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ON ONE DIMENSIONAL INVERSE PROBLEMS ARISING FROM POLARIMETRIC MEASUREMENTS OF NEMATIC LIQUID CRYSTALS

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ABSTRACT. We revisit the problem of determining dielectric parameters in layered nematic liquid crystals from polarimetric measurements originally introduced by Lionheart & Newton. After a detailed analysis of the model, of the scales involved, and of natural obstacles to the reconstruction of more than one dielectric parameters, we produce two simple one-dimensional inverse problems which can be studied without any expertise in liquid crystals. We then confirm that very little can be recovered about the internal configuration of smooth dielectric parameters from these measurements, and give a uniqueness result for one of the two problem, when the unknown parameter satisfies a monotonicity property. In that case, the available data can be expressed in terms of Laplace and Hankel transforms.

1. INTRODUCTION

Consider the experiment sketched in Figure 1. A polarized focused laser beam is split in two beams. One beam propagates in an homogeneous medium (in light blue), and travels through a slab containing the liquid crystal (LC) cell. The transmitted beam exits on the opposite side of the slab, and enters an analyser,
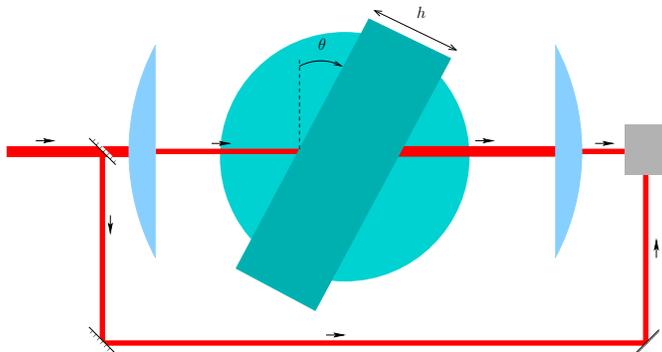

FIGURE 1. Sketch of the polarimetric measurement experiment. A polarized laser beam is split in two, the main part is focused, sent through a thin slab containing the liquid crystal. The slab is placed in an isotropic, homogeneous medium whose index is close to the average index of the slab. The transmitted beam is refocused, and compared to the initial beam by an analyser. The planar cross-section sketched above is referred to as the $(x_1$–$x_3)$ plane in the text.

where it is compared to the other part of the split beam, which did not pass through the slab. The analyser delivers four real numbers, called the Stokes parameters. The LC cell is placed on a cylindrical mount, to allow variations of the incident angle of the laser in the slab. The Stokes parameters, appropriately normalized, are collected for a large range of incident angles. The purpose of this article is to study the dependence of the data on the dielectric tensor of the liquid crystal, and discuss its possible reconstruction.

The liquid crystal contained in the slab has a layered structure: the dielectric permittivity is constant and anisotropic in each layer. The magnetic permeability is a constant number. The liquid crystal is coated with thin films (polyimide, indium tin oxide...) but these thin films will not be taken into account in our study. This experiment was performed at Hewlett-Packard Laboratories, Bristol and was analysed by Lionheart & Newton [1] using a singular value decomposition approach.

To model this problem we adopt the approximation introduced in Berreman [2], following Lionheart & Newton [1]. The incident polarized laser beam in the isotropic medium in front of the liquid crystal is modelled by monochromatic plane waves incident obliquely in the $(x_1 - x_3)$ plane (the plane in which the experiment is sketched), with direction $\overrightarrow{s} = (\sin(\theta), 0, \cos(\theta))$. The corresponding electric field has two components,

$$\mathcal{E}_1^i = E_1 \exp\left(\mathbf{i}\omega\left(\frac{n_0}{c}\overrightarrow{s} \cdot \overrightarrow{x}\right)\right)$$

which is parallel to $\overrightarrow{s}_\perp^1 = (\cos(\theta), 0, -\sin(\theta))$, and

$$\mathcal{E}_2^i = E_2 \exp\left(\mathbf{i}\omega\left(\frac{n_0}{c}\overrightarrow{s} \cdot \overrightarrow{x}\right)\right)$$

along $\overrightarrow{s}_\perp^2 = (0, 1, 0)$. To take into account possible phase differences, the amplitudes of E_1 and E_2 are complex numbers. When the incident field is polarized linearly, E_1 and E_2 are of the form $A\cos(\phi)$ and $A\sin(\phi)$, with $A \in \mathbb{C}$ and $\phi \in [0, 2\pi)$.

When the incident field has a circular polarization, $E_2 = \pm iE_1$. The central frequency of the laser is denoted by ω, c is the speed of light, and n_0 is the index of the isotropic medium around the slab on the wave path. We assume that within the slab, the propagating electric and magnetic fields $\mathcal{E}$ and $\mathcal{H}$ stay alike plane waves along the x_1 direction, and do not depend on x_2. Thus $\mathcal{E}$ and $\mathcal{H}$ are modelled by the following ansatz

$$(1) \qquad \mathcal{E} = E(x_3)\exp\{\mathbf{i}\omega(\frac{n_0}{c}s_1 x_1)\}, \quad \mathcal{H} = H(x_3)\exp\{\mathbf{i}\omega(\frac{n_0}{c}s_1 x_1)\},$$

and satisfy the time-harmonic Maxwell's equations

$$(2) \qquad \nabla \wedge \mathcal{E} = \mathbf{i}\omega\mu_0\mathcal{H}, \quad \nabla \wedge \mathcal{H} = -\mathbf{i}\omega\varepsilon_0\varepsilon(x_3)\mathcal{E}.$$

Where ε is the relative dielectric tensor, with constant entries outside the slab and varying as a function of x_3 within the LC cell, and μ_0 and ε_0 are universal constants. See Lionheart & Newton [1], Tsering-Xiao [3] and references therein for more details and discussions on this model, and Lavrentovich [4] for further insight on optical measurements of liquid crystals. Substituting (1) into (2), Maxwell's system becomes a 4×4 system of ordinary differential equations in x_3 where the unknown is the vector $X = (\mathcal{E}_1, \mathcal{H}_2, \mathcal{E}_2, -\mathcal{H}_1)^T$ which satisfies

$$(3) \qquad \frac{dX}{dx_3} = \mathbf{i}\frac{\omega}{c}M(x_3)X(x_3).$$

The Berreman vector X is the tangential part of the electromagnetic fields and represents the optical field. The Berreman matrix M depends on the dielectric tensor ε. In the outer isotropic medium, the Berreman matrix has a block diagonal structure, corresponding to the two possible polarizations of the electric field, along $\overrightarrow{s}^1_\perp$ and $\overrightarrow{s}^2_\perp$. The Berreman matrix has four eigenvectors $(V_i)_{i=1,..,4}$. The vectors V_1 and V_3 correspond to the incoming electric and magnetic fields, whereas the vectors V_2 and V_4 correspond to the outgoing electric and magnetic fields, see Section 2.1. The direct problem is formulated in Lionheart & Newton [1] as a transmission problem. The incident field and the transmitted field are represented by the input vector $X^i = \mathcal{E}_1^i V_1 + \mathcal{E}_2^i V_3$ and the output vector $X^t = \mathcal{E}_1^t V_1 + \mathcal{E}_2^t V_3$. The reflected field X^r is a combination of the outgoing eigenvectors V_2 and V_4. If h is the thickness of the slab, and $P(h)$ is the transmission matrix coming from (3), the transmission problem takes the form

(4) $$P(h)(X_i + X_r) = X_t.$$

The Stokes parameters (the measured data) are

(5) $$S(E_1^i, E_2^i, \theta, \varepsilon) = \left(\left|\mathcal{E}_1^t\right|^2 + \left|\mathcal{E}_2^t\right|^2, \left|\mathcal{E}_1^t\right|^2 - \left|\mathcal{E}_2^t\right|^2, 2\mathcal{E}_1^t \overline{\mathcal{E}_2^t} \right).$$

The questions investigated numerically in Lionheart & Newton [1] are

(1) Can the dielectric tensor through the LC cell be deduced from the data?
(2) How is the solution affected by the range of incident angles and input polarisations used?
(3) Given the limited accuracy of the polarimeter how much information can be deduced from the data?

The goal of this paper is to address these questions analytically. In section 2 we reformulate the problem in an equivalent, non-dimensional form, taking into account the scale of the various parameters of the problem. In section 2.1 we detail the derivation of the Berreman model (3), given in a non-dimensional form by (7).

We consider two types of LC cells, orthorhombic and nematic uniaxial. In section 2.2 we investigate the case of an orthorhombic medium with principle axes aligned with the coordinate axes, that is, we assume that ε is a diagonal matrix-valued function. This first model was also discussed in Lionheart & Newton [1], with a different approach. Orthorhombic crystals are one of the typical anisotropic materials. Within the class of liquid crystals, orthorhombic symmetry was considered as a convenient theoretical possibility in early developments, see e.g. Freiser [5]. Since then, it has been observed for specific bi-axial liquid crystals by Hegmann et al. [6, 7], but it is deemed to be very rare see e.g. Karahaliou et al. [8]. In section 2.3, we turn to the case of a nematic uniaxial LC cell. This is a very common model for LC cells. In that case, ε can be expressed in terms of a director profile, that is

(6) $$\varepsilon = \varepsilon_\perp I_d + (\varepsilon_\parallel - \varepsilon_\perp)\mathbf{n} \otimes \mathbf{n},$$

where, for each x_3, $\mathbf{n}(x_3) \in \mathbb{S}^2$ is a unit vector, the so-called director vector of the LC cell. The two constant eigenvalues of ε are usually written $\varepsilon_\perp$ and $\varepsilon_\parallel$. The numbers $\sqrt{\varepsilon_\perp}$ and $\sqrt{\varepsilon_\parallel}$ are known as the refractive indices of ordinary and extraordinary waves. The subscripts $\parallel$ and $\perp$ refer to field directions respectively parallel and perpendicular to the director vector $\mathbf{n}$. Formula (6) implies that the electric field energy propagates separately along the ordinary and extraordinary

direction. The derivation of this model and further background can be found in Virga [9], de Gennes & Prost [10] and Chandrasekhar [11].

