpha=i}\big)|\alpha| & \text{if } q_i(\eta)<0,\\ \big(\sum_{\alpha\in\mathcal{J}(V),\eta_\alpha<\eta,k_\alpha=i} + \sum_{\alpha\in\mathcal{J}(U),\eta_\alpha>\eta,k_\alpha=i}\big)|\alpha| & \text{if } q_i(\eta)>0,\end{cases}
\end{aligned}$$

where $\mathcal{J}(U)$ and $\mathcal{J}(V)$ are the sets of fronts at "time" ξ in U and V, respectively, η_α is the position of the front α, and k_α is the characteristic family associated to the front α. In the following, we sometimes drop the dependence of ξ and write simply $q_i(\eta), W_i(\eta), \mathcal{A}_i(\eta)$, etc. when no confusion arises.

Since, for any $U(\xi,\cdot)$, $V(\xi,\cdot)\in \mathrm{BV}$ with $TV\big(U(\cdot)\big)+TV\big(V(\cdot)\big)$ sufficiently small (depending only on κ_1,κ_2, and c_i^a), we have

$$\begin{aligned}
&C_0^{-1}\,\|U(\xi,\cdot)-V(\xi,\cdot)\|_{L^1(\mathbb{R}^+)} \le \sum_{i=1}^{3}\int_0^\infty |q_i(\eta)|\,d\eta \le C_0\|U(\xi,\cdot)-V(\xi,\cdot)\|_{L^1(\mathbb{R}^+)},\\
&1\le W_i(\eta)\le 2, \qquad i=1,2,3,
\end{aligned}$$

for some constant C_0 depending essentially only on the system and $\overline{U}_+$. Therefore, for any $\xi\ge 0$,

$$C_1^{-1}\,\|U(\xi,\cdot)-V(\xi,\cdot)\|_{L^1(\mathbb{R}^+)} \le \Phi(U,V) \le C_1\|U(\xi,\cdot)-V(\xi,\cdot)\|_{L^1(\mathbb{R}^+)}, \tag{3.8}$$

where C_1 is independent of ξ, U, and V.

3.2. **Decreasing of the Lyapunov functional.** From now on, we examine how the Lyapunov functional $\Phi(U^{\delta_1}(\xi,\cdot),V^{\delta_2}(\xi,\cdot))$ evolves in the flow direction $\xi>0$, where U^{δ_1} and V^{δ_2} are approximate solutions obtained by the front tracking method from the initial data U_0 and V_0, respectively, and we also set $\delta=\max(\delta_1,\delta_2)$ to control the errors. For simplicity, in the following, we also write U^{δ_1} and V^{δ_2} as U and V, respectively.

Denote $\lambda_i(\eta)$ the speed of the i-wave $q_i(\eta)$ along the Hugoniot curve in the phase space (it is not the characteristic speed except for the characteristic discontinuity). At $\xi>0$ which is not an interaction "time" of the waves either in U or V, for $\mathcal{J}=\mathcal{J}(U)\cup\mathcal{J}(V)$ (it is a finite set), it holds that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\xi}\Phi\left(U(\xi),V(\xi)\right)\\
&=\sum_{\alpha\in\mathcal{J}}\sum_{i=1}^{3}\Big(\left|q_i(\eta_\alpha^-)\right|W_i(\eta_\alpha^-)-\left|q_i(\eta_\alpha^+)\right|W_i(\eta_\alpha^+)\Big)\dot\eta_\alpha+\sum_{i=1}^{3}\left|q_i(b)\right|W_i(b)\dot\eta_b\\
&=\sum_{\alpha\in\mathcal{J}}\sum_{i=1}^{3}\Big(\left|q_i(\eta_\alpha^-)\right|W_i(\eta_\alpha^-)\left(\dot\eta_\alpha-\lambda_i(\eta_\alpha^-)\right)-\left|q_i(\eta_\alpha^+)\right|W_i(\eta_\alpha^+)\left(\dot\eta_\alpha-\lambda_i(\eta_\alpha^+)\right)\Big)\\
&\quad+\sum_{i=1}^{3}\left|q_i(b)\right|W_i(b)\big(\dot\eta_b+\lambda_i(b)\big),
\end{aligned}$$

where $\dot\eta_\alpha$ is the speed of the Hugoniot wave $\alpha\in\mathcal{J}$, $b=0^+$ stands for the points near the boundary $\eta=0$, and $\dot\eta_b$ is the slope of the boundary, which is actually zero. For the second equality, we used the fact that, when $U_0-V_0\in L^1(\mathbb{R}^+)$, if α' is the uppermost front in $\mathcal{J}$, then $\lambda_i(\eta_{\alpha'}^+)=0$ for $i=1,2,3$. Also, suppose that the front $\beta'\in\mathcal{J}$ is the closest to $\eta=0$ so that $\lambda_i(b)=\lambda_i(\eta_{\beta'}^-)$. Define

