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The focus of this paper is the study of the regularity properties of the time harmonic Maxwell's equations with anisotropic complex coefficients, in a bounded domain with $C^{2,1}$ boundary. We assume that at least one of the material parameters is $W^{1,3+\delta}$ for some $\delta > 0$. Using regularity theory for second order elliptic partial differential equations, we derive $W^{1,p}$ estimates and Hölder estimates for electric and magnetic fields up to the boundary. We also derive interior estimates in bi-anisotropic media.

Keywords: Maxwell's equations, Hölder estimates, L^p regularity, anisotropic media, bi-anisotropic media.

35Q61, 35J57, 35B65, 35Q60

1. Introduction

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded and connected open set in $\mathbb{R}^3$, with $C^{2,1}$ boundary. Let $\varepsilon, \mu \in L^\infty\left(\Omega; \mathbb{C}^{3\times 3}\right)$ be two bounded complex matrix-valued functions with uniformly positive definite real parts and symmetric imaginary parts. In other words, there exists a constant $\Lambda > 0$ such for any $\lambda \in \mathbb{C}^3$ there holds

$$2\Lambda \left|\lambda\right|^2 \leq \overline{\lambda} \cdot \left(\varepsilon + \overline{\varepsilon}^T\right) \lambda,\ 2\Lambda \left|\lambda\right|^2 \leq \overline{\lambda} \cdot \left(\mu + \overline{\mu}^T\right) \lambda, \text{ and } \left|\mu\right| + \left|\varepsilon\right| \leq \Lambda^{-1} \text{ a.e. in } \Omega. \tag{1.1}$$

where a^T is the transpose of a, $\overline{a} = \Re(a) - \mathbf{i}\Im(a)$, where $\mathbf{i}^2 = -1$, and $|x| = \mathrm{Trace}(\overline{x}^T x)$ is the Euclidean norm. The 3×3 matrix ε represents the complex electric permittivity of the medium Ω: its real part is the physical electric permittivity, whereas its imaginary part is proportional to the electric conductivity, by Ohm's Law. The 3×3 matrix μ stands for the complex magnetic permeability: the imaginary part may model magnetic dissipation or lag time.

For a given frequency $\omega \in \mathbb{C} \setminus \{0\}$ and current sources J_e and J_m in $L^2\left(\Omega; \mathbb{C}^3\right)$ we are interested in the regularity of the time-harmonic electromagnetic fields E and H, that is, the weak solutions E and H in $H(\mathrm{curl}, \Omega)$ of the time-harmonic anisotropic Maxwell's equations

$$\begin{cases} \mathrm{curl} H = \mathbf{i}\omega\varepsilon E + J_e & \text{in } \Omega, \\ \mathrm{curl} E = -\mathbf{i}\omega\mu H + J_m & \text{in } \Omega, \\ E \times \nu = G \times \nu \text{ on } \partial\Omega. \end{cases} \tag{1.2}$$

The boundary constraint is to be understood in the sense of traces, with $G \in H(\mathrm{curl}, \Omega)$. Our focus is the dependence of the regularity of E and H on the coefficients ε and μ, the current sources J_e and J_m, and the boundary condition G. The dependence on the regularity of the boundary of Ω (see e.g. [1,10,5]) is beyond the scope of this work. For $k \in \mathbb{N}^*$ and $p > 1$ we denote by $W^{k,p}(\mathrm{curl}, \Omega)$ and $W^{k,p}(\mathrm{div}, \Omega)$ the Banach spaces

$$W^{k,p}(\mathrm{curl}, \Omega) = \{v \in W^{k-1,p}(\Omega; \mathbb{C}^3) : \mathrm{curl} v \in W^{k-1,p}(\Omega; \mathbb{C}^3)\},$$
$$W^{k,p}(\mathrm{div}, \Omega) = \{v \in W^{k-1,p}(\Omega; \mathbb{C}^3) : \mathrm{div} v \in W^{k-1,p}(\Omega; \mathbb{C})\},$$

equipped with the canonical norms. The space $H(\mathrm{curl}, \Omega)$ mentioned above is $W^{1,2}(\mathrm{curl}, \Omega)$, whereas $W^{1,2}(\mathrm{div}, \Omega)$ is commonly denoted by $H(\mathrm{div}, \Omega)$. Throughout this paper, $H^1(\Omega) = W^{1,2}(\Omega; \mathbb{C}^3)$ and $L^2(\Omega) = L^2(\Omega; \mathbb{C}^3)$.

It is very well known that when the domain is a cylinder $\Omega' \times (0, L)$, the electric field E has only one component, $E = (0, 0, u)^T$, the physical parameters are real, scalar and do not depend on the third variable, then u satisfies a second order elliptic equation in the first two variables

$$\mathrm{div}\left(\mu^{-1} \nabla u\right) + \omega^2 \varepsilon u = 0 \text{ in } \Omega'.$$

In such a case, the regularity of u follows from the classical elliptic regularity theory. In particular, u is Hölder continuous due to the De Giorgi–Nash Theorem (at least in the interior). The regularity of E and H is less clear when the material parameters are anisotropic and/or complex valued. For general non diagonal elliptic systems with non regular coefficients, Müller and Šverák [19] have shown that the solutions may not to be in $W^{1,2+\delta}$ for any $\delta > 0$. Assuming that the coefficients are real, anisotropic, suitably smooth matrices, Leis [16] established well-posedness in $H^1(\Omega)$. The regularity of the coefficients was reduced to globally Lipschitz in Weber [23], for a C^2 smooth boundary, and C^1 for a $C^{1,1}$ domain in Costabel [9].

As far as the authors are aware, neither the H^1 nor the Hölder regularity of the electric and magnetic fields for complex anisotropic less than Lipschitz media have been addressed so far. Anisotropic dielectric parameters have received a renewed attention in the last decades. They appear for example in the mathematical theory of liquid crystals, in optically chiral media, and in meta-materials. In this work we show that the theory of elliptic boundary value problems can be used to study the general case of complex anisotropic coefficients.