For both models, we highlight intrinsic obstacles to the determination of the dielectric parameters: non-uniqueness is endemic, regardless of practical limitations such as the range and precision of the measurements involved. In section 2.4, we express the scales of the various coefficients involved in the experiment described in Lionheart & Newton [1] in terms of a non-dimensional parameter.

We conclude this section by producing three one dimensional inverse problems coming from both the orthorhombic LC cell and nematic uniaxial LC cell model, Problem A given by (29), Problem B given by (30) and Problem C given by (31). Problem A and Problem B concern the reconstruction of one coefficient from the amplitude of the transmission data. This coefficient would be ε_{22} for the orthorhombic LC cell example, and one component of director profile **n** *a priori* assumed to lie in the $(x_1$–$x_3)$ incident plane. Problem C presents the independent phase data available from the nematic uniaxial LC cell model, which is just one numerical value.

Section 3 is devoted to Problem A. Under a smoothness assumption of the coefficient to be reconstructed, we describe what can be extracted from the data corresponding to moderate angles of incidence. We then explain how Problem B connects to Problem A, as a first order approximation, when the full range of angles of incidence are considered.

Section 4 is devoted to Problem B. For this model problem, we can describe the class of equivalent parameters, and give a uniqueness result with a monotonicity assumption. We show that this problem can be reformulated in terms of the classical Laplace and Hankel transforms.

Section 5 contains the proofs of various technical intermediate results given in the previous sections. We summarize our findings and discuss possible extensions in section 6. We use some facts concerning systems of ordinary differential equations in this paper, they are given in appendix for the reader's convenience.

2. MODELLING AND SCALING ASSUMPTIONS

2.1. **An alternative equivalent formulation.** In this section, we prove the following proposition.

Proposition 1. *The transmission problem* (3)–(4) *can be written in a non-dimensional form as follows. Given I_1 and I_2 in $\mathbb{C}$, find T_1, T_2, R_1 and R_2 in $\mathbb{C}$ such that there exists a continuous solution of*

$$
\begin{aligned}
\frac{dZ}{dt} &= \mathbf{i}\frac{1}{\eta}B(t)Z(t), \quad t \in [0,1], \\
Z(0) &= [I_1 - R_1, I_1 + R_1, I_2 - R_2, I_2 + R_2]^T, \\
Z(1) &= [T_1, T_1, T_2, T_2]^T,
\end{aligned}
\tag{7}
$$

where B is a matrix-valued piecewise continuous function of $t \in [0,1]$ given by
(8)
$$
B = \frac{1}{\tilde{\varepsilon}_{33}} \times
\begin{bmatrix}
-\sin(\theta)\tilde{\varepsilon}_{13} & \frac{\lambda^2 + \tilde{\varepsilon}_{33} - 1}{\lambda} & 0 & -\sin(\theta)\lambda^{-1}\tilde{\varepsilon}_{23} \\
\lambda\left(\tilde{\varepsilon}_{11}\tilde{\varepsilon}_{33} - \tilde{\varepsilon}_{13}^2\right) & -\sin(\theta)\tilde{\varepsilon}_{13} & 0 & \tilde{\varepsilon}_{12}\tilde{\varepsilon}_{33} - \tilde{\varepsilon}_{13}\tilde{\varepsilon}_{23} \\
\tilde{\varepsilon}_{12}\tilde{\varepsilon}_{33} - \tilde{\varepsilon}_{13}\tilde{\varepsilon}_{23} & -\sin(\theta)\lambda^{-1}\tilde{\varepsilon}_{23} & 0 & \frac{(\lambda^2 + \tilde{\varepsilon}_{22} - 1)\tilde{\varepsilon}_{33} - \tilde{\varepsilon}_{23}^2}{\lambda} \\
0 & 0 & \lambda\tilde{\varepsilon}_{33} & 0
\end{bmatrix},
$$

with

(9) $$t = \frac{x_3}{h}, \quad \tilde{\varepsilon} = \frac{1}{(n_0)^2}\varepsilon, \quad \frac{\omega n_0 h}{c} = \eta^{-1}, \ \text{and} \ \cos(\theta) = \lambda.$$

The length h is the thickness of the LC slab, θ is the angle of incidence, and n_0 is the index of the homogeneous isotropic medium surrounding the slab.

The Stokes parameter data given by (5) *is equivalent to*

(10) $$\tilde{\mathcal{S}}(I_1, I_2, \theta) = (T_1 \exp(\mathbf{i}f(I_1, I_2, \theta)), T_2 \exp(\mathbf{i}f(I_1, I_2, \theta))),$$

where the real function f is unknown, and may depend on I_1, I_2 and θ.

Problem (7) has only partial conditions on the solution at the start point and partial conditions at the end point. It is therefore not a standard initial eigenvalue problem for system of ordinary differential equations (14). It is nevertheless well-posed, as the following proposition shows, proved in A for completeness.

Proposition 2. *Given I_1 and I_2, there exist a unique pair of reflection parameters R_1 and R_2, a unique pair of transmission parameters T_1 and T_2, and a unique Z solution of* (7).

Proof of proposition 1. We rescale the space variable x_3 by $t = x_3/h$. Problem 3 becomes

(11) $$\frac{dX}{dt} = \frac{\mathbf{i}}{\eta}\tilde{M}(t)X(t),$$

and $\tilde{M}$ is given by
(12)

$$\tilde{M} = \begin{bmatrix} -\frac{\tilde{\varepsilon}_{13}\sin(\theta)}{\tilde{\varepsilon}_{33}} & \frac{\tilde{\varepsilon}_{33}-\sin(\theta)^2}{n_0\varepsilon_0 c\tilde{\varepsilon}_{33}} & -\frac{\tilde{\varepsilon}_{23}\sin(\theta)}{\tilde{\varepsilon}_{33}} & 0 \\ n_0\frac{\varepsilon_0 c(\tilde{\varepsilon}_{11}\tilde{\varepsilon}_{33}-\tilde{\varepsilon}_{13}^2)}{\tilde{\varepsilon}_{33}} & -\frac{\tilde{\varepsilon}_{13}\sin(\theta)}{\tilde{\varepsilon}_{33}} & n_0\frac{\varepsilon_0 c(\tilde{\varepsilon}_{12}\tilde{\varepsilon}_{33}-\tilde{\varepsilon}_{13}\tilde{\varepsilon}_{23})}{\tilde{\varepsilon}_{33}} & 0 \\ 0 & 0 & 0 & \frac{1}{n_0\varepsilon_0 c} \\ n_0\frac{\varepsilon_0 c(\tilde{\varepsilon}_{12}\tilde{\varepsilon}_{33}-\tilde{\varepsilon}_{13}\tilde{\varepsilon}_{23})}{\tilde{\varepsilon}_{33}} & -\frac{\tilde{\varepsilon}_{23}\sin(\theta)}{\tilde{\varepsilon}_{33}} & n_0\frac{\varepsilon_0 c(\tilde{\varepsilon}_{22}\tilde{\varepsilon}_{33}-\tilde{\varepsilon}_{23}^2-\sin(\theta)^2\tilde{\varepsilon}_{33})}{\tilde{\varepsilon}_{33}} & 0 \end{bmatrix}.$$

In the outer isotropic medium, the matrix $\tilde{M}$ simplifies to

$$M_1 = \begin{bmatrix} 0 & \frac{1}{n_0\varepsilon_0 c}\lambda & 0 & 0 \\ n_0\varepsilon_0 c & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{n_0\varepsilon_0 c} \\ 0 & 0 & n_0\varepsilon_0 c\lambda & 0 \end{bmatrix},$$

which has a block diagonal structure, corresponding to the independence of two possible polarisations of the electric field. The matrix M_1 has four eigenvectors, two corresponding to the incoming electric and magnetic fields, and two corresponding to the outgoing electric and magnetic fields. Ordering them in agreement with the block structure, these are

$$V = \begin{bmatrix} \lambda & -\lambda & 0 & 0 \\ \varepsilon_0 c n_0 & \varepsilon_0 c n_0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & \varepsilon_0 c n_0\lambda & -\varepsilon_0 c n_0\lambda \end{bmatrix}$$

corresponding to the eigenvalues $\lambda, -\lambda, \lambda, -\lambda$. We note that

$$X^i = \mathcal{E}_1^I V [1, 0, 0, 0]^T + \mathcal{E}_2^I V [0, 0, 1, 0]^T.$$

The transmission problem takes a simpler form if the matrix $\tilde{M}$ is written in the eigenbasis of M_1, that is,

$$(13) \qquad\qquad\qquad \hat{M} = V^{-1}\tilde{M}V.$$

The transmission problem (3)–(4) then becomes

> *Given an initial condition $I = [I_1, 0, I_2, 0]^T$, find $T = [T_1, 0, T_2, 0]^T$*
> *and $R = [0, R_1, 0, R_2]^T$ such that the solution of*

$$(14) \qquad\qquad\qquad \frac{dY}{dt} = \frac{\mathbf{i}}{\eta}\hat{M}(t)Y(t), \quad t \in [0, 1],$$

> *satisfies*

$$Y(0) = I + R, \quad Y(1) = T.$$

The stokes parameter are then given by

$$S(I_1, I_2, \theta) = \left(|T_1|^2 + |T_2|^2, |T_1|^2 - |T_2|^2, 2T_1\overline{T_2} \right).$$

A computation shows that the available data is equivalent to the knowledge of the vector $\mathcal{S}$ given by (10). Finally, the structure of the matrix $\hat{M}$, is simpler after a block $-\pi/4$ a rotation. We define

$$B = R_{\frac{\pi}{4}}(\hat{M})R_{-\frac{\pi}{4}},$$

where

$$R_{\frac{\pi}{4}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

and the entries of B are given by (8). After these simplifications, problem (14) transforms into problem (7). □

2.2. **Orthorhombic LC cell model.** The following proposition summarizes our findings concerning the orthorhombic model.