$$E_{\alpha,i}=\left|q_i^+\right|W_i^+\left(\lambda_i^+-\dot\eta_\alpha\right)-\left|q_i^-\right|W_i^-\left(\lambda_i^--\dot\eta_\alpha\right),\tag{3.9}$$

$$E_{b,i}=\left|q_i(b)\right|W_i(b)\big(\dot\eta_b+\lambda_i(b)\big),\tag{3.10}$$

where $q_i^\pm=q_i(\eta_\alpha^\pm)$, $W_i^\pm=W_i(\eta_\alpha^\pm)$, and $\lambda_i^\pm=\lambda_i(\eta_\alpha^\pm)$. Then

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\Phi\left(U(\xi),V(\xi)\right)=\sum_{\alpha\in\mathcal{J}}\sum_{i=1}^{3}E_{\alpha,i}+\sum_{i=1}^{3}E_{b,i}.\tag{3.11}$$

Our main goal is to establish the following bounds:

$$\sum_{i=1}^{3}E_{b,i}\le 0\qquad\text{(near the boundary)},\tag{3.12}$$

$$\sum_{i=1}^{3}E_{\alpha,i}\le\mathcal{O}(1)\delta\left|\alpha\right|\qquad\text{(when α is a weak wave in $\mathcal{J}$)}.\tag{3.13}$$

If these are established, from (3.12)–(3.13), we conclude

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\Phi\left(U(\xi),V(\xi)\right)\le\mathcal{O}(1)\delta.\tag{3.14}$$

If the constant κ_2 in the Lyapunov functional is chosen large enough, by the Glimm interaction estimates, all the weight functions $W_i(\eta)$ decrease at each "time" where two fronts of U or two fronts of V interact. By the self-similarity of the Riemann solutions, Φ decreases at this "time". Integrating (3.14) over the interval $[0,\xi]$, we obtain

$$\Phi\left(U(\xi),V(\xi)\right)\le\Phi\left(U(0),V(0)\right)+\mathcal{O}(1)\delta\xi\tag{3.15}$$

as desired. Actually, (3.13) has been proved in [4] or [12] via a case-by-case analysis.

We also need to consider what happens to the Lyapunov functional Φ if a weak wave reflected off the boundary. Even in this case, the weights $W_i(\eta)$ can be made to decrease across "time" $\xi=\tau$ when a 1-wave interacts with the boundary. Using the modified Glimm interaction potential $\mathcal{Q}$ and Lemma 2.2 above, it holds that, across "time" $\xi=\tau$ when a weak 3-wave reflected off the characteristic boundary $\{\eta = 0\}$, both $\mathcal{G}$ and $\mathcal{Q}$ decrease, if k_+ is chosen sufficiently large (see Section 2). One sees that the result holds if $\kappa_2 \gg \kappa_1$ in (3.6). What remains to be proved is (3.12) near the boundary.

3.3. **Estimates near the boundary.** In this section, we focus on estimate (3.12) near the boundary. It is different from those for the Cauchy problem. We exploit the exclusive property of the boundary condition in (2.17): The flows U and V have the same pressure near the boundary. Then we construct a piecewise constant weak solution only along the Hugoniot curves determined by the Riemann data $U(b)$ and $V(b)$, which are the states of U and V near the boundary (that is, the point b is some fixed point (ξ_b, η_b) near the positive ξ-axis with $\eta_b > 0$). Let $h_i(b)$ be the strength of the i-th shock in the Riemann problem determined by $U(b)$ and $V(b)$, and let λ_i be the corresponding wave speed. Then, as the second-family is linearly degenerate, we infer that $\lambda_2 \equiv 0$. Recalling that $\dot{\eta}_b = 0$, we conclude

$$E_{b,2} \equiv 0. \tag{3.16}$$

To estimate $E_{b,1} + E_{b,3}$, we consider the following two cases.