Our first result addresses the $H^1(\Omega)$ regularity of E.

Theorem 1.1. *Assume that* (1.1) *holds, and that ε also satisfies*

$$\varepsilon \in W^{1,3+\delta}\left(\Omega; \mathbb{C}^{3\times 3}\right) \text{ for some } \delta > 0. \tag{1.3}$$

Suppose that the source terms J_m, J_e and G satisfy

$$J_m \in L^p\left(\Omega; \mathbb{C}^3\right),\ J_e \in W^{1,p}(\mathrm{div}, \Omega) \text{ and } G \in W^{1,p}\left(\Omega; \mathbb{C}^3\right), \tag{1.4}$$

for some $p \geq 2$. If $E, H \in H(\mathrm{curl}, \Omega)$ are weak solutions of (1.2)*, then $E \in H^1(\Omega)$ and*

$$\|E\|_{H^1(\Omega)} \leq C\big(\|E\|_{H(\mathrm{curl},\Omega)} + \|G\|_{H^1(\Omega)} + \|J_m\|_{L^2(\Omega)} + \|J_e\|_{H(\mathrm{div},\Omega)}\big), \tag{1.5}$$

for some constant C depending on Ω, Λ given in (1.1)*, ω and $\|\varepsilon\|_{W^{1,3+\delta}(\Omega;\mathbb{C}^{3\times3})}$ only.*

Note that no regularity assumption is made on μ, apart from (1.1). Our second result is devoted to the $H^1(\Omega)$ regularity of H.

Theorem 1.2. *Assume that* (1.1) *holds, and that μ also satisfies*

$$\mu \in W^{1,3+\delta}\left(\Omega;\mathbb{C}^{3\times3}\right) \text{ for some } \delta > 3. \tag{1.6}$$

Suppose that the source terms J_e, J_m and G satisfy

$$J_e \in L^p\left(\Omega;\mathbb{C}^3\right),\ J_m \in W^{1,p}(\operatorname{div},\Omega),\ J_m\cdot\nu \in W^{1-\frac{1}{p},p}(\partial\Omega,\mathbb{C}) \text{ and } G \in W^{1,p}\left(\Omega;\mathbb{C}^3\right), \tag{1.7}$$

for some $p \geq 2$. If $E, H \in H(\mathrm{curl},\Omega)$ are weak solutions of (1.2)*, then $H \in H^1(\Omega)$ and*

$$\|H\|_{H^1(\Omega)} \leq C\big(\|H\|_{H(\mathrm{curl},\Omega)}+\|G\|_{H^1(\Omega)}+\|J_e\|_{L^2(\Omega)}+\|J_m\|_{H(\mathrm{div},\Omega)}+\|J_m\cdot\nu\|_{H^{1/2}(\partial\Omega;\mathbb{C})}\big),$$

for some constant C depending on Ω, Λ given in (1.1)*, ω and $\|\mu\|_{W^{1,3+\delta}(\Omega;\mathbb{C}^{3\times3})}$ only.*

Naturally, interior regularity for H follows from the interior regularity of E, due to the (almost) symmetrical role of the pairs (E,ε) and (H,μ) in Maxwell's equations. The difference between Theorem 1.1 and Theorem 1.2 comes from the fact that (1.2) involves a boundary condition on E, not on H. Combining both results, we then show that when ε and μ are both $W^{1,3+\delta}$ with $\delta > 3$, then E and H enjoy the regularity inherited from the source terms, up to $W^{1,3+\delta}$.

Theorem 1.3. *Suppose that the hypotheses of Theorems 1.1 and 1.2 hold.*

If E and H in $H(\mathrm{curl},\Omega)$ are weak solutions of (1.2)*, then $E, H \in W^{1,q}\left(\Omega;\mathbb{C}^3\right)$ with $q = \min(p, 3+\delta)$ and*

$$\begin{aligned}\|E\|_{W^{1,q}(\Omega;\mathbb{C}^3)} + \|H\|_{W^{1,q}(\Omega;\mathbb{C}^3)} \leq C\big(&\|E\|_{L^2(\Omega)} + \|H\|_{L^2(\Omega)} + \|G\|_{W^{1,p}(\Omega;\mathbb{C}^3)} \\ &+ \|J_e\|_{W^{1,p}(\mathrm{div},\Omega)} + \|J_m\|_{W^{1,p}(\mathrm{div},\Omega)} + \|J_m\cdot\nu\|_{W^{1-\frac{1}{p},p}(\partial\Omega;\mathbb{C})}\big),\end{aligned}$$

for some constant C depending on Ω, Λ, ω, q, $\|\varepsilon\|_{W^{1,3+\delta}(\Omega;\mathbb{C}^{3\times3})}$ and $\|\mu\|_{W^{1,3+\delta}(\Omega;\mathbb{C}^{3\times3})}$ only. In particular, if $p > 3$, then $E, H \in \mathcal{C}^{0,\alpha}\left(\overline{\Omega};\mathbb{C}^3\right)$ with $\alpha = \min\left(1-\frac{3}{p}, \frac{\delta}{3+\delta}\right)$.

As an extension of this work, we show in Section 3 that, as far as interior regularity is concerned, the analog of Theorem 1.3 holds for more general constitutive relations, for which Maxwell's equations read

$$\begin{cases} \mathrm{curl}H = \mathbf{i}\omega\left(\varepsilon E + \xi H\right) + J_e & \text{in } \Omega, \\ \mathrm{curl}E = -\mathbf{i}\omega\left(\zeta E + \mu H\right) + J_m & \text{in } \Omega, \end{cases} \tag{1.8}$$

provided that $\zeta, \xi \in L^\infty\left(\Omega;\mathbb{C}^{3\times3}\right)$ are small enough to preserve the underlying elliptic structure of the system – see condition (3.2). These constitutive relations are commonly used to model the so called bi-anisotropic materials.