Proposition 3. *If ε is a diagonal matrix with three independent piecewise contin-uous functions $(\varepsilon_{ii})_{i=1,2,3}$ as diagonal entries, not all three can be determined from the Stokes parameters: ε_{11} and ε_{33} cannot be determined independently from each other. On the other hand, the transmission coefficient T_1 is independent of ε_{22}, whereas T_2 is uniquely determined by the system*

$$
\begin{aligned}
\eta\lambda\frac{dz_3}{dt} &= \mathbf{i}(\lambda^2 + \tilde{\varepsilon}_{22} - 1)z_4 \ in \ [0, 1], \\
(15) \qquad \eta\frac{dz_4}{dt} &= \mathbf{i}\lambda z_3 \ in \ [0, 1], \\
z_3(0) &= 1 - R_2, \quad z_4(0) = 1 + R_2, \quad z_3(1) = z_4(1) = T_2.
\end{aligned}
$$

If ε_{11} and ε_{33} are unknown, the available data to determine ε_{22} is $|T_2|$, where $|\cdot|$ denotes the complex modulus. If the phase of T_1 is known, and this is the case when both ε_{11} and ε_{33} are known, the available data is T_2.

Proof. If we assume that ε is a diagonal matrix-valued function with $\varepsilon_{ii} > C > 0$ for $i = 1, 2, 3$ and C is a positive constant, the Berreman matrix B becomes block diagonal:

$$B = \begin{bmatrix} 0 & \dfrac{\lambda^2 + \tilde{\varepsilon}_{33} - 1}{\tilde{\varepsilon}_{33}\lambda} & 0 & 0 \\ \lambda\tilde{\varepsilon}_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{\lambda^2 + \tilde{\varepsilon}_{22} - 1}{\lambda} \\ 0 & 0 & \lambda & 0 \end{bmatrix}.$$

By inspection we note that the unknowns (R_1, T_1) only depend on I_1, and the unknowns (R_2, T_2) only depend on I_2: the two transmission modes are decoupled. By linearity, we can thus set $I_1 = I_2 = 1$. The transmission problem relating R_2 and T_2 becomes (15).

To write the transmission problem relating R_1 and T_1 in a similar form, we write $\Lambda_1 = \int_0^1 \tilde{\varepsilon}_{11}(s)\,ds$, and introduce the change of variable $\Lambda_1\nu(x) = \int_0^x \tilde{\varepsilon}_{11}(s)\,ds$, and write the inverse change of variable $\mu = \nu^{-1}$. The first block of (7) then becomes

$$(16) \quad \begin{aligned} \frac{\eta\lambda}{\Lambda_1}\frac{dz_1}{d\nu} &= \mathbf{i}\frac{\lambda^2 + \tilde{\varepsilon}_{33}(\mu(\nu)) - 1}{\tilde{\varepsilon}_{11}(\mu(\nu))\tilde{\varepsilon}_{33}(\mu(\nu))}z_2 \text{ in } [0,1], \\ \frac{\eta}{\Lambda_1}\frac{dz_2}{d\nu} &= \mathbf{i}\lambda z_1 \text{ in } [0,1], \\ z_1(0) &= 1 - R_1, \quad z_2(0) = 1 + R_1, \quad z_1(1) = z_2(1) = T_1. \end{aligned}$$

It is clear that problem (16) contains too many unknown functions and parameters for each of them to be uniquely determined by the map $\lambda \to T_1$. Furthermore, formula (10) given in proposition 1 shows that for each λ the pair (T_1, T_2) is only known up to an arbitrary phase shift. Since T_1 and T_2 depend on different unknown functions, in general only $|T_1|$ and $|T_2|$ are available. $\qquad\square$

Proposition 4. *The transmission coefficient T_2 is unchanged if $t \to \varepsilon_{22}(t)$ is replaced by $t \to \varepsilon_{22}(1 - t)$.*

We check this classical reversibility property in B.

2.3. **Uniaxial nematic LC cell model.** Let us now consider the nematic uniaxial model (6). The index of the surrounding homogeneous medium is given by

$$(17) \qquad\qquad n_0^2 = \sqrt{\varepsilon_\perp \varepsilon_\parallel}.$$

We parametrize the director vector by

$$(18) \quad \begin{aligned} \mathbf{n} &= (\cos(\psi)\cos(\varphi), \cos(\psi)\sin(\varphi), \sin(\psi)), \\ &(\psi, \varphi) \in (-\frac{\pi}{2}, \frac{\pi}{2}] \times [0, \pi). \end{aligned}$$

The tilt angle ψ and the azimuthal angle φ parametrize a half-sphere only, since ε is unchanged if $\mathbf{n}$ is changed into $-\mathbf{n}$. Uniaxial configurations are *a priori* less complex than the orthorhombic ones, since two functions instead of three are to be determined. We simplify the problem even further, and impose that $\varphi = 0$: the director vector $\mathbf{n}$ lies in the incident plane. This is still not sufficient for uniqueness, as shown by the following proposition.

Proposition 5. *If ε is the dielectric permittivity matrix of a nematic uniaxial LC cell modelled by (6), the Stokes parameter data are not sufficient to determine uniquely the tilt angle ψ and the azimuthal angle φ of the director vector defined in (18). When the director vector stays in the incident plane, that is $\varphi \equiv 0$, the available data are*

$$T(\lambda)F(\lambda) \ \text{and} \ T(\lambda)/F(\lambda),$$

where

$$(19) \qquad F(\lambda) = \exp\left(\mathbf{i}\eta^{-1}\delta\sqrt{1-\lambda^2} \int_0^1 \frac{\sin(2\psi(s))}{1+\delta\cos(2\psi(s))} ds \right).$$

The parameters with λ, and η are defined in (9), whereas δ is given by

$$(20) \qquad\qquad \delta = \frac{\varepsilon_\perp - \varepsilon_\parallel}{\varepsilon_\perp + \varepsilon_\parallel}.$$

The map $\lambda \to T(\lambda)$ is defined by

$$\tilde{\eta}\lambda\frac{du_1}{dt} = \mathbf{i}\left(\lambda^2 + \frac{\delta}{\sqrt{1-\delta^2}}\left(\cos(2\psi(\mu)) + \frac{\delta}{1+\sqrt{1-\delta^2}} \right) \right) u_2 \ in \ [0,1],$$

$$(21) \ \ \tilde{\eta}\frac{du_2}{dt} = \mathbf{i}\lambda u_1 \ in \ [0,1],$$

$$u_1(0) = 1 - R(\lambda), \quad u_2(0) = 1 + R(\lambda), \quad u_1(1) = u_2(1) = T(\lambda).$$

where $\mu(\tau(t)) = t$ for all $t \in [0,1]$, with

$$\tau(t) = \frac{1}{\mathcal{N}} \int_0^t \frac{1}{1+\delta\cos(2\psi(s))} ds, \quad \mathcal{N} = \int_0^1 \frac{1}{1+\delta\cos(2\psi(s))} ds, \quad \tilde{\eta} = \frac{\eta}{\sqrt{1-\delta^2}\mathcal{N}}.$$

The map $t \to \psi(t)$ is not uniquely determined by the available data.

Proof. When $\varphi = 0$, the Berreman matrix B is block diagonal, with

$$(B)_{1..2,1..2} = \begin{bmatrix} \dfrac{\delta\sin(\theta)\sin(2\psi)}{\delta\cos(2\psi)+1} & \dfrac{1}{\lambda}\left(1 + \dfrac{\sqrt{1-\delta^2}(\lambda^2-1)}{\delta\cos(2\psi)+1} \right) \\ \dfrac{\sqrt{1-\delta^2}\lambda}{\delta\cos(2\psi)+1} & \dfrac{\delta\sin(\theta)\sin(2\psi)}{\delta\cos(2\psi)+1} \end{bmatrix},$$

and

$$(B)_{3..4,3..4} = \begin{bmatrix} 0 & \dfrac{n_0^2(\lambda^2-1)+\varepsilon_\perp}{n_0^2\lambda} \\ \lambda & 0 \end{bmatrix}.$$

The second transmission parameter, T_2, can be computed explicitly independently of the tilt angle ψ. Thus, using formula (10) we see that the available data to determine ψ is T_1, and not just its modulus. Performing the change of unknown

$$(u_1(t), u_2(t)) = \exp\left(-\mathbf{i}\frac{\sin(\theta)}{\eta} \int_0^t \frac{\delta\sin(2\psi(s))}{\delta\cos(2\psi(s))+1} ds \right) (z_1(t), z_2(t)),$$
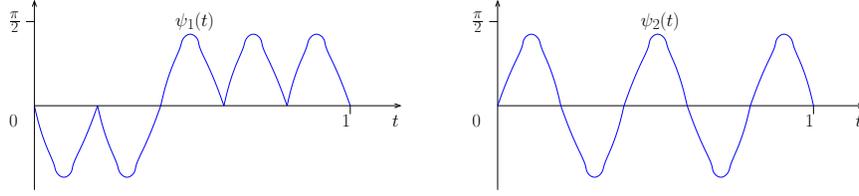
FIGURE 2. Example of non-uniqueness for nematic crystals. If the azimuthal angle φ is constantly zero, then $T_1(\psi_1) = T_1(\psi_2)$ for any incident angle θ .

we see that T_1 is given by $T(\lambda)F(\lambda)$ or $T(\lambda)/F(\lambda)$ with $T(\lambda)$ determined by

$$
\eta\lambda\frac{du_1}{dt} = \mathbf{i}\left(1 + (\lambda^2 - 1)\frac{\sqrt{1-\delta^2}}{1+\delta\cos(2\psi)}\right)u_2 \text{ in } [0,1],
$$

$$
(22) \qquad \eta\frac{du_2}{dt} = \mathbf{i}\lambda\frac{\sqrt{1-\delta^2}}{1+\delta\cos(2\psi)}u_1 \text{ in } [0,1],
$$

$$
u_1(0) = 1 - R(\lambda), \quad u_2(0) = 1 + R(\lambda), \quad u_1(1) = u_2(1) = T(\lambda).
$$

Since both positive and negative incident angles are measured, but $T(\lambda)$ only depends on $\lambda = \cos(\theta)$, the data T_1 is both $T(\lambda)F(\lambda)$ and $T(\lambda)/F(\lambda)$.