Case 1: $h_1(b) = 0$. If $h_1(b) = 0$, then there is no first-family wave in the Riemann solution to the Riemann problem $(U(b), V(b))$. Since the second-family waves are the characteristic discontinuities, the pressure of the middle-state U_m keeps unchanged, *i.e.* $p_m = \overline{p}$. As the third-family is genuinely nonlinear, there must be a jump of pressure across a wave of the third-family. Thus, in this case, there must be no wave of the third-family, that is, $h_3(b) = 0$. Hence, $q_1(b) = 0 = q_3(b)$. We conclude in this case that

$$E_{b,i} = 0, \qquad i = 1, 3.$$

Case 2: $h_1(b) \neq 0$. For this case, as analysed in Case 1, one concludes that $h_3(b) \neq 0$ as well. Starting from $U(b)$, go along the 1-Hugoniot curve to reach U_1, then possibly along the 2-characteristic Hugoniot curve to reach U_2, and the 3-Hugoniot curve to reach $V(b)$. Furthermore, among the first-family and third-family waves, they must be of distinct type, that is, there can be a 1-compressive shock (Lax shock) and a 3-decompressive shock (non Lax shock) or vice versa in the Riemann solution to keep the pressure unchanged in the approximate solutions U and V. We need to show that

$$|h_3(b)| = \mathcal{O}(1)|h_1(b)|. \tag{3.17}$$

If this is true, observing that $\lambda_1 < 0$ and $\lambda_3 > 0$ at $\overline{U}_+$, so that $q_i(b) = c_i^a h_i(b)$ with the constants $c_i^a > 0$ to be chosen, we have

$$\begin{aligned}
E_{b,1} + E_{b,3} &= c_1^a|h_1(b)|W_1(b)\lambda_1(b) + c_3^a|h_3(b)|W_3(b)\lambda_3(b) \\
&= |h_1(b)|\big(- c_1^a W_1(b)|\lambda_1(b)| + c_3^a\mathcal{O}(1)W_3(b)|\lambda_3(b)|\big) \\
&\leq |h_1(b)|\big(- c_1^a|\lambda_1(b)| + 2c_3^a\mathcal{O}(1)|\lambda_3(b)|\big) \\
&\leq 0,
\end{aligned}$$

by using the fact that $\left|\frac{\lambda_3(b)}{\lambda_1(b)}\right|$ is bounded (depending only on $\overline{U}_+$) and then choosing c_1^a quite large. This completes the proof of (3.12).

Then what left is to prove (3.17). We know $U_1 = S_1(h_1)(U_b)$ and $V_b = S_3(h_3)(U_2)$. For U close to $\overline{U}_+$, since $\frac{\mathrm{d}S_1(\alpha)(U)}{\mathrm{d}\alpha}|_{\alpha=0} = r_1(U)$ and especially the third argument of r_1 is $\kappa_1(-\bar{\lambda}_1 u) \neq 0$ (it is nonzero at the reference state $\overline{U}_+$), by the inverse function theorem, we infer that $|h_1(b)| \backsim |p_1 - \overline{p}|$, and similarly for the third-family, $|h_3(b)| \backsim |p_2 - \overline{p}|$. Since $p_2 = p_1$, we conclude that $|h_1(b)| \backsim |h_3(b)|$. Here, for positive quantities a and b, we use $a \backsim b$ to mean that there is a positive constant C_1 depending only on $\overline{U}_+$ so that $C_1^{-1} b \leq a < C_1 b$.

4. Existence of a Semigroup

Using the existence and uniqueness results established in the earlier sections, we can now establish the existence of the semigroup $\mathcal{S}$ of solutions generated by the wave-front tracking algorithm.

Proposition 4.1. *Assume that* $\mathrm{TV}(\overline{U}(\cdot))$ *is sufficiently small. Then, for* $\delta > 0$*, the map* $(\overline{U}(\cdot), \xi) \mapsto U^\delta(\xi, \cdot) := \mathcal{S}^\delta_\xi(\overline{U}(\cdot))$ *produced by the wave-front tracking algorithm is a uniformly Lipschitz continuous semigroup such that*

(i) $\mathcal{S}^\delta_0 \overline{U} = \overline{U}, \quad \mathcal{S}^\delta_{\xi_1} \mathcal{S}^\delta_{\xi_2} \overline{U} = \mathcal{S}^\delta_{\xi_1+\xi_2} \overline{U}$;

(ii) $\left\| \mathcal{S}^\delta_\xi \overline{U} - \mathcal{S}^\delta_\xi \overline{V} \right\|_{L^1(\mathbb{R}^+)} \leq C \left\| \overline{U} - \overline{V} \right\|_{L^1(\mathbb{R}^+)} + C\delta\xi.$

Proof. Property (i) follows immediately because $\mathcal{S}^\delta$ is generated by the wave-front tracking method. We now prove property (ii).