Our approach is classical and fundamentally scalar. It is oblivious of the fact that Maxwell's equations is posed on vectors, as we consider the problem component per component, just like it is done in Leis [17]. A general L^p theory for vector potentials has been developed very recently by Amrouche & Seloula [2,3]. Applying their results would lead to similar regularity results for scalar coefficients. It seems our approaches are completely independent, even though both are based on the L^p theory for elliptic equations.

Finally, Section 4 is devoted to the case when only one of the two coefficients is complex-valued. We consider the case when $\varepsilon \in W^{1,3+\delta}\left(\Omega;\mathbb{C}^{3\times 3}\right)$, with $\delta > 3$, and $\mu \in L^\infty(\Omega,\mathbb{R}^{3\times 3})$. In that situation, a Helmholtz decomposition of the magnetic field into $H = T + \nabla p$, where $T \in H^1(\Omega)$ is divergence free, provides additional insight on the regularity of H. Indeed, the potential p then satisfies a real scalar second order elliptic equation, and therefore enjoys additional regularity properties.

Theorem 1.4. *Suppose that the hypotheses of Theorem 1.1 hold for some $p > 3$. Assume additionally that Ω is simply connected and that $\Im\mu = 0$.*

If E and H in $H(\mathrm{curl},\Omega)$ are weak solutions of (1.2), then there exists $0 < \alpha \leq \min(1-\frac{3}{p}, \frac{\delta}{3+\delta})$ depending only on Ω and Λ given in (1.1) such that $E \in C^{0,\alpha}(\overline{\Omega};\mathbb{C}^3)$ with

$$\|E\|_{C^{0,\alpha}(\overline{\Omega};\mathbb{C}^3)} \leq C\big(\|E\|_{L^2(\Omega)} + \|G\|_{W^{1,p}(\Omega;\mathbb{C}^3)} + \|J_e\|_{W^{1,p}(\mathrm{div},\Omega)} + \|J_m\|_{L^p(\Omega;\mathbb{C}^3)}\big),$$

for some constant C depending on Ω, Λ, ω and $\|\varepsilon\|_{W^{1,3+\delta}(\Omega;\mathbb{C}^{3\times 3})}$ only.

This is a generalization of the result proved by Yin [24] who assumed $\varepsilon \in W^{1,\infty}(\Omega;\mathbb{C})$ and $\mu \in L^\infty(\Omega;\mathbb{R})$: this is not the minimal regularity requirement to prove Hölder continuity of the electric field.

We do not claim that requiring that (one of) the parameters is in $W^{1,3+\delta}$ for some $\delta > 0$ is optimal. We are confident that it is sufficient to assume that the derivatives are in the Campanato space $L^{3,\lambda}$ with $\lambda > 0$, for example. However, as we do not know that these are necessary conditions, it seemed that such a level of sophistication was unjustified in this work. Assuming simply $W^{1,3}$ regularity (i.e., $\lambda = 0$) does not seem to work with our proof: the bootstrap argument we use stalls in this case. A completely different approach would be required to handle the case of coefficients with less than VMO regularity.

Our paper is structured as follows. Section 2 is devoted to the proof of Theorems 1.1, 1.2 and 1.3. Section 3 is devoted to the statement of our result for the generalized bi-anisotropic Maxwell's equations; the proof of this result is given in the appendix. Section 4 focuses on the particular case when μ is real-valued and is devoted to the proof of Theorem 1.4.

2. $W^{1,p}$ regularity for E and H

In this section, we investigate the $W^{1,p}$ regularity of E and H. Our strategy is to consider a coupled elliptic system satisfied by each component of the electric and magnetic field, where in each equation, only one component appears in the leading order term. In a first step, we show that the electric and magnetic fields are very weak solutions of such a system. This system was already introduced, in its strong form, in Leis [17], and was used recently in Nguyen & Wang [20].

Proposition 2.1. *Assume that* (1.1) *holds. Let* $E = (E_1, E_2, E_3)^T$ *and* $H = (H_1, H_2, H_3)^T$ *in* $H(\mathrm{curl}, \Omega)$ *be weak solutions of* (1.2).

- *If* (1.3) *and* (1.4) *hold, for each* $k = 1, 2, 3$, E_k *is a very weak solution of*
$$-\mathrm{div}\left(\varepsilon \nabla E_k\right) = \mathrm{div}\left(\left(\partial_k \varepsilon\right) E - \varepsilon\left(\mathbf{e}_k \times \left(J_m - \mathbf{i}\omega\mu H\right)\right) - \mathbf{i}\omega^{-1}\mathbf{e}_k \mathrm{div} J_e\right) \text{ in } \Omega, \tag{2.1}$$
where $\mathbf{e}_k$ *is the unit vector in the* k*-th direction. More precisely,* E_k *satisfies for any* $\varphi \in W^{2,2}(\Omega; \mathbb{C})$
$$\int_\Omega E_k \mathrm{div}\left(\varepsilon^T \nabla \overline{\varphi}\right) dx = \int_{\partial\Omega} (\partial_k \overline{\varphi}) \varepsilon E \cdot \nu \, ds - \int_{\partial\Omega} \left(\mathbf{e}_k \times (E \times \nu)\right) \cdot \left(\varepsilon^T \nabla \overline{\varphi}\right) d\sigma$$
$$+ \int_\Omega \left((\partial_k \varepsilon) E - \varepsilon\left(\mathbf{e}_k \times \left(J_m - \mathbf{i}\omega\mu H\right)\right) - \mathbf{i}\omega^{-1}\mathbf{e}_k \mathrm{div} J_e\right) \cdot \nabla \overline{\varphi} \, dx. \tag{2.2}$$