We then change t to τ in (22) and obtain (21) (where we named t instead τ the dummy variable).

The coefficient $T(\lambda)$ depends on ψ, through $\cos(2\psi)$: positive and negative tilt angles are not distinguishable. The phase difference between $T(\lambda)$ and $T(\lambda)F(\lambda)$ or $T(\lambda)/F(\lambda)$ depends on ψ via the number

$$
K = \int_0^1 \frac{\sin(2\psi(s))}{1+\delta\cos(2\psi(s))}ds.
$$

In Figure 2 we show the graph of two tilt angle functions which would have identical transmission parameter $T(\lambda)$ for all λ. To construct these examples, we chose to repeat a given compactly supported pattern five times. In two instances out of five, the pattern is flipped with respect to the t axis. There are ten possibilities: we chose two different ones arbitrarily. The parameter $T(\lambda)$ is equal for both graphs, as it is invariant under arbitrary changes of the tilt angle sign. The constant K sums the contribution of the five patterns, three with a plus sign and two with a minus sign. The way these patterns are ordered does not change this integral. This non-uniqueness comes in addition to the one already highlighted in proposition 4, namely that $T(\lambda)$ is unchanged if $t \to \psi(t)$ is replaced by $t \to \psi(1-t)$, see B. $\square$

2.4. **Scaling assumptions.** In this section, we discuss the scales of the various quantities involved. The numerical values of the physical parameters used in this section are taken from Lionheart & Newton [1].

Frequency. For a He-Ne laser of wavelength 0.633 μm and a slab of thickness 5 μm, we find $\eta = \eta_e \approx 8.4 \times 10^{-2}$. In what follows, we will therefore assume that

$$
(23) \qquad \eta \text{ is a small dimensionless parameter.}
$$

Dielectric parameters. In the case of a orthorhombic (diagonal) dielectric tensor, we will assume that

$$(24) \qquad \tilde{\varepsilon}_{22} = 1 + \eta^\alpha q, \text{ where } \sup_{t \in [0,1]} |q(t)| \leq 1, \text{ and } \alpha \text{ near } 1.$$

A similar assumption for the nematic uniaxial LC cell to give a simple form to (19) is

$$(25) \qquad \qquad \delta = \eta^\alpha.$$

For problem (21) to match the orthorhombic assumption (24), we can choose

$$(26) \qquad \qquad \frac{\delta}{\sqrt{1-\delta^2}} \left(1 + \frac{\delta}{1 + \sqrt{1-\delta^2}} \right) = \tilde{\eta}^\alpha.$$

For the nematic uniaxial LC cell the values are $|\sqrt{\varepsilon_\perp} - \sqrt{\varepsilon_\parallel}| \approx 0.15$, $n_0 \approx 1.52$. The scaling (25) yields $\alpha \approx 0.94$ whereas (26) leads to $\alpha \approx 0.92$, both values are indeed close to one.

Incident angle. We write the range of the incident angle as follows:

$$(27) \qquad \qquad \lambda \in [\sqrt{\tau \eta^\alpha}, 1], \text{ and } \tau > 1.$$

In the experiment considered, θ varies between from $0°$ to about $70°$: for larger angles the measurements become unreliable. The extremal value $(\cos(70°))^2$ corresponds to $\tau = \tau_e \approx 1.2$.

Measurement error. We will assume that the measured data is accurate up to errors of order

$$(28) \qquad \qquad \mathcal{O}\left(\frac{\eta^{5+\alpha}}{\lambda^5} \right)$$

Where for any $x > 0$, $|\mathcal{O}(x)| \leq Cx$, where C is a constant independent of x. In the experimental case considered, this corresponds to a precision of the order of 10^{-7} for a normal incidence, and of the order of 10^{-4} for the most slanted incidence. This assumption models the fact that the measurements become less accurate as slant of the slab increases.

Sampling rate. We suppose that λ is measured with a fine sampling rate, e.g. η^2. Experimentally, 200 incident angles θ between $-70°$ and $70°$ are used, corresponding to sampling rate $\eta^{1.86}$.

2.5. **Model Problems.** To summarize the discussion of the previous section, we now write down three traceable reconstruction problems pertaining to the orthorhombic LC cell model and the nematic uniaxial LC cell model.

> **Problem A.** Let $\tau > 1$ be a constant, α a parameter close to 1, and $\eta > 0$ a small parameter. Let $q : [0,1] \to [-1,1]$ be a piecewise continuous function. For every $\lambda \in [\sqrt{\tau \eta^\alpha}, 1]$, let $T(\lambda)$ and $R(\lambda)$ be the solutions of
>
> $$\eta \lambda \frac{du_1}{dt} = \mathbf{i} \left(\lambda^2 + \eta^\alpha q \right) u_2 \text{ in } [0,1]$$
>
> $$(29) \qquad \eta \frac{du_2}{dt} = \mathbf{i} \lambda u_1 \text{ in } [0,1]$$
>
> $$u_1(0) = 1 - R_2, \quad u_2(0) = 1 + R_2, \quad u_1(1) = u_1(1) = T(\lambda).$$
>
> What can be determined about q from $\mathcal{D}_A(\lambda) = |T(\lambda)| + \mathcal{O}\left(\eta^{5+\alpha} \lambda^{-5} \right)$?

Remark 6. We note that the lower bound on λ, coming from experimental considerations, has a natural analytic interpretation. When $\lambda^2 \gg \eta^\alpha q$, Problem (29) corresponds to a wave propagation problem, slightly perturbed by q. We are therefore in an 'optical' regime: the transmitted wave is very similar to the incident wave. When λ^2 and $\eta^\alpha q$ are of the same order: the parameter q is no longer a perturbation but a leading order term. Problem (29) becomes diffusive, and one can therefore expect the polarimetric measurements to become unreliable.

Problem A is directly inspired by proposition 3 for the orthorhombic LC cell model. We argued that when the dielectric tensor is diagonal, with principle axes aligned with the coordinate axes, we cannot hope to reconstruct all three diagonal coefficients. In particular, the entries ε_{11} and ε_{33} cannot be determined independently. According to proposition 3 the second diagonal entry ε_{22} determines uniquely $|T_2|$, independently of ε_{11} and ε_{33}. The scaling assumption were discussed in section 2.4.

Problem A is also relevant for the in-plane nematic uniaxial LC cell model, that is when $\mathbf{n} = (\cos(\psi), 0, \sin(\psi))$. In that case $q = \cos(2\psi)$. More precisely, proposition 5 shows that $|T(\lambda)| = |T(\lambda)F(\lambda)|$ is measurable, and determined by Problem A using the scaling assumption discussed in section 2.4 with $\tilde{\eta}$ instead of η and

$$q = \left(\cos(2\psi(\mu)) + \frac{\delta}{1 + \sqrt{1 - \delta^2}} \right) \left(1 + \frac{\delta}{1 + \sqrt{1 - \delta^2}} \right)^{-1},$$

with δ and $\tilde{\eta}$ related by (26), and μ given by proposition 5. At first order, μ is the identity and $\delta \approx \eta^\alpha$. Problem (26) is therefore a close variant of Problem (29).

We already know from of the various non-uniqueness examples presented in section 2.2 and 2.3 that little can be determined about the interior values of q from Problem A. We discuss it further, assuming q is smooth in section 3. We will see in section 4 that we can give a more precise answer to the related problem

> **Problem B.** Let $\tau > 1, \alpha > 0$ be constants, $\eta > 0$ a small dimensionless parameter, and q a piecewise continuous function such that $|q| < 1$. For every $\lambda \in [\sqrt{\tau \eta^\alpha}, 1]$, let $\mathcal{D}_B[q]$ be given by
>
> $$\mathcal{D}_B[q] : \quad [\sqrt{\tau \eta^\alpha}, 1] \quad \to \mathbb{R}$$
> (30)
> $$\lambda \quad \to \int_0^1 \sqrt{1 + \frac{\eta^\alpha}{\lambda^2} q(s)} ds.$$
>
> What are sufficient conditions so that $\mathcal{D}_B[q_1] = \mathcal{D}_B[q_2]$ implies $q_1 = q_2$?

Finally, let us consider the phase information available from the nematic uniaxial LC cell data.

> **Problem C.** Let $\mathcal{D}_C[\psi]$ be given by
>
> (31)
> $$\mathcal{D}_C[\psi] = \int_0^1 \frac{\sin \psi(s)}{1 + \delta \cos \psi(s)} ds + \mathcal{O}(\delta^2),$$
>
> where δ is a small parameter. Assuming that $\psi : [0, 1] \to [-\pi, \pi]$ changes sign once, at $s^* \in (0, 1)$ and $|\psi|$ is known, find possible values for s^*.

As both $T(\lambda)F(\lambda)$ and $T(\lambda)/F(\lambda)$ are measured, $F^2(\lambda)$ is available. This data is measured precisely for close to normal incident angles, that is, $\eta^{-1}\delta\sqrt{1 - \lambda^2}$ small,

we can extract from (19) the constant depending on ψ, which is $\mathcal{D}_C[2\psi]$. This problem is naturally very simple to solve: we highlight it here to point out how this information can be extracted from the data. On the other hand, not much more can be obtained from this problem, since the data in this case is just one value.