Let U^δ and V^δ be the front tracking δ-approximate solutions of (2.17) with the initial data $\overline{U} = U_0(\cdot)$ and $\overline{V} = V_0(\cdot)$, respectively. By (3.8) and (3.15), we obtain that, for any $\xi \geq 0$,

$$\begin{aligned} \left\| U^\delta(\xi) - V^\delta(\xi) \right\|_{L^1(\mathbb{R}^+)} &\leq C\Phi(U^\delta(\xi), V^\delta(\xi)) \\ &\leq C\Phi(U^\delta(0), V^\delta(0)) + C_1 \mathcal{O}(1)\delta\xi \\ &\leq C_1 C_2 \left\| \overline{U} - \overline{V} \right\|_{L^1(\mathbb{R}^+)} + C_1 \mathcal{O}(1)\delta\xi. \end{aligned}$$

This establishes the Lipschitz continuity of the δ-semigroup with respect to the initial data and time. □

Definition 4.1. Given $\delta_0 > 0$, define a domain $\mathscr{D}$ as the closure of the set consisting of the functions $U : \mathbb{R}_+ \mapsto \mathbb{R}^3$ such that U belongs to $L^1(\mathbb{R}_+; \mathbb{R}^3)$ by modulo a constant and $\mathrm{BV}(U - \overline{U}_+)(\mathbb{R}^+) \leq \varepsilon_0$.

The semigroup $\mathcal{S}$ generated by the wave-front tracking method is given by the subsequent theorem.

Theorem 4.1. *Let* $\mathrm{TV}(\overline{U}(\cdot))$ *be sufficiently small. Then the sequence* $\mathcal{S}^\delta$ *generated from the front tracking algorithm is a Cauchy sequence in the* L^1*–norm, and the sequence* $\mathcal{S}^\delta_\xi(\overline{U})$ *converges to a unique limit* $\mathcal{S}_\xi(\overline{U})$ *as* $\delta \to 0$*. The map* $\mathcal{S} : [0, \infty) \times \mathscr{D} \mapsto \mathscr{D}$ *is a uniformly continuous semigroup. In particular, there exists a constant L such that, for all* $\xi_1, \xi_2 > 0$ *and* $\overline{U}, \overline{V} \in \mathscr{D}$,

(i) ***Semigroup Property:*** $\mathcal{S}_0 \overline{U} = \overline{U}, \quad \mathcal{S}_{\xi_1} \mathcal{S}_{\xi_2} \overline{U} = \mathcal{S}_{\xi_1+\xi_2} \overline{U}$;

(ii) ***Lipschitz Continuity:*** $\left\| \mathcal{S}_\xi \overline{U} - \mathcal{S}_\xi \overline{V} \right\|_{L^1(\mathbb{R}^+)} \leq L \left\| \overline{U} - \overline{V} \right\|_{L^1(\mathbb{R}^+)}$;

(iii) *Each trajectory* $\xi \mapsto \mathcal{S}_\xi \overline{U}$ *yields a weak solution to the initial-boundary value problem* (2.17);

(iv) ***Consistency with Riemann Solver:*** *For any piecewise constant initial data* $\overline{U} \in \mathscr{D}$*, there exists a small* $\theta > 0$ *such that, for all* $\xi \in [0, \theta]$*, the trajectory* $U(\xi, \cdot) = \mathcal{S}_\xi \overline{U}$ *agrees* *with the solution of* (2.17) *obtained by piecing together the standard entropy solutions of the Riemann problems.*

Theorem 4.1 implies that a sequence of δ-approximate front tracking solutions to (2.17) converges to a unique entropy solution whose value is in $\mathscr{D}$, and this unique limit is L^1–stable. More precisely, we have the following immediate consequence from Theorem 4.1:

Corollary 4.1. *Let* $\mathrm{TV}(\overline{U}(\cdot))$ *be sufficiently small. Then the entropy solution to the initial-boundary value problem* (2.17) *generated by the wave-front tracking algorithm is* L^1*-stable and unique.*

The proof of Theorem 4.1 relies on the subsequent crucial estimate (also used in [3]) about the approximation of Lipschitz flows:

Suppose that $\mathcal{S} : [0,\infty) \times \mathscr{D} \to \mathscr{D}$ is a global semigroup with a Lipschitz constant L. Let $T > 0$, $\overline{Z} \in \mathscr{D}$, and $Z : [0,T] \mapsto \mathscr{D}$ be a continuous mapping taking values in piecewise constant functions, with jumps along finitely many polygonal lines in the (ξ,η)–plane. Then

$$\left\|Z(T) - \mathcal{S}_T\overline{Z}\right\|_{L^1} \leq L\left\{\left\|Z(0) - \overline{Z}\right\|_{L^1} + \int_0^T \varliminf_{\mu\to 0^+} \frac{\|Z(\xi+\mu) - \mathcal{S}_\mu Z(\xi)\|_{L^1}}{\mu}\,\mathrm{d}\xi\right\}. \tag{4.1}$$