- *If* (1.6) *and* (1.7) *hold, for each* $k = 1, 2, 3$, H_k *is a very weak solution of*
$$-\mathrm{div}\left(\mu \nabla H_k\right) = \mathrm{div}\left((\partial_k \mu) H - \mu\left(\mathbf{e}_k \times \left(J_e + \mathbf{i}\omega\varepsilon E\right)\right) + \mathbf{i}\omega^{-1}\mathbf{e}_k \mathrm{div} J_m\right) \text{ in } \Omega. \tag{2.3}$$
More precisely, H_k *satisfies for any* $\varphi \in W^{2,2}(\Omega; \mathbb{C})$
$$\int_\Omega H_k \mathrm{div}\left(\mu^T \nabla \overline{\varphi}\right) dx = \int_{\partial\Omega} (\partial_k \overline{\varphi}) \mu H \cdot \nu \, ds - \int_{\partial\Omega} \left(\mathbf{e}_k \times (H \times \nu)\right) \cdot \left(\mu^T \nabla \overline{\varphi}\right) d\sigma$$
$$+ \int_\Omega \left((\partial_k \mu) H - \mu\left(\mathbf{e}_k \times \left(J_e + \mathbf{i}\omega\varepsilon E\right)\right) + \mathbf{i}\omega^{-1}\mathbf{e}_k \mathrm{div} J_m\right) \cdot \nabla \overline{\varphi} \, dx. \tag{2.4}$$

Proof. We detail the derivation of (2.2) for the sake of completeness. The derivation of (2.4) is similar, thanks to the intrinsic symmetry of Maxwell's equations (1.2).

We multiply the identity $\mathrm{curl} E = -\mathbf{i}\omega\mu H + J_m$ by $\overline{\Phi} = \overline{g}\mathbf{e}_l$ for some $g \in W^{1,2}(\Omega; \mathbb{C})$ integrate by parts and multiply the result by $\mathbf{e}_l$. We obtain
$$\mathbf{e}_l \int_\Omega \overline{g}\left(-\mathbf{i}\omega\mu H + J_m\right) \cdot \mathbf{e}_l \, dx = \mathbf{e}_l \int_\Omega E \cdot (\nabla \times \overline{\Phi}) \, dx - \mathbf{e}_l \int_{\partial\Omega} (E \times \nu) \cdot \overline{\Phi} \, d\sigma,$$
which can be written also as
$$\int_\Omega \overline{g}\left(-\mathbf{i}\omega\mu H + J_m\right) dx + \int_{\partial\Omega} \overline{g}\,(E \times \nu)\, d\sigma = \int_\Omega E \times \nabla \overline{g} \, dx$$
Note that since $E \in H(\mathrm{curl}, \Omega)$ by assumption, $E \times \nu$ is well defined in $H^{-\frac{1}{2}}\left(\partial\Omega; \mathbb{C}^3\right)$ and this formulation is valid. Next, we cross product this identity with $\mathbf{e}_k$, and take the scalar product with $\mathbf{e}_i$. Using the vector identity $a \times (b \times c) = (a \cdot c)\, b - (a \cdot b)\, c$ on the right-and side, we obtain
$$\mathbf{e}_i \cdot \int_\Omega \overline{g}\mathbf{e}_k \times \left(-\mathbf{i}\omega\mu H + J_m\right) dx + \mathbf{e}_i \cdot \int_{\partial\Omega} \overline{g}\mathbf{e}_k \times (E \times \nu)\, d\sigma = \int_\Omega E_i \partial_k \overline{g} - E_k \partial_i \overline{g} \, d\sigma, \tag{2.5}$$

for any i and k in $\{1,2,3\}$ and $g \in W^{1,2}(\Omega;\mathbb{C})$. In view of (1.3), we have that $\overline{\varepsilon}^T\nabla\varphi \in H^1(\Omega)$ for any $\varphi \in W^{2,2}(\Omega;\mathbb{C})$. Thus, applying (2.5) with $g = \left(\overline{\varepsilon}^T\nabla\varphi\right)_i$ for any $i = 1,2,3$ and $\varphi \in W^{2,2}(\Omega;\mathbb{C})$ we find that

$$\int_\Omega E_i\,\partial_k\left(\varepsilon^T\nabla\overline{\varphi}\right)_i\,dx = \int_\Omega E_k\,\partial_i\left(\varepsilon^T\nabla\overline{\varphi}\right)_i\,dx + \mathbf{e}_i\cdot\int_{\partial\Omega}\left(\varepsilon^T\nabla\overline{\varphi}\right)_i\mathbf{e}_k\times(E\times\nu)\,d\sigma$$
$$+\,\mathbf{e}_i\cdot\int_\Omega\left(\varepsilon^T\nabla\overline{\varphi}\right)_i\mathbf{e}_k\times(-\mathbf{i}\omega\mu H + J_m)\,dx.$$

Summing over i, this yields

$$\int_\Omega E\cdot\partial_k\left(\varepsilon^T\nabla\overline{\varphi}\right)\,dx = \int_\Omega E_k\,\mathrm{div}\left(\varepsilon^T\nabla\overline{\varphi}\right)\,dx$$
$$+\int_{\partial\Omega}(\mathbf{e}_k\times(E\times\nu))\cdot(\varepsilon^T\nabla\overline{\varphi})\,d\sigma + \int_\Omega\varepsilon\,(\mathbf{e}_k\times(-\mathbf{i}\omega\mu H + J_m))\cdot\nabla\overline{\varphi}\,dx. \quad (2.6)$$