Define

$$G(t) = \int_0^t \frac{\sin |\psi(s)|}{1 + \delta \cos |\psi(s)|} ds.$$

Since $0 \leq |\psi(s)| \leq \pi$, G is non decreasing, and strictly increasing if ψ does not equal 0 or π on a set of positive measure. There are two possibilities, depending on whether ψ is positive and then negative, or negative and then positive.

$$G(s_0) = \frac{1}{2}G(1) + \frac{1}{2}(\mathcal{D}_C[\psi] + \mathcal{O}(\delta^2)) \text{ or } G(s_1) = \frac{1}{2}G(1) - \frac{1}{2}\mathcal{D}_C[\psi] + \mathcal{O}(\delta^2).$$

If δ is small enough so that $-G(1) < \mathcal{D}_C[\psi] + \mathcal{O}(\delta^2) < G(1)$ both s_1 and s_0 always exists, are unique if G is strictly increasing.

3. On Problem A under a smoothness assumption.

mIn this section, we show that if q smooth, namely $C^4([0, 1])$, moderate angles of incidence provide information about the endpoint values of q only. The internal configuration of q is not decidable as it is stable under suitable re-arrangements. We then explain how Problem A leads to Problem B.

Our strategy is to find an explicit approximate formula for

$$(32) \qquad \mathcal{D}'_A = \frac{4}{|T|^2} - 2.$$

Notation. In this section, given $x > 0$ and $y > 0$, $\mathcal{O}(x^y)$ means $|\mathcal{O}(x^y)x^{-y}| \leq K$, where K is a constant depending on $\|q\|_{C^4([0,1])}$, τ given by (27) and y only.

Proposition 7. *Assume that $q \in C^4([0, 1])$. There exist $\eta_0 > 0$ such that for all $0 < \eta \leq \eta_0$ and $0 < \alpha < 2$, Problem A given by (29) is equivalent to the reconstruction of q from*

$$(33) \quad \mathcal{D}'_A = \left(\frac{\eta}{\lambda}\frac{dv_1}{dt}(1)\right)^2 + \left(\frac{\eta}{\lambda}\frac{dv_2}{dt}(1)\right)^2 + (v_1(1))^2 + (v_2(1))^2 + \mathcal{O}\left(\frac{\eta^{5+\alpha}}{\lambda^5}\right),$$

where v_1 and v_2 are given by

$$v_1(t) = \sqrt{\frac{C(\lambda, 0)}{C(\lambda, t)}} \cos\left(\frac{\lambda}{\eta}\int_0^t C(\lambda, s)ds\right) + \frac{\eta}{2\lambda}\frac{dC}{dt}(\lambda, 0)\frac{1}{C(\lambda, 0)}v_2(t),$$

$$v_2(t) = \sqrt{\frac{1}{C(\lambda, 0)C(\lambda, t)}} \sin\left(\frac{\lambda}{\eta}\int_0^t C(\lambda, s)ds\right),$$

with

$$(34) \qquad A = \left(1 + \frac{\eta^\alpha}{\lambda^2}q\right)^{-1/4}, \quad B = -\frac{1}{4}A^3\frac{d^2A}{dt^2}, \text{ and } C = (A(1 + \frac{\eta^2}{\lambda^2}B))^{-2}.$$

We have

$$\eta_0 > \max\{\eta \ : \ 1 + \eta^{2-\alpha}B > 0 \text{ for all } t \in [0, 1]\}.$$

Remark 8. This is an approximation of WKB-type. The uniform pointwise error estimate involves second order derivatives of v_1 and v_2 and requires $q \in C^4([0, 1])$.
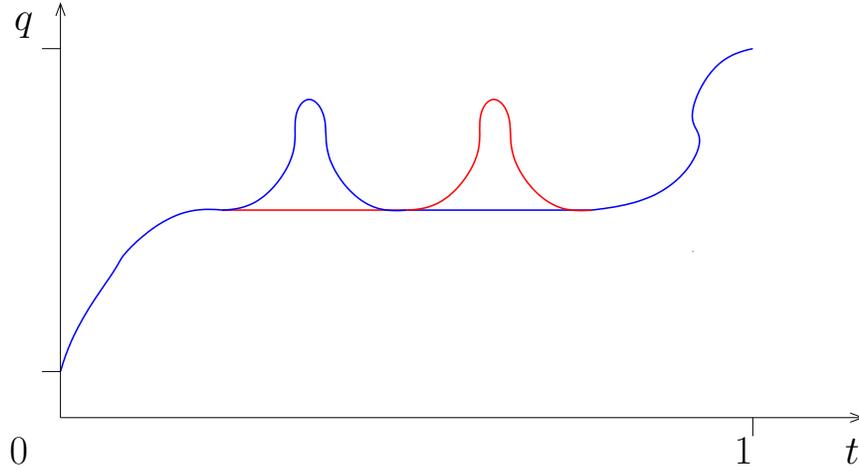
FIGURE 3. Example of non-unique determination from the data. If the dielectric permittivity q follows the blue curve, or the red variant, the available data $\mathcal{D}'_A$ is the same, for any $\lambda > 0$.

We will prove this proposition in section 5. Note that because up to error terms, $\mathcal{D}'_A$ depends only on q via the value of $(C, \frac{dC}{dt})$ at $t = 0$ and $t = 1$ and on $\int_0^1 C(\lambda, s)ds$, only a large class of equivalent q can be determined in Problem A, at best.

Corollary 9. *The usable data contains only some information of the first three derivatives of q at $t = 0$ and $t = 1$, and on*

$$(35) \qquad \int_0^1 F\left(\frac{\eta^\alpha}{\lambda^2}q, \frac{\eta^\alpha}{\lambda^2}\frac{dq}{dt}, \frac{\eta^\alpha}{\lambda^2}\frac{d^2q}{dt^2}\right) dt,$$

where F is an explicit algebraic function, independent of λ. Since (35) is stable under sufficiently smooth rearrangements of q (see e.g. Example 10) the determination of q from the data is not possible without additional a priori information on the variations of q.

Example 10. Let $q_0 \in C^4([0,1])$ be a positive function which is constant on the interval $[t_0, t_1]$ with $1 < t_0 < t_1 < 0$. Let $q_1 \in C^4(\mathbb{R})$ be positive function with compact support in $[0, (t_1 - t_0)/2]$. Consider the family of functions q_s given by

$$q_s(t) = q_0(t) \text{ for } t \in [0, t_0] \cup [t_1, 1], \text{ and } q_s(t) = q_0(t) + q_1(t - s) \text{ for } t \in (t_0, t_1)$$

where s is a parameter in $(t_0, (t_0 + t_1)/2)$. Then for all such s, $q_s \in C^4([0,1])$ has the same end-point values, and for any smoooth functional F,

$$\frac{d}{ds} \int_0^1 F\left(\frac{\eta^\alpha}{\lambda^2}q_s, \frac{\eta^\alpha}{\lambda^2}\frac{dq_s}{dt}, \frac{\eta^\alpha}{\lambda^2}\frac{d^2q_s}{dt^2}\right) dt = 0.$$

Figure 3 represents such a construction for two different values of s. Note that the order of accuracy is not the issue when q is smooth, as the WKB-type ansatz can be continued to obtain any order of accuracy, for a suitable η_0. Counter examples to uniqueness such as the one depicted in Figure 3 would still hold.

The main result in this section is devoted to the extraction of a few properties of q at $t = 0$ and $t = 1$ from moderate angles of incidence, for a sufficiently smooth q.

Proposition 11. *Suppose that $q \in C^4([0,1])$, and that α is close to one. Using the data available for $\frac{1}{2} \leq \lambda \leq 1$, in Problem A given by (29), the parameters $A_i, i = 1, \ldots, 5$ given by*

$$
\begin{aligned}
A_1 &= q(0) + q(1) + \mathcal{O}(\eta^2), \\
A_2 &= q(0)q(1) + \mathcal{O}(\eta^2), \\
A_3 &= \frac{d \ln q}{dt}(1) - \frac{d \ln q}{dt}(0) + \mathcal{O}(\eta), \\
A_4 &= \int_0^1 q(s)ds + \mathcal{O}(\eta^{3-\alpha}), \\
A_5 &= \int_0^1 q^2(s)ds + \mathcal{O}(\eta^{4-2\alpha}).
\end{aligned}
$$

can be determined.

The proof of this proposition is given in section 5.

Remark 12. This first order data respects the invariance by mirror symmetry discussed in Proposition 4. If the LC cell is flipped, its index becomes $\tilde{q}(t) = q(1-t)$, and the parameters $A_i, i = 1, \ldots, 5$ are unchanged. The fact that only a handful of moments can be recovered from moderate incidence is known for related more general problems, see Sharafutdinov [12].

To investigate the problem further, let us assume that the map $\lambda \to C(\lambda, t)$ defined in (34) is known at the endpoints $t = 0$ and $t = 1$ up to the order of approximation. The usable data $\mathcal{D}'_A$ takes the form

$$
\mathcal{D}'_A = K_1 \cos\left(\frac{2\lambda}{\eta} \int_0^1 C(\lambda, s)ds + K_2\right) + K_3 + \mathcal{O}\left(\frac{\eta^{5+\alpha}}{\lambda^5}\right),
$$

where K_1, K_2 and K_3 are known parameters. Since the data is sampled in λ at a fine η^2 rate, the leading order λ dependent data is

$$
\tag{36} \int_0^1 C(\lambda, s)ds
$$

which, at leading order, is $\mathcal{D}_B[q](\lambda)$ given by (30). Note that this problem also arises naturally if T, and not simply its amplitude, was considered in Problem A, as (36) is precisely the rate of change of the phase of T.

4. On Problem B : uniqueness under a monotonicity assumption

We now turn to the reconstruction of q from $\mathcal{D}_B[q]$.