Using the essential estimates shown in the earlier sections, Theorem 4.1 is shown along a similar line of arguments to the one followed by Bressan-Colombo in [3]. The only significant difference here between the wave-front tracking by Bressan (*cf.* [2]) and the version of the method (*cf.* Holden and Risebro [12]) we employ is on how to control the number of fronts from growing to infinity within finite time. In [3], the accurate Riemann solver (**ARS**) and simplified Riemann solver (**SRS**) are used to construct the approximate solutions. With (**ARS**), waves of each family can be possibly introduced in the Riemann solution where every rarefaction wave is divided into equal parts to obtain a rarefaction fan of wave-fronts, while with (**SRS**), all new waves are lumped together as a single non-physical front, traveling faster than all wave speeds. Furthermore, in [3], the simplified Riemann solver, rather than the accurate Riemann solver, is employed when the interaction term is less than a cut-off function in the order of $\sqrt{\delta}$. In the front tracking method we use, the approximate Riemann solution is changed by removing weak fronts. The point of this procedure is that the fronts of high generation are quite weak. Particularly, in our arguments, when the interaction term is less than δ, this corresponds to new and removed fronts with generation higher than N (here N is a suitably large positive integer computed using the initial approximation parameter δ, see [5] and [12] for further details). We also refer to Chen-Li [6] for a related analysis with the full Euler system where (**SRS**) is used in place of (**ARS**) when the interaction term is less than δ.

Thus, using an argument similar to [3], we conclude that $\mathcal{S}_\xi^{\delta_n}\overline{Z}_n$ is a Cauchy sequence, converging in the L^1–sense as long as the error from removing weak fronts tends to zero as the initial error parameter $\delta \to 0$ (this has been proved in [5]). Consequently, the map $\mathcal{S} : [0,\infty) \times \mathscr{D} \mapsto \mathscr{D}$ defined as the limit of the approximate solutions generated by the front tracking algorithm is well-defined.

Now, the proofs of statements (i) to (iv) in Theorem 4.1 are as follows. Statements (i), (ii), and (iv) are immediate since $\mathcal{S}$ is the limit of front tracking approximate solutions $\mathcal{S}^\delta$. It is similar to prove (iii) as [3], but, as discussed above, the wave-front tracking method we employ here is slightly different. Lastly, we observe that the entropy solution fulfills the boundary condition of the initial-boundary value problem (2.17) due to the construction of our approximate solutions. This completes the proof of Theorem 4.1.

5. Uniqueness in the class of viscosity solutions

In this section, we analyse an arbitrary Lipschitz semigroup defined on the domain $\mathscr{D}$ of BV functions. In particular, we prove that the semigroup $\mathcal{S}$ is uniquely determined when the local flow is assigned in connection with the initial data, a piecewise constant function. Based on the results in Section 4, we first show that the semigroup $\mathcal{S}$ generated by the wave-front tracking method is the canonical trajectory of the standard Riemann semigroup (SRS). Then the uniqueness of entropy solutions is shown to extend to a broader class, namely the class of viscosity solutions as introduced by Bressan in [1]. The essential part here is to show that, in the viscosity class, the entropy solution matches the semigroup trajectory generated by the front tracking method.

Definition 5.1. Problem (2.17) is said to admit a *standard Riemann semigroup* if, for some small $\varepsilon_0 > 0$, there is a continuous map $\mathcal{R} : [0,\infty) \times \mathscr{D} \mapsto \mathscr{D}$ and a constant L such that, for every $\overline{U}, \overline{V} \in \mathscr{D}$ and $\xi_1, \xi_2 \geq 0$, we have

(i) $\mathcal{R}_0\overline{U} = \overline{U}, \quad \mathcal{R}_{\xi_1}\mathcal{R}_{\xi_2}\overline{U} = \mathcal{R}_{\xi_1+\xi_2}\overline{U}$;

(ii) $\left\|\mathcal{R}_\xi\overline{U} - \mathcal{R}_\xi\overline{V}\right\|_{L^1} \leq L\left\|\overline{U} - \overline{V}\right\|_{L^1}$;

(iii) If $\overline{U} \in \mathscr{D}$ is piecewise constant, then, for all $\xi \in [0, \varepsilon_0]$ sufficiently small, the trajectory $U(\xi,\cdot) = \mathcal{S}_\xi\overline{U}$ coincides with the solution of (2.17) obtained by patching together the standard entropy solutions of the Riemann problems produced by the discontinuities of $\overline{U}$.