We then use the second part of Maxwell's equations. We test $\mathrm{curl}H - J_e = i\omega\varepsilon E$ against $\nabla\left(\partial_k\overline{\varphi}\right)\frac{1}{\mathbf{i}\omega}$ for $\varphi \in W^{2,2}(\Omega;\mathbb{C})$ and obtain

$$\int_\Omega\varepsilon E\cdot\partial_k\left(\nabla\overline{\varphi}\right)\,dx = -\mathbf{i}\omega^{-1}\int_\Omega\mathrm{curl}(H)\cdot\nabla\left(\partial_k\overline{\varphi}\right)\,dx + \mathbf{i}\omega^{-1}\int_\Omega J_e\cdot\nabla\left(\partial_k\overline{\varphi}\right)\,dx$$
$$= -\mathbf{i}\omega^{-1}\left(\int_{\partial\Omega}(\partial_k\overline{\varphi})\mathrm{curl}H\cdot\nu\,ds - \int_{\partial\Omega}\left(\partial_k\overline{\varphi}\right)J_e\cdot\nu\,ds + \int_\Omega\mathrm{div}J_e\partial_k\overline{\varphi}\,dx\right)$$
$$= -\mathbf{i}\omega^{-1}\left(\mathbf{i}\omega\int_{\partial\Omega}(\partial_k\overline{\varphi})\varepsilon E\cdot\nu\,ds + \int_\Omega\mathrm{div}J_e\partial_k\overline{\varphi}\,dx\right).$$

Since $J_e \in H\left(\mathrm{div},\Omega\right)$, the boundary term is well defined. Writing the left-hand side of the above identity in the form

$$\int_\Omega\varepsilon E\cdot\partial_k\left(\nabla\overline{\varphi}\right)\,dx = \int_\Omega E\cdot\partial_k\left(\varepsilon^T\nabla\overline{\varphi}\right)\,dx - \int_\Omega\left(\partial_k\varepsilon\right)E\cdot\nabla\overline{\varphi}\,dx,$$

we obtain

$$-\int_\Omega(\partial_k\varepsilon)E\cdot\nabla\overline{\varphi}\,dx + \int_\Omega E\cdot\partial_k\left(\varepsilon^T\nabla\overline{\varphi}\right)\,dx = \int_{\partial\Omega}(\partial_k\overline{\varphi})\varepsilon E\cdot\nu\,ds - \mathbf{i}\omega^{-1}\int_\Omega\mathrm{div}J_e\partial_k\overline{\varphi}\,dx$$

Inserting this identity in (2.6) we obtain (2.2). □

To transform the very weak identities given by Proposition 2.1 into regular weak formulations, we shall use the following lemma. Given $r \in (1,\infty)$, we write r' the solution of $\frac{1}{r}+\frac{1}{r'} = 1$.

Lemma 2.1. *Assume that* (1.1) *and* (1.3) *hold. Given* $r \geq 6/5$, $u \in L^2(\Omega;\mathbb{C}) \cap L^r(\Omega;\mathbb{C})$, $F \in (W^{1,r'}(\Omega;\mathbb{C}))'$, *let* B *the trace operator given either by* $B\varphi = \varphi$ *on* $\partial\Omega$ *or by* $B\varphi = \varepsilon^T\nabla\overline{\varphi}\cdot\nu$ *on* $\partial\Omega$ *for* $\varphi \in W^{2,2}(\Omega;\mathbb{C})$.

If for all $\varphi \in W^{2,2}(\Omega;\mathbb{C})$ *such that* $B\varphi = 0$ *there holds*

$$\int_\Omega u\,\mathrm{div}(\varepsilon^T\nabla\overline{\varphi})\,dx = \langle F,\varphi\rangle, \quad (2.7)$$

then $u \in W^{1,r}(\Omega;\mathbb{C})$ *and*

$$\|\nabla u\|_{L^r(\Omega;\mathbb{C}^3)} \leq C\,\|F\|_{(W^{1,r'}(\Omega;\mathbb{C}))'}\,, \tag{2.8}$$

for some constant C *depending on* Ω, Λ *given in* (1.1), $\|\varepsilon\|_{W^{1,3}(\Omega;\mathbb{C}^{3\times 3})}$ *and* r *only.*

Proof. We first observe that, since $r \geq 6/5$, then both terms of the identity (2.7) are well defined as $W^{2,2}(\Omega;\mathbb{C}) \subset W^{1,6}(\Omega;\mathbb{C})$ and $\frac{1}{6} + \frac{1}{6/5} = 1$. Let $\psi \in \mathcal{D}(\Omega)$ be a test function and fix $i = 1, 2$ or 3. Let $\varphi^* \in W^{1.2}(\Omega;\mathbb{C})$ be the unique solution to the problem

$$\begin{cases} \operatorname{div}(\varepsilon^T \nabla \overline{\varphi^*}) = \partial_i \psi & \text{in } \Omega, \\ B\varphi^* = 0 & \text{on } \partial\Omega. \end{cases}$$

In the case of the Neumann boundary condition, we add the normalization condition $\int_\Omega \varphi^*\,dx = 0$. Since $\varepsilon \in W^{1,3}\left(\Omega,\mathbb{C}^{3\times 3}\right)$, it is known [4] that for any $q \in (1,\infty)$ there holds

$$\|\varphi^*\|_{W^{1,q}(\Omega;\mathbb{C})} \leq C\,\|\psi\|_{L^q(\Omega;\mathbb{C})} \tag{2.9}$$

for some $C = C(q,\Omega,\Lambda,\|\varepsilon\|_{W^{1,3}(\Omega;\mathbb{C}^{3\times 3})}) > 0$. In particular, $\varphi^* \in W^{1,q}(\Omega;\mathbb{C})$ for all $q < \infty$. The usual difference quotient argument (see e.g. [15],[13]) shows in turn that $\varphi^* \in W^{2,2}(\Omega;\mathbb{C})$, as ψ is regular. Thus, by assumption we have

$$\left|\int_\Omega u\partial_i\psi\,dx\right| = \left|\int_\Omega u\operatorname{div}(\varepsilon^T\nabla\overline{\varphi^*})\,dx\right| = |\langle F,\varphi^*\rangle| \leq \|F\|_{(W^{1,r'}(\Omega;\mathbb{C}))'}\,\|\varphi^*\|_{W^{1,r'}(\Omega;\mathbb{C})}\,,$$

which in view of (2.9) gives

$$\left|\int_\Omega u\partial_i\psi\,dx\right| \leq C\,\|F\|_{(W^{1,r'}(\Omega;\mathbb{C}))'}\,\|\psi\|_{L^{r'}(\Omega;\mathbb{C})}\,,$$

as required. □

We now are equipped to write the main regularity proposition for E, which will lead to the proof of Theorem 1.1 by a bootstrap argument.