Theorem 13. *Consider $\lambda \to \mathcal{D}_B[q](\lambda)$ for all $\lambda \in [\sqrt{\tau \eta^\alpha}, 1]$ as defined in Problem B given by (30).*

- *The map $q \to \mathcal{D}_B[q]$ is invariant under level-set preserving rearrangements of q. That is, given two functions q_1 and q_2 defined on $[0, 1]$ such that $\sup |q_i| \leq 1$ which satisfy*

  $meas\{q_1(t) = y, t \in [0, 1]\} = meas\{q_2(t) = y, t \in [0, 1]\} \quad \forall y \in [-1, 1],$

we have $\mathcal{D}_B[q_1] \equiv \mathcal{D}_B[q_2]$.

- If $q \in C^1([0,1])$ is strictly increasing, $\sup |q| \leq 1$, and $q^{-1} \in C^1([q(0), q(1)])$, then $\mathcal{D}_B[q]$ determines q uniquely. The reconstruction problem can be formulated in terms of classical transforms in this case. There holds

(37)
$$\frac{1}{\lambda}\mathcal{D}_B[q](\lambda) + \frac{d\mathcal{D}_B[q]}{d\lambda}(\lambda) = \frac{2}{\sqrt{\eta^\alpha}}\mathcal{L}F_0Q\left(\frac{\lambda}{\sqrt{\eta^\alpha}}\right),$$

where $\mathcal{L}$ is the Laplace transform, $\mathcal{L}f(s) = \int_0^\infty e^{-st}f(t)dt$, and F_0 is the first Hankel transform, $F_0f(s) = \int_0^\infty tf(t)J_0(st)dt$, where J_0 is the Bessel function of the first kind of order 0, and Q is given by

$$Q := x \to \mathbf{1}_{(q(0),q(1))}(x^2)\frac{dq^{-1}}{dx}(x^2).$$

Remark 14. The Hankel Transform is self-invertible, and stable. Inverting the Laplace transform is well known to be an exponentially ill-posed task see e.g. Epstein & Schotland [13]. Even more so when, as it is the case here, the Laplace data is limited to an interval $(\sqrt{\tau}, \eta^{-\alpha/2})$ located away from zero. The data $\mathcal{D}_B[q]$ can be written as a linear integral operator acting on $\frac{dq^{-1}}{dy}$ in the form

$$\mathcal{D}_B(\lambda) = K\frac{dq^{-1}}{dy}(\frac{\eta^\alpha}{\lambda^2}) \quad \text{with } Kf(x) = \int_{q(0)}^{q(1)}\sqrt{1+xy}f(y)dy \text{ for } 0 \leq x \leq 1.$$

It is clear that $K : C^0([q(0), q(1)]) \to C^0([\eta^\alpha, \tau^{-1}])$ is a compact operator : its inverse is therefore unbounded. Including grazing angles measures, corresponding to $\tau = 1$ would still lead to an unstable problem.

To prove this result, we note that the available data takes a much simpler form after an integral transformation.

Lemma 15. *Given a function q such that defined on $[0,1]$ such that $\sup_{x \in [0,1]} |q(t)| \leq 1$, we can define $\hat{\mathcal{D}}_B[q]$ as the complex valued holomorphic extension of $\mathcal{D}_B[q]$ on the annulus $\sqrt{\eta^\alpha\tau} < |\zeta| < 1$.*

Let $\hat{\mathcal{K}}$ represent the holomorphic extension of

$$\mathcal{K} : x \to 1 + 2\sqrt{x}e^x \int_0^{\sqrt{x}} e^{-t^2}dt$$

to the same annulus.

Let $\mathcal{D}'_B[q] \in C^\infty(\mathbb{R})$ be given by

$$\mathcal{D}'_B[q](t) = \int_0^1 e^{-\mathbf{i}tq(s)}ds \quad \text{for all } t \in \mathbb{R}.$$

Then, for any $t \in \mathbb{R}$ and $\rho \in (\sqrt{\tau\eta^\alpha}, 1)$, we have

$$\mathcal{D}'_B[q](t) = \frac{1}{2\pi}\int_0^{2\pi} \hat{\mathcal{D}}_B[q]\left(\rho e^{\mathbf{i}s}\right)\hat{K}\left(\mathbf{i}\eta^{-\alpha}\rho^2 te^{\mathbf{i}2s}\right)ds.$$

The correspondence between $\mathcal{D}_B[q]$ and $\mathcal{D}'_B[q]$ is one to one.

We prove this result in section 5. This in turn shows that $\mathcal{D}_B[q]$ determines q uniquely within the class of smooth monotone functions, albeit in a very unstable manner.

Proof of theorem 13. The first statement is a straightforward consequence of the coarea formula applied to the map $t \to \mathcal{D}'_B(t)$. The second statement follows from the change of variable formula as follows

$$\mathcal{D}'_B[q](t) = \int_0^1 e^{-\mathbf{i}tq(s)} ds = \int_{\mathbb{R}} e^{-\mathbf{i}ty} \mathbf{1}_{(q(0),q(1))}(y) \frac{dq^{-1}}{dy}(y) dy,$$

Thus $\mathcal{D}'_B[q]$ is the Fourier transform of $\frac{dq^{-1}}{dy}$ on the range of q. Applying the inverse Fourier transform we therefore recover the values of $q(0)$, $q(1)$ and $\frac{dq^{-1}}{dy}$, and therefore q. We do not claim that the requirement $q^{-1} \in C^1([q(0), q(1)])$ is sharp.

Let us now turn to the equation satisfied by $\mathcal{D}_B[q]$. We write,

$$\frac{1}{\lambda} \mathcal{D}_B[q](\lambda) + \frac{d\mathcal{D}_B[q]}{d\lambda}(\lambda) = \int_0^1 \frac{1}{\sqrt{\eta^\alpha}} \frac{1}{\sqrt{\left(\frac{\lambda}{\sqrt{\eta^\alpha}}\right)^2 + q(y)}} dy.$$

Using the same change of variable as above, it follows that

$$\frac{1}{\lambda} \mathcal{D}_B[q](\lambda) + \frac{d\mathcal{D}_B[q]}{d\lambda}(\lambda) = \frac{2}{\sqrt{\eta^\alpha}} \int_0^\infty \frac{1}{\sqrt{\left(\frac{\lambda}{\sqrt{\eta^\alpha}}\right)^2 + v^2}} f(v) v dv$$

$$= \frac{2}{\sqrt{\eta^\alpha}} \int_0^\infty e^{-\frac{\lambda}{\sqrt{\eta^\alpha}} r} F_0 f(r) dr$$

$$= \frac{2}{\sqrt{\eta^\alpha}} \mathcal{L} F_0 f \left(\frac{\lambda}{\sqrt{\eta^\alpha}}\right),$$

as announced.   □

5. Proofs for Proposition 7, Proposition 11 and Lemma 15

We start by expressing the available data in terms of the fundamental solutions of the ordinary differential system (29). Let u_1 and u_2 be the two solution of

(38) $$\eta^2 \frac{d^2 u_i}{dt^2} + (\lambda^2 + \eta^\alpha q) u_i = 0, \qquad i = 1, 2,$$

$$u_1(0) = 1, \quad \frac{du_1}{dt}(0) = 0, \quad u_2(0) = 0, \quad \eta \frac{du_2}{dt}(0) = \lambda.$$

We have the following identity

Proposition 16. *. The transmission parameter T_2 determined by (29) satisfies*

$$\frac{2}{T_2} = \mathbf{i} \frac{\eta}{\lambda} \frac{du_1}{dt}(1) + \frac{\eta}{\lambda} \frac{du_2}{dt}(1) + u_1(1) - \mathbf{i} u_2(1).$$

where u_1 and u_2 are determined by (38). Furthermore,

$$\frac{4}{|T_2|^2} = \left(\frac{\eta}{\lambda} \frac{du_1}{dt}(1)\right)^2 + \left(\frac{\eta}{\lambda} \frac{du_2}{dt}(1)\right)^2 + (u_1(1))^2 + (u_2(1))^2 + 2.$$

This is a simple calculation performed in B.

Proof of Proposition 7. We introduce a perturbed problem closely related to the original problem when q is smooth. We note C given by (34) is well defined and bounded provided η is small enough, namely when $1 + \frac{\eta^2}{\lambda^2} B$ is bounded below by

a positive constant. Assume that $\eta \leq \eta_0$, with η_0 chosen so that C is bounded. A tedious but straightforward computation shows that v_1 and v_2 are solutions of

$$(39) \qquad \eta^2 \frac{d^2 v_i}{dt^2} + (\lambda^2 + \eta^\alpha \tilde{q}) v_i = 0, \qquad i = 1, 2,$$

$$v_1(0) = 1, \quad \frac{dv_1}{dt}(0) = 0, \quad v_2(0) = 0, \quad \eta \frac{dv_2}{dt}(0) \; = \; \lambda,$$

with

$$(40) \qquad \|\tilde{q} - q\|_{C^0([0,1])} \leq K \frac{\eta^4}{\lambda^4},$$

where K depends on $\|q\|_{C^4([0,1])}$. Introducing the function $G(x, y) \in C^4([0,1] \times [0,1])$ given by

$$
\begin{aligned}
G(x, y) \;&=\; \left(v_2(x) v_1(y) - v_1(x) v_2(y) \right) \left(\tilde{q}(y) - q(y) \right), \\
&=\; \sqrt{\frac{1}{C(\lambda, x) C(\lambda, y)}} \sin\left(\frac{\lambda}{\eta} \int_y^x C(\lambda, s) ds \right) \left(\tilde{q}(y) - q(y) \right),
\end{aligned}
$$

we find, comparing (38) and (39) and using Duhamel's Formula, that for $i = 1, 2$ and $t \in [0,1]$ there holds

$$u_i(t) - v_i(t) - \frac{\eta^{1+\alpha}}{\lambda} \int_0^1 G(t, y) \left(u_i(y) - v_i(y) \right) dy = \frac{\eta^{1+\alpha}}{\lambda} \int_0^1 G(t, y) v_i(y) dy.$$

And from the bound (40) we deduce that

$$\frac{\eta}{\lambda} \left\| \frac{du_i}{dt} - \frac{dv_i}{dt} \right\|_{C^0([0,1])} + \| u_i - v_i \|_{C^0([0,1])} \leq K \frac{\eta^{5+\alpha}}{\lambda^5}.$$

Proposition 16 then shows that

$$\frac{2}{T} = \mathbf{i} \frac{\eta}{\lambda} \frac{dv_1}{dt}(1) + \frac{\eta}{\lambda} \frac{dv_2}{dt}(1) + v_1(1) - \mathbf{i} v_2(1) + \mathcal{O}\left(\frac{\eta^{5+\alpha}}{\lambda^5} \right),$$

and since $T \approx 1$, this approximation also gives an approximation for T of the same order, which is the precision up to which T is known. This in turn leads to the formula for $\mathcal{D}'_A$ announced in Proposition 7. $\qquad \square$

To prove proposition 11, we focus on what can be recovered from $\mathcal{D}'_A$ around the normal incidence.