Theorem 5.1. *Suppose that the initial-boundary value problem* (2.17) *admits a standard Riemann semigroup* $\mathcal{R} : [0,\infty)\times\mathscr{D} \mapsto \mathscr{D}$. *Consider* $\mathcal{S}$ *the semigroup generated by the front tracking algorithm, that is,* $\mathcal{S}_\xi(\overline{U}) = \lim_{\delta\to 0} S^\delta_\xi(\overline{U})$ *with* $\overline{U} \in \mathscr{D}$. *Then, for all* $\xi \geq 0$ *and* $\overline{U} \in \mathscr{D}$, $\mathcal{R}_\xi\overline{U} = \mathcal{S}_\xi\overline{U}$.

Theorem 5.1 is proved by using similar arguments as in [1] based on the fundamental estimate (4.1) and the essential feature of the local flow in the ξ-direction, that is, the wave-front tracking method and the standard Riemann semigroup both have the structure of the Riemann solutions.

In what follows, we discuss some necessary and sufficient conditions for a function $\xi \mapsto U(\xi) \in \mathscr{D}$ to coincide with a semigroup trajectory. Following Bressan [1], there are two types of local approximate parametrices for system (2.7).

The first approximate parametrice comes from the self-similar solution of the Riemann problem. To that end, consider a function $U : [0,\infty) \times \mathbb{R}_+ \mapsto \mathbb{R}^3$ and a fixed point (τ,ζ) in the domain of U. Suppose that $U(\tau,\cdot) \in \mathscr{D}$. The boundedness of the total variation implies that the following limits exist:

$$U^- = \lim_{\eta\to\zeta^-} U(\tau,\eta), \qquad U^+ = \lim_{\eta\to\zeta^+} U(\tau,\eta).$$

Let $\vartheta = \vartheta(\xi,\eta)$ be the solution of the Riemann problem with piecewise constant data U^- and U^+ and, for $\xi > \tau$, define the following function:

$$H^\sharp_{(U,\tau,\zeta)}(\xi,\eta) = \begin{cases} \vartheta(\xi-\tau, \eta-\zeta) & \text{if } |\eta-\zeta| \leq \hat{\lambda}(\xi-\tau), \\ U(\tau,\eta) & \text{if } |\eta-\zeta| > \hat{\lambda}(\xi-\tau). \end{cases}$$

Here $\hat{\lambda}$ is an upper bound for all wave speeds,

$$\sup_U |\lambda_k(U)| < \hat{\lambda}, \qquad k = 1, 2, 3. \tag{5.1}$$

The other required parametrice is obtained from the corresponding quasilinear hyperbolic system (2.8)

$$A(U)\partial_\xi U + B(U)\partial_\eta U = 0$$

by "freezing" the coefficients of the matrices $A(U)$ and $B(U)$ in a neighbourhood of the state $U(\tau, \zeta)$.

For $\xi > \tau$, define $H^\flat_{(U,\tau,\zeta)}$ as the solution of the linear Cauchy problem with constant coefficients:

$$\tilde{A}\partial_\xi M + \tilde{B}\partial_\eta M = 0, \qquad M(\tau, \eta) = U(\tau, \eta). \tag{5.2}$$

where $\tilde{A} := A(U(\tau, \zeta))$ and $\tilde{B} := B(U(\tau, \zeta))$ the matrices evaluated at the fixed state $U(\tau, \zeta)$. Note that the functions $H^\#$ and $H^\flat$ depend on the values $U(\tau, \zeta)$ and $U(\tau, \zeta\pm)$. In the remaining, we introduce the class of viscosity solutions and discuss how these solutions share the same local characterization as $H^\#$ and $H^\flat$.

Definition 5.2. Assume that $U : [0, T] \mapsto \mathscr{D}$ is a continuous map with respect to the L^1–norm. Then the function U is called a viscosity solution of problem (2.17) if there are constants C and $\hat{\lambda}$ providing bound (5.1) such that, with β and δ small enough, we have, for every $(\tau, \zeta) \in [0, T) \times \mathbb{R}_+$,

$$\begin{aligned}
\frac{1}{\delta}\int_{\zeta-\beta+\delta\hat{\lambda}}^{\zeta+\beta-\delta\hat{\lambda}} |U(\tau+\delta, \eta) - H^\#_{(U,\tau,\zeta)}(\xi, \eta)|\, d\xi &\leq C\,\mathrm{TV}\{U(\tau) : (\zeta-\beta, \zeta) \cup (\zeta, \zeta+\beta)\},\\
\frac{1}{\delta}\int_{\zeta-\beta+\delta\hat{\lambda}}^{\zeta+\beta-\delta\hat{\lambda}} |U(\tau+\delta, \eta) - H^\flat_{(U,\tau,\zeta)}(\xi, \eta)|\, d\xi &\leq C\big(\mathrm{TV}\{U(\tau) : (\zeta-\beta, \zeta+\beta)\}\big)^2.
\end{aligned}$$