Proposition 2.2. *Assume that* (1.1)*,* (1.3) *and* (1.4) *hold. Assume that* $E, H \in H(\operatorname{curl},\Omega)$ *are solutions of* (1.2) *with* $G = 0$.

Suppose that $E \in L^q(\Omega;\mathbb{C}^3)$ *and* $H \in L^s(\Omega;\mathbb{C})$*, with* $2 \leq q, s < \infty$ *and write* $r = \min((3q+q\delta)(q+3+\delta)^{-1}, p, s)$. *Then* $E \in W^{1,r}(\Omega;\mathbb{C}^3)$ *and*

$$\|E\|_{W^{1,r}(\Omega;\mathbb{C}^3)} \leq C\big(\|E\|_{L^q(\Omega;\mathbb{C}^3)} + \|H\|_{L^s(\Omega;\mathbb{C}^3)} + \|J_e\|_{L^2(\Omega)} + \|J_m\|_{L^p(\Omega;\mathbb{C}^3)} + \|\operatorname{div}J_e\|_{L^p(\Omega;\mathbb{C})}\big), \tag{2.10}$$

for some constant C *depending on* Ω, Λ *given in* (1.1), ω, $\|\varepsilon\|_{W^{1,3+\delta}(\Omega;\mathbb{C}^{3\times 3})}$ *and* r *only.*

The corresponding proposition regarding H is as follows.

Proposition 2.3. *Assume that* (1.1)*,* (1.6) *and* (1.7) *hold. Assume that* $E, H \in H(\operatorname{curl},\Omega)$ *are solutions of* (1.2) *with* $G = 0$.

Suppose that $E \in L^s(\Omega;\mathbb{C}^3)$ and $H \in L^q(\Omega;\mathbb{C})$, with $2 \le q, s < \infty$ and write $r = \min((3q+q\delta)(q+3+\delta)^{-1}, p, s)$. Then $H \in W^{1,r}(\Omega;\mathbb{C}^3)$ and

$$\|H\|_{W^{1,r}(\Omega;\mathbb{C}^3)} \le C\big(\|H\|_{L^q(\Omega;\mathbb{C}^3)} + \|E\|_{L^s(\Omega;\mathbb{C}^3)} + \|J_m\|_{L^2(\Omega)} + \|J_e\|_{L^p(\Omega;\mathbb{C}^3)} + \|\mathrm{div} J_m\|_{L^p(\Omega;\mathbb{C})} + \|J_m\cdot\nu\|_{W^{1-\frac{1}{p},p}(\partial\Omega;\mathbb{C})}\big), \quad (2.11)$$

for some constant C depending on Ω, Λ given in (1.1)*, ω, $\|\mu\|_{W^{1,3+\delta}(\Omega;\mathbb{C}^{3\times 3})}$ and r only.*

We prove both propositions below. We are now ready to prove Theorems 1.1, 1.2 and 1.3.

Proof. [Proof of Theorems 1.1, 1.2 and 1.3] Let us prove Theorem 1.1 first. Considering the system satisfied by $E - G$ and H, we may assume $G = 0$. Since $H \in L^2\left(\Omega;\mathbb{C}^3\right)$, we may apply Proposition 2.2 with $p = s = 2$ a finite number of times with increasing values of q. For $q_n \ge 2$ we obtain $E \in W^{1,r_n}\left(\Omega;\mathbb{C}^3\right)$, with $r_n = \min(q_n(3+\delta)\left(q_n+3+\delta\right)^{-1}, 2)$. If $r_n = 2$, the result is proved. If $r_n < 2$, Sobolev embeddings show that $E \in L^{q_{n+1}}\left(\Omega;\mathbb{C}^3\right)$ with

$$q_{n+1} = q_n + \frac{\delta q_n^2}{9+\delta\left(3-q_n\right)} \ge q_n + \frac{4\delta}{9+\delta},$$

using the bounds $q_n \ge 2$ and $9 + \delta\left(3 - q_n\right) > 0$, which follows from $r_n < 2$. Thus the sequence r_n converges to 2 in a finite number of steps. Note that in estimate (1.5), H is bounded in terms of E and J_m using the simple bound

$$\Lambda\left|\omega\right|\|H\|_{L^2(\Omega;\mathbb{C}^3)} \le \|\mathrm{curl} E\|_{L^2(\Omega;\mathbb{C}^3)} + \|J_m\|_{L^2(\Omega;\mathbb{C}^3)},$$

which follows from (1.2). The proof of Theorem 1.2 is similar, using Proposition 2.3 in lieu of Proposition 2.2 to bootstrap.

Let us now turn to Theorem 1.3. Suppose first $p \le 3$ and $\delta < 3$. From Theorem 1.1 (resp. Theorem 1.2) and Sobolev Embeddings, we have $E \in L^6(\Omega;\mathbb{C}^3)$ (resp. $H \in L^6(\Omega;\mathbb{C}^3)$). We apply Propositions 2.2 and 2.3 a finite number of times, with $q = s$. Starting with $q_n \ge 6 = q_0$ we obtain E (and H) $\in W^{1,r_n}(\Omega;\mathbb{C}^3)$, with $r_n = \min(q_n(3+\delta)(3+\delta+q_n)^{-1}, p)$. If $r_n = p$, the result is proved. If $r_n < p$, Sobolev embeddings imply that E and H belong to $L^{q_{n+1}}(\Omega;\mathbb{C}^3)$, with

$$q_{n+1} = q_n + \frac{\delta q_n^2}{9+\delta\left(3-q_n\right)} \ge q_n + \frac{q_0^2\delta}{9+\delta\left(3-q_n\right)} \ge q_n + \frac{12\delta}{3-\delta},$$

since $q_n \ge 6$, $\delta < 3$ and $9 + \delta\left(3 - q_n\right) > 0$ (as $r_n < 3$). Thus the sequence r_n converges to p in a finite number of steps.