Proposition 17. *Suppose that $\frac{1}{2} \leq \lambda \leq 1$, and α is close to one. Then, the parameters $A_i, i = 1, \ldots, 5$ given by*

$$
\begin{aligned}
A_1 \;&=\; q(0) + q(1) + \mathcal{O}(\eta^2), \\
A_2 \;&=\; q(0) q(1) + \mathcal{O}(\eta^2), \\
A_3 \;&=\; \frac{d \ln q}{dt}(1) - \frac{d \ln q}{dt}(0) + \mathcal{O}(\eta), \\
A_4 \;&=\; \int_0^1 q(s) ds + \mathcal{O}(\eta^{3-\alpha}), \\
A_5 \;&=\; \int_0^1 q^2(s) ds + \mathcal{O}(\eta^{4-2\alpha}).
\end{aligned}
$$

can be extracted from the data $\mathcal{D}'_A$ in this moderate range of incidences.

Proof. A Taylor expansion of $\mathcal{D}'_A$ given (33) around $\eta = 0$, using the fact that $\alpha \geq 1$ shows that

$$
\begin{aligned}
\mathcal{D}'_A(\lambda) &= 2 + \frac{\eta^{2\alpha}}{4\lambda^4}\left(q(1)^2 + q(0)^2 - \left(q(1)^3 + q(0)^3\right)\frac{\eta^\alpha}{\lambda^2}\right) \\
&+ q(0)q(1)\frac{\eta^{2\alpha}}{4\lambda^4}\left((q(0)+q(1))\frac{\eta^\alpha}{\lambda^2} - 2\right)\cos(2\frac{\lambda}{\eta} + \phi(\lambda)) \\
&+ \left(\frac{dq}{dt}(1)q(0) - \frac{dq}{dt}(0)q(1)\right)\frac{\eta^{2\alpha+1}}{\lambda^5}\sin(2\frac{\lambda}{\eta} + \phi(\lambda)) + \mathcal{O}(\eta^{2+2\alpha}),
\end{aligned}
$$

where

$$
\phi(\lambda) = 2\frac{\lambda}{\eta}(\int_0^1 C(\lambda, s)ds - 1).
$$

Using the formula for $C(\lambda, s)$ given by (34) we find that for $\lambda \in [1 - \pi N\eta, 1]$, for a given N, we have

$$
\phi(\lambda) - \phi(1) = \frac{1-\lambda}{\lambda}\eta^{\alpha-1}\int_0^1 q(s)ds + \frac{\lambda^3 - 1}{4\lambda^3}\eta^{2\alpha-2}\int_0^1 q^2(s)ds + \mathcal{O}(\eta^2).
$$

Without changing the order of approximation, since η and α are known parameters, the available data thus becomes

$$
\begin{aligned}
\tilde{\mathcal{D}}_2(\lambda) &= \frac{4\lambda^4}{\eta^{2\alpha}}\left(\mathcal{D}_2(\lambda) - 2\right) \\
&= \left(A_1^2 - 2A_2 - A_1\left(A_1^2 - 3A_2\right)\frac{\eta^\alpha}{\lambda^2}\right) \\
&+ A_2\left(A_1\frac{\eta^\alpha}{\lambda^2} - 2\right)\cos\left(2\frac{\lambda}{\eta} + \frac{1-\lambda}{\lambda}\eta^{\alpha-1}A_4 + \frac{\lambda^3-1}{4\lambda^3}\eta^{2\alpha-2}A_5\right) \\
&+ A_2 A_3\frac{\eta}{\lambda}\sin\left(2\frac{\lambda}{\eta} + \frac{1-\lambda}{\lambda}\eta^{\alpha-1}A_4 + \frac{\lambda^3-1}{4\lambda^3}\eta^{2\alpha-2}A_5\right) + \mathcal{O}(\eta^2),
\end{aligned}
$$

The conclusion follows, as we have $\mathcal{O}(\eta^{-2})$ data points available for $\tilde{\mathcal{D}}_2$. We could recover these five parameters either by a least square fit, or by deriving explicit formulas. $\qquad\square$

Finally, we prove lemma 15.

Proof of lemma 15. Since $|\eta^\alpha q(t)/\lambda^2| < \tau < 1$ for every $t \in [0, 1]$, we can write using the binomial formula

$$
\mathcal{D}_B[q](\lambda) := \sum_{n=0}^{\infty}(-1)^n\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}n!}\frac{\eta^{\alpha n}}{\lambda^{2n}}\int_0^1(q(s))^n ds
$$

Introducing the holomorphic function $\hat{\mathcal{D}}_3 q$ of the complex variable ζ on the annulus $\sqrt{\eta^\alpha\tau} < |\zeta| < 1$ given by

$$
\hat{\mathcal{D}}_B[q](\zeta) := \sum_{n=0}^{\infty}(-\eta^\alpha)^n\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}n!}\zeta^{-2n}\int_0^1(q(s))^n ds,
$$

we observe that this quantity is well defined, as the series is absolutely convergent for $|\zeta| > \sqrt{\eta^\alpha}$. As λ varies in $[\sqrt{\tau\eta^\alpha}, 1]$, $\mathcal{D}_B[q](\lambda)$ determines $\hat{\mathcal{D}}_B[q]$ on a non-empty real interval and therefore also on the full annulus. We may therefore define $\hat{\mathcal{D}}_B[q]$ as the holomorphic extension of $\mathcal{D}_B[q](\lambda)$.

We compute, using the residue formula on the ring of radius $\sqrt{\tau\eta^\alpha} < \rho < 1$, that for any $t \in \mathbb{R}$,

$$\frac{(-\mathbf{i}\eta^\alpha t)^n}{n!} \int_0^1 (q(s))^n ds = \frac{1}{2\sqrt{\pi}} \frac{(\mathbf{i}\rho^2 t)^n}{\Gamma\left(n+\frac{1}{2}\right)} \int_0^{2\pi} \hat{\mathcal{D}}_B[q]\left(\rho e^{\mathbf{i}s}\right) e^{2\mathbf{i}ns} ds.$$

This in turn shows, using the fact that all sums are well behaved, that

$$
\begin{aligned}
\int_0^1 \exp\left(-\mathbf{i}\eta^\alpha t q(s)\right) ds &= \sum_{n=0}^\infty \frac{(-\mathbf{i}\eta^\alpha t)^n}{n!} \int_0^1 (q(s))^n ds \\
&= \frac{1}{2\sqrt{\pi}} \sum_{n=0}^\infty \frac{(\mathbf{i}\rho^2 t)^n}{\Gamma\left(n+\frac{1}{2}\right)} \int_0^{2\pi} \hat{\mathcal{D}}_B[q]\left(\rho e^{\mathbf{i}s}\right) e^{2\mathbf{i}ns} ds, \\
&= \frac{1}{2\pi} \int_0^{2\pi} \hat{\mathcal{D}}_B[q]\left(\rho e^{\mathbf{i}s}\right) \hat{\mathcal{K}}\left(\mathbf{i}\rho^2 t e^{2\mathbf{i}s}\right) ds,
\end{aligned}
$$

where

$$\hat{\mathcal{K}}(\zeta) = \sqrt{\pi} \sum_{n=0}^\infty \frac{\zeta^n}{\Gamma\left(n+\frac{1}{2}\right)}.$$

It turns out that $\hat{\mathcal{K}}$ has a closed form expression on the positive real line, in terms of the Error Function. For any $x > 0$,

$$\mathcal{K}(x) = 1 + \sqrt{\pi x}e^x \operatorname{erf}\sqrt{x} = 1 + 2\sqrt{x}e^x \int_0^{\sqrt{x}} e^{-t^2} dt.$$

The converse transformation, using the same steps backwards, gives $\mathcal{D}_B[q]$ in terms of $\mathcal{D}'_B[q]$. □

6. Conclusion

We have analyzed the polarimetric measurement with variable incident angle experiment described in Lionheart & Newton [1]. After a detailed inspection of the various scales involved in the problem, we produced two related one-dimensional reconstruction problems, Problem A given by (29) and Problem B given by (30). We argued that these problems are relevant for the two Liquid Crystal Cell configurations we considered, orthorhombic and nematic uniaxial. Both problems have a simple formulation, and can be studied by applied analysts with no prior exposure to Liquid Crystals.

We partially addressed Problem A, under a smoothness assumption. Further investigation of this problem would be of practical and theoretical interest. In the smooth case, it would be interesting to see how the boundary data can be recovered in an optimal manner numerically. The non-smooth case is left open. While we do not believe that much more can be done with very general parameters, it is very possible the ill-posedness of this problem is dramatically reduced when additional constraints are imposed, e.g. when the medium is piecewise linear.

We addressed Problem B, by characterising a class of equivalent parameters, and by providing a uniqueness result when the coefficient is strictly increasing and C^1. In that case, we show that the reconstruction amounts to of the inversion of the Laplace Transform of the Hankel Transform of a target function. We have not attempted to address the numerical resolution of such a notoriously ill-posed problem: it is very possible that it is manageable when restricted to a particular class of functions.