Theorem 5.2. *Suppose that problem* (2.17) *admits a standard Riemann semigroup $\mathcal{R}$. Then a continuous mapping $U : [0, T] \mapsto \mathscr{D}$ is said to be a viscosity solution of* (2.17) *if and only if*

$$U(\xi, \cdot) = \mathcal{R}_\xi \overline{U} \quad \textit{at every } \xi \in [0, T]. \tag{5.3}$$

Using Theorem 5.2, one concludes that, for problem (2.17), the entropy solution is unique in the viscosity class of solutions, which coincides with the semigroup trajectory $\mathcal{S}_\xi \overline{U}$ generated by the front tracking method. More precisely, we have

Corollary 5.1. *A continuous mapping $U : [0, T] \mapsto \mathscr{D}$ is a viscosity solution if and only if*

$$U(\xi, \cdot) = \mathcal{S}_\xi \overline{U} \quad \textit{for any } \xi \in [0, T]. \tag{5.4}$$

With the estimates in Sections 2–3, the proof here follows along a similar line of reasoning to the one presented in [1]. The only difference is a straight boundary and physical domain restricted to the positive η-axis; nonetheless, we can still proceed with the proof provided that the convergence of the wave-front tracking method is obtained which has been outlined in Section 2 and carried out in Eulerian coordinates in [5].

Remark 5.1. Note that, in simpler cases such as the potential flow, isentropic or isothermal Euler flow, as far as the L^1–stability problem is concerned, we achieve the same results as the full Euler equations.

6. Uniqueness of Solutions to the Free Boundary Problem in Eulerian Coordinates

In this section we apply the following Wagner's theorem [15, Theorem 2] to show the uniqueness of entropy solution to the free boundary problem (1.7).

Theorem 6.1 (Wagner [15]). *Consider*

$$\partial_t U + \partial_x F(U) = 0, \tag{6.1}$$

where $(t,x) \in \mathbb{R}^2$, $U(t,x) = (u_1, \cdots, u_n)^\top$, *and* $F(U) = (f_1(U), \cdots, f_n(U))^\top \in \mathbb{R}^n$ *is a smooth vector-function. For any bounded measurable solutions of* (6.1), *with* $u_1(t,x) \geq 0$, *let* (t,y) *satisfy*

$$\frac{\partial y}{\partial x} = u_1(t,x), \qquad \frac{\partial y}{\partial t} = -f_1(U(t,x))$$

in the sense of distributions. Then $T : (t,x) \mapsto (t, y(t,x))$ *is a Lipschitz-continuous transformation, which induces a one-to-one correspondence between* L^∞ *weak solutions of* (6.1) *on* $\mathbb{R}^+ \times \mathbb{R}$ *satisfying* $0 < \varepsilon \leq u_1(t,x) \leq M < \infty$ *for some* ε *and* M, *and* L^∞ *weak solutions of*

$$\begin{aligned} &\partial_t\big(\frac{1}{u_1}\big) - \partial_y\big(\frac{f_1(U)}{u_1}\big) = 0, \\ &\partial_t\big(\frac{1}{u_1}(u_2, \cdots, u_n)^\top\big) + \partial_y\big((f_2(U), \cdots, f_n(U))^\top - \frac{1}{u_1} f_1(U)(u_2, \cdots, u_n)^\top\big) = 0, \end{aligned} \tag{6.2}$$

on $\mathbb{R}^+ \times \mathbb{R}$ *satisfying* $\varepsilon \leq u_1(t,x) \leq M$.

If $\eta(U)$ *is any convex extension of* (6.1), *i.e. there is a flux* $q(U)$ *such that* $\nabla\eta\nabla F = \nabla q$, *so that* $\partial_t \eta + \partial_x q = 0$ *for classical solutions, then any solution of* (6.1) *satisfying*

$$\partial_t \eta(U) + \partial_x q(U) \leq 0 \tag{6.3}$$

in the sense of distributions in (t,x) *corresponds to a solution of* (6.2), *satisfying*

$$\partial_t \tilde{\eta}(V) + \partial_y \tilde{q}(V) \leq 0, \tag{6.4}$$

in the sense of distributions in (t,y), *where* $V = (v_1, \cdots, v_n)^\top$, $\tilde{\eta}(V) = \frac{\eta(U)}{u_1}$, *and* $\tilde{q}(V) = q(U) - f_1(U)\tilde{\eta}(V)$. *Furthermore,* η *is convex if and only if* $\tilde{\eta}$ *is convex as a function of* V. *Thus, the Lax entropy inequality holds for a solution of* (6.1) *if and only if it also holds for the corresponding solution of* (6.2).