Suppose now $p > 3$ and $\delta \in (0, \infty)$. The previous argument shows that E and H are in $W^{1,3}(\Omega;\mathbb{C}^3)$. One more bootstrap concludes the proof if $p < 3 + \delta$, and shows otherwise that E and H are in $L^\infty(\Omega;\mathbb{C}^3)$, and the result is obtained by a final application of Propositions 2.2 and 2.3. □

We now prove Proposition 2.2.

Proof. [Proof of Proposition 2.2]We subdivide the proof in four steps.

Step 1. Variational formulation. Since $E \times \nu = 0$ on $\partial\Omega$, identity (2.2) shows that for every $\varphi \in W^{2,2}(\Omega;\mathbb{C})$ and $k = 1,2,3$ there holds

$$\int_\Omega E_k \mathrm{div}(\varepsilon^T \nabla\overline{\varphi})\, dx = \int_\Omega F_k \cdot \nabla\overline{\varphi}\, dx + \int_{\partial\Omega} (\partial_k \overline{\varphi})\varepsilon E \cdot \nu\, ds, \tag{2.12}$$

where we set

$$F_k = (\partial_k \varepsilon)\, E - \varepsilon\, (\mathbf{e}_k \times (J_m - \mathbf{i}\omega\mu H)\,) - \mathbf{i}\omega^{-1}\mathbf{e}_k\, \mathrm{div} J_e.$$

Since $(\partial_k\varepsilon)E \in L^{q(3+\delta)(q+3+\delta)^{-1}}(\Omega;\mathbb{C}^3)$, we have that $F_k \in L^r(\Omega;\mathbb{C}^3)$.

Step 2. Interior regularity. Given a smooth subdomain $\Omega_0 \Subset \Omega$, we consider a cut-off function $\chi \in C_0^\infty(\Omega;\mathbb{R})$ such that $\chi = 1$ in Ω_0. A direct calculation gives for $\varphi \in W^{2,2}(\Omega;\mathbb{C})$

$$\int_\Omega \chi E_k \mathrm{div}(\varepsilon^T \nabla\overline{\varphi})\, dx = \int_\Omega E_k \mathrm{div}(\varepsilon^T \nabla(\chi\overline{\varphi}))\, dx + T_k(\varphi),$$

where $T_k(\varphi) = -\int_\Omega E_k (\mathrm{div}(\varepsilon^T \overline{\varphi}\nabla\chi) + \varepsilon\nabla\chi \cdot \nabla\overline{\varphi})\, dx$. Thus, by (2.12) we obtain

$$\int_\Omega \chi E_k \mathrm{div}(\varepsilon^T \nabla\overline{\varphi})\, dx = \int_\Omega F_k \cdot \nabla(\chi\overline{\varphi})\, dx + T_k(\varphi),$$

since χ is compactly supported. Using Sobolev embeddings and the fact that F_k is in $L^r\left(\Omega;\mathbb{C}^3\right)$, we verify that $\varphi \mapsto \int_\Omega F_k \cdot \nabla(\chi\overline{\varphi})\, dx + T_k(\varphi)$ is in $(W^{1,r'}(\Omega;\mathbb{C}))'$ using Sobolev embeddings. Thanks to Lemma 2.1 we conclude that $\chi E_k \in W^{1,r}(\Omega;\mathbb{C})$, namely $E \in W^{1,r}(\Omega_0;\mathbb{C}^3)$.

Step 3. Boundary regularity. Take now $x_0 \in \partial\Omega$ and denote the tangent space to $\mathbb{R}^3$ in x_0 by $T_{x_0}\mathbb{R}^3$. Since $\partial\Omega$ has $C^{2,1}$ regularity, there exist a ball B centred in x_0 and a $\mathcal{C}^{1,1}$ field A of orthonormal matrices defined on B such that for every $x \in B \cap \partial\Omega$, $A(x)$ transforms the canonical basis of $T_x\mathbb{R}^3$ into an orthonormal basis formed by the outward normal $\nu(x)$ to $\partial\Omega$ and two tangential vectors $\mathbf{t}_1(x)$ and $\mathbf{t}_2(x)$. Namely, $A \in \mathcal{C}^{1,1}(B; SO(3))$ is such that for every $x \in B \cap \partial\Omega$ we have $A(x)v = v'$ if $v, v' \in \mathbb{R}^3$ and satisfy $\sum_{i=1}^3 v_i \mathbf{e}_i = v_1'\mathbf{t}_1(x) + v_2'\mathbf{t}_2(x) + v_3'\nu(x)$ in $T_x\mathbb{R}^3$. Take $\chi \in \mathcal{D}(B;\mathbb{R})$ such that $\chi = 1$ in a neighborhood $\tilde{B}$ of x_0 and write $\tilde{E} = \chi A E$ in $B \cap \Omega$.