Acknowledgements

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Appendix A. Proof of Proposition 2

Proposition 18. [14] *For any two solutions Z_1 and Z_2 of* (7), *$\bar{Z}_1^T D Z_2$ is independent of t, where* $D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

Proof. We compute that

$$
\begin{aligned}
\frac{d}{dt}\left(\bar{Z}_1^T D Z_2\right) &= \frac{d}{dt}(\bar{Z}_1^T) D Z_2 + \bar{Z}_1^T D \frac{d}{dt}(Z_2) \\
&= -\frac{i}{\epsilon}\bar{Z}_1^T B^T D Z_2 + \bar{Z}_1^T D \frac{i}{\epsilon} B Z_2 \\
&= \mathbf{i}\frac{i}{\epsilon}\bar{Z}_1^T(DB - B^T D)Z_2,
\end{aligned}
$$

and since $DB = B^T D$, we obtain that $\bar{Z}_1^T D Z_2$ does not depend on t.   □

We now prove Proposition 2.

Proof of Proposition 2. Since the entries of the matrix B is are piecewise continuous functions, global existence and uniqueness of the fundamental solution of the system is well-known see e.g. [15]. Now we prove the existence and uniqueness of solutions to the transmission problem via the fundamental solution. Let us define the fundamental solution of equation (7) as follows

$$\Phi(t) = (\phi_1, \phi_2, \phi_3, \phi_4),$$

where $\phi_j = [\phi_{1j}, \phi_{2j}, \phi_{3j}, \phi_{4j}]^T$, $j = 1, 2, 3, 4$ with $\Phi(0) = \text{Id}$.

We have $\Phi(1)\tilde{Z}(0) = \tilde{Z}(1)$ therefore

(41)
$$A_1(1) \begin{bmatrix} R_1 \\ R_2 \\ T_1 \\ T_2 \end{bmatrix} = A_2(1) \begin{bmatrix} 0 \\ 0 \\ I_1 \\ I_2 \end{bmatrix}$$

where

$$A_1 = \begin{bmatrix} \phi_{12} - \phi_{11} & \phi_{14} - \phi_{13} & -1 & 0 \\ \phi_{22} - \phi_{21} & \phi_{24} - \phi_{23} & -1 & 0 \\ \phi_{32} - \phi_{31} & \phi_{34} - \phi_{33} & 0 & -1 \\ \phi_{42} - \phi_{41} & \phi_{44} - \phi_{43} & 0 & -1 \end{bmatrix}$$

and

$$A_2 = \begin{bmatrix} -1 & 0 & -(\phi_{11} + \phi_{12}) & -(\phi_{13} + \phi_{14}) \\ 1 & 0 & -(\phi_{21} + \phi_{22}) & -(\phi_{23} + \phi_{24}) \\ 0 & -1 & -(\phi_{31} + \phi_{32}) & -(\phi_{33} + \phi_{34}) \\ 0 & 1 & -(\phi_{41} + \phi_{42}) & -(\phi_{43} + \phi_{44}) \end{bmatrix}.$$

To establish the announced existence and uniqueness property, let us now show that for any $t \in [0, 1]$, $|\det A_1| \geq 2$. Let $U = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}$ and $V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}$ be given by

$U_i = \phi_{i2} - \phi_{i1}$ and $V_i = \phi_{i4} - \phi_{i3}$.

Then the determinant of A_1 satisfies

$$\det(A_1) = \begin{vmatrix} U_1 - U_2 & V_1 - V_2 \\ U_3 - U_4 & V_3 - V_4 \end{vmatrix}.$$

Set $C_1 = \begin{bmatrix} U_1 - U_2 \\ U_3 - U_4 \end{bmatrix}$ and $C_2 = \begin{bmatrix} V_1 - V_2 \\ V_3 - V_4 \end{bmatrix}$. We compute that

$$C_1 \cdot \bar{C}_2 = (U_1 \bar{V}_1 + U_2 \bar{V}_2 - U_1 \bar{V}_2 - U_2 \bar{V}_1) + (U_3 \bar{V}_3 + U_4 \bar{V}_4 - U_3 \bar{V}_4 - U_4 \bar{V}_3).$$

From the identity $\Phi(0) = \text{Id}$, we deduce that $\bar{V}^T(0)DU(0) = 0$, Since U and V are two solutions of (7), Proposition 18 shows that this implies $\bar{V}^T DU = 0$ for all $t \in [0, 1]$. Expressing this identity in terms of the components of U and V, we obtain

$$(U_1 \bar{V}_2 + U_2 \bar{V}_1) + (U_3 \bar{V}_4 + U_4 \bar{V}_3) = 0,$$

and in turn

(42)
$$C_1 \cdot \bar{C}_2 = U_1 \bar{V}_1 + U_2 \bar{V}_2 + U_3 \bar{V}_3 + U_4 \bar{V}_4 = U \cdot \bar{V}.$$

Cauchy-Schwarz inequality thus shows that

$$C_1 \cdot \bar{C}_2 = \lambda N(U) N(V),$$

with $\lambda \in \mathbb{C}$ and $|\lambda| \leq 1$. Similarly, starting from the identity $\bar{U}^T(0)DU(0) = -2$ and $\bar{V}^T(0)DV(0) = -2$ we obtain $C_1 \cdot \bar{C}_1 = N(U)^2 + 2$ and $C_2 \cdot \bar{C}_2 = N(V)^2 + 2$. Therefore

$$|\det(A_1)|^2 \qquad = \det(\bar{A_1}^T)\det(A_1) = \begin{vmatrix} C_1 \cdot \bar{C}_1 & C_2 \cdot \bar{C}_1 \\ C_1 \cdot \bar{C}_2 & C_2 \cdot \bar{C}_2 \end{vmatrix}$$

$$= (1 - |\lambda|^2)N(U)^2 N(V)^2 + 4 + 2N(U)^2 + 2N(V)^2$$

$$\geq 4.$$

$\square$

Appendix B. Proof of Proposition 16 and Proposition 4

Problem 29 can be formulated slightly more generally as follows. Given an initial condition

$$u(0) = 1 + r, v(0) = 1 - r,$$

find $r \in \mathbb{C}$, $t \in \mathbb{C}$ such that $u(1) = v(1) = t$, when u and v satisfy

$$\frac{d}{dx}u = \mathbf{i}f(x)v, \qquad \frac{d}{dx}v = \mathbf{i}g(x)u,$$

with f and g two positive functions, bounded above and below by positive constants. This applies to system (15) with $v = z_3$, $u = z_4$, $\eta f = \lambda$, and $\eta \lambda g = \lambda^2 + \varepsilon_{22} - 1$. Note u is a solution of the real valued elliptic equations

$$(43) \qquad \frac{d}{dx}\left(\frac{1}{f}\frac{d}{dx}w\right) + g\,w = 0,$$

The general uniqueness results for ordinary differential equations show that u is a linear combination of the two fundamental solutions w_1 and w_2, of (43) corresponding to the initial conditions $w_1(0) = 1 \frac{1}{f(0)}\frac{d}{dx}w_1(0) = 0$ and $w_2(0) = 0$, $\frac{1}{f(0)}\frac{d}{dx}w_2(0) = 1$.

$$u(t) = Aw_1(t) + Bw_2(t).$$

Note that the Wronskian of the problem is constant:

$$\frac{1}{f(x)}\left(w_1\frac{d}{dx}w_2 - w_2\frac{d}{dx}w_1\right) \equiv 1.$$

Using the initial and final conditions, we obtain

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1+r \\ \mathbf{i}(1-r) \end{bmatrix} \text{ and } \begin{bmatrix} w_1(1) & w_2(1) \\ \frac{1}{f(1)}\frac{d}{dx}w_1(1) & \frac{1}{f(1)}\frac{d}{dx}w_2(1) \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} t \\ \mathbf{i}t \end{bmatrix}$$

which leads to the following formulae for t and r,

$$t = \frac{2}{e_2(1) + \mathbf{i}e_1(1)} \text{ and } r = \frac{\mathbf{i}e_1(1) - e_2(1)}{e_2(1) + \mathbf{i}e_1(1)}.$$

where $e_i = \frac{1}{f(x)}\frac{d}{dx}w_i - \mathbf{i}w_i$ for $i = 1, 2$. Using the Wronskian identity, we note that t is always well defined, as a simple calculation shows that

$$\frac{4}{|t|^2} \qquad = |e_1(1)|^2 + |e_2(1)|^2 + 2$$

$$= \left(\frac{1}{f(1)}\frac{d}{dx}w_2(1)\right)^2 + \left(\frac{1}{f(1)}\frac{d}{dx}w_1(1)\right)^2 + (w_2(1))^2 + (w_1(1))^2 + 2.$$

One can check that $|t|^2 + |r|^2 = 1$, therefore the amplitude of the reflected field, if it was available, would be redundant.

To prove that t is unchanged if f and g are replaced by $\tilde{f}(x) = f(1-x)$ and $\tilde{g}(x) = g(1-x)$, note the solution $\tilde{u}$ corresponding to $\tilde{f}$ and $\tilde{g}$ can be written as linear combination of $w_1(1-x)$ and $w_2(1-x)$. The associated reflection and transmission coefficients $\tilde{r}$ and $\tilde{t}$ satisfy

$$\left[\begin{array}{c} A \\ -B \end{array} \right] = \left[\begin{array}{c} t \\ \mathbf{i}t \end{array} \right] \text{ and } \left[\begin{array}{cc} w_1(1) & w_2(1) \\ -\frac{1}{f(1)}\frac{d}{dx}w_1(1) & -\frac{1}{f(1)}\frac{d}{dx}w_2(1) \end{array} \right] \left[\begin{array}{c} A \\ B \end{array} \right] = \left[\begin{array}{c} 1+r \\ \mathbf{i}(1-r) \end{array} \right]$$

which shows that $t = \tilde{t}$ and $r = \tilde{r}$.

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