As indicated by Wagner [15, p. 123], this theorem may also be applied to initial-boundary value problems by slight modification. Thus, we can also use it in our case. Suppose that $U_0^1(y)$ and $U_0^2(y)$ are two initial data functions of problem (1.7) in Eulerian coordinates, and (U^1, g^1) and (U^2, g^2) are the corresponding weak entropy solutions obtained by the front tracking method. We have known that U^1 and U^2 must be bounded. We now show that $U^1 = U^2$ and $g^1 = g^2$, provided $U_0^1 = U_0^2$.

Suppose that V^1 and V^2 are the solutions of (2.17) in Lagrangian coordinates that correspond to U^1 and U^2, respectively. By Theorem 6.1, it suffices to show that $V^1 = V^2$. Then, from (2.18), we may find $g^1 = g^2$. Therefore, by the uniqueness results of problem (2.17) (Corollary 4.1), it suffices to prove the following lemma.

Lemma 6.1. *If $U_0^1(y) = U_0^2(y)$, then $V^1(0,\eta) = V^2(0,\eta)$.*

Proof. For $j = 1, 2$, let $T^j : (x, y) \mapsto (\xi, \eta)$ be the Lagrangian transform (2.4) associated with the solution U^j. They are Lipschitz-continuous and one-to-one. Furthermore, we know that $U^j(x,\cdot)$ is of bounded variation and $\lim_{x\to x_0} \left\|U^j(x,\cdot) - U^j(x_0,\cdot)\right\|_{L^1} = 0$. Thus, not only we can solve η from (2.2), but also those equations make sense on each line $x =$ constant.

We first show that, if $U_0^1(y) = U_0^2(y)$, then $T^1|_{x=0} = T^2|_{x=0}$.

Suppose $(\xi_j, \eta_j) = T^j(x, y)$. Recall by definition that $\xi = x$. Then, at $x = 0$, we have both $\xi_1 = \xi_2 = 0$. Note that η_j satisfies $\eta_j(0, 0) = 0$ and solves

$$\frac{\partial \eta_j(x,y)}{\partial y}\Big|_{x=0} = (\rho u)^j(0,y).$$

Since $(\rho u)^1(0,y) = (\rho u)^2(0,y)$ from $U_0^1(y) = U_0^2(y)$, we conclude that $\eta_1(0,y) = \eta_2(0,y)$ as well. This shows that $T^1(0,y) = T^2(0,y)$, which implies that $(T^1)^{-1}(0,\eta) = (T^2)^{-1}(0,\eta)$ and lies on $\{x = 0\}$. Since

$$V^j(\xi,\eta) = U^j((T^j)^{-1}(\xi,\eta)),$$

we conclude

$$V^1(0,\eta) = U^1((T^1)^{-1}(0,\eta)) = U^1((T^2)^{-1}(0,\eta)) = U^2((T^2)^{-1}(0,\eta)) = V^2(0,\eta).$$

□

Finally, summarising all the analysis above, we state the main theorem of this paper.

Theorem 6.2. *For the initial data U_0 close to the reference state $\overline{U}_+$ in the sense that $\left\|U_0 - \overline{U}_+\right\|_{\mathrm{BV}}$ is sufficiently small, there is one and only one weak entropy solution (U, g) to problem (1.7) constructed by the front tracking method. Furthermore, reformulating this problem in Lagrangian coordinates as problem (2.17), then the L^1–stability holds in the sense that*

$$\left\|V^1(\xi) - V^2(\xi)\right\|_{L^1(\mathbb{R}^+)} \le C \left\|V^1(0) - V^2(0)\right\|_{L^1(\mathbb{R}^+)}$$

for any two solutions V^1 and V^2 of (2.17). The solution is also unique in the class of viscosity solutions in Lagrangian coordinates.

Acknowledgments. The research of Gui-Qiang Chen was supported in part by the National Science Foundation under Grant DMS-0807551, the UK EPSRC Science and Innovation Award to the Oxford Centre for Nonlinear PDE (EP/E035027/1), the NSFC under a joint project Grant 10728101, and the Royal Society-Wolfson Research Merit Award (UK). Vaibhav Kukreja was supported in part by the National Science Foundation under Grant DMS-0807551 and the UK EPSRC Science and Innovation Award to the Oxford Centre for Nonlinear PDE (EP/E035027/1). Hairong Yuan is supported in part by National Natural Science Foundation of China under Grant No. 10901052, China Scholarship Council (No. 2010831365), Chenguang Program (No. 09CG20) sponsored by Shanghai Municipal Education Commission and Shanghai Educational Development Foundation, and a Fundamental Research Funds for the Central Universities.

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