Let us first consider the two tangential components of E, that is, $\tilde{E}_j$ with $j \in \{1,2\}$. We obtain that for every $\varphi \in W^{2,2}(\Omega;\mathbb{C}) \cap W_0^{1,2}(\Omega;\mathbb{C})$

$$\int_\Omega \tilde{E}_j \mathrm{div}(\varepsilon^T \nabla\overline{\varphi})\, dx = \sum_{k=1}^3 \int_\Omega E_k \mathrm{div}(\varepsilon^T \nabla(\chi A_{jk}\overline{\varphi}))\, dx + R(\varphi),$$

where $R(\varphi) = -\sum_{k=1}^3 \int_\Omega E_k \big(\mathrm{div}(\varepsilon^T \overline{\varphi}\nabla(\chi A_{jk})) + \varepsilon\nabla(\chi A_{jk}) \cdot \nabla\overline{\varphi}\big)\, dx$. Inserting the identity (2.12) we obtain

$$\int_\Omega \tilde{E}_j \mathrm{div}(\varepsilon^T \nabla\overline{\varphi})\, dx = \sum_{k=1}^3 \int_\Omega F_k \cdot \nabla(\chi A_{jk}\overline{\varphi})\, dx + \int_{\partial\Omega} \mathrm{div}\left(\chi A^T \mathbf{e}_j \overline{\varphi}\right) \varepsilon E \cdot \nu\, ds + R(\varphi). \tag{2.13}$$

Since $\overline{\varphi} = 0$ on $\partial\Omega$ and $\mathbf{t}_j = A^T\mathbf{e}_j$ on $B \cap \partial\Omega$, we have $0 = \nabla\overline{\varphi} \cdot A^T\mathbf{e}_j$ on $\partial\Omega \cap B$. In particular, $\operatorname{div}(\chi A^T\mathbf{e}_j\overline{\varphi}) = 0$ on $\partial\Omega$. Thus (2.13) becomes

$$\int_\Omega \tilde{E}_j \operatorname{div}(\varepsilon^T\nabla\overline{\varphi})\, dx = \sum_{k=1}^{3}\int_\Omega F_k \cdot \nabla(\chi A_{jk}\overline{\varphi})\, dx + R(\varphi), \qquad \varphi \in W^{2,2}(\Omega;\mathbb{C}) \cap H^1_0(\Omega;\mathbb{C}).$$

The functional $\varphi \mapsto \sum_{k=1}^3 \int_\Omega F_k \cdot \nabla(\chi A_{jk}\overline{\varphi})\, dx + R(\varphi)$ is in $(W^{1,r'}(\Omega;\mathbb{C}))'$, as A is $\mathcal{C}^{1,1}$. Thus, thanks to Lemma 2.1 we obtain that $\tilde{E}_1$ and $\tilde{E}_2$ are bounded in $W^{1,r}(\Omega;\mathbb{C})$.

Let us now turn to the normal component, $\tilde{E}_3$. Consider the second part of Maxwell's equations (1.2), $\operatorname{curl}E = -\mathbf{i}\omega\mu H + J_m$ in the quotient space $W^{-1,2}(\tilde{B};\mathbb{C}^3)/L^r(\tilde{B};\mathbb{C}^3)$, that is, identifying every element of $L^r(\tilde{B};\mathbb{C}^3)$ in that space with nought in the same space. We find

$$0 = -\operatorname{curl}E = -\operatorname{curl}\big(A^T\tilde{E}\big) = -\operatorname{curl}\big(A^T\mathbf{e}_3\tilde{E}_3\big) = \big(A^T\mathbf{e}_3\big) \times \nabla\tilde{E}_3,$$

since $-\mathbf{i}\omega\mu H + J_m \in L^r\left(\Omega;\mathbb{C}^3\right)$, $\tilde{E}_1, \tilde{E}_2 \in W^{1,r}(\Omega;\mathbb{C})$ and A is C^1. In other words,

$$\nabla\tilde{E}_3 = A^T\mathbf{e}_3\left(A^T\mathbf{e}_3 \cdot \nabla\tilde{E}_3\right) \text{ in } W^{-1,2}\big(\tilde{B};\mathbb{C}^3\big)/L^r\big(\tilde{B};\mathbb{C}^3\big). \tag{2.14}$$

Taking now the divergence of the first identity in Maxwell's equations (1.2), and using the fact that $\operatorname{div}J_e \in L^r(\Omega;\mathbb{C})$ and $E \in L^r\left(\Omega;\mathbb{C}^3\right)$ we obtain, in $W^{-1,2}\big(\tilde{B};\mathbb{C}\big)/L^r\big(\tilde{B};\mathbb{C}\big)$,

$$0 = \operatorname{div}(\varepsilon E) = \operatorname{div}\big(\big(\varepsilon A^T e_3\big)\,\tilde{E}_3\big) = \big(\varepsilon A^T e_3\big) \cdot \nabla\tilde{E}_3.$$

Thanks to (2.14) this implies

$$0 = \left(\varepsilon\left(A^T\mathbf{e}_3\right) \cdot \left(A^T\mathbf{e}_3\right)\right)\left(A^T\mathbf{e}_3 \cdot \nabla\tilde{E}_3\right) \text{ in } W^{-1,2}\big(\tilde{B};\mathbb{C}\big)/L^r\big(\tilde{B};\mathbb{C}\big).$$

Since by the ellipticity assumption (1.1), $\left(\varepsilon\left(A^T\mathbf{e}_3\right) \cdot \left(A^T\mathbf{e}_3\right)\right) \neq 0$, we have obtained $\nabla\tilde{E}_3 \in L^r\big(\tilde{B};\mathbb{C}^3\big)$, and therefore $E \in W^{1,r}\big(\tilde{B};\mathbb{C}^3\big)$.

Step 4. Global regularity. Combining the interior and the boundary regularities, a standard ball covering argument shows $E \in W^{1,r}(\Omega;\mathbb{C}^3)$. The estimate given in (2.10) follows from Lemma 2.1. □

We now turn to the proof of Proposition 2.3. Naturally, the interior estimates can be obtained in the exact same way, substituting the very weak formulations for the components of E by the corresponding identities for the components of H. The boundary estimates require additional care, and we detail this step below.

Proof. [Proof of Proposition 2.3. Boundary regularity] To avoid additional technicalities, we consider the situation $J_m \cdot \nu = 0$ on $\partial\Omega$. The general case follows from a lifting argument.

We first observe that, as $E \times \nu = 0$ on $\partial\Omega$, we may write

$$0 = \operatorname{div}_{\partial\Omega}(E \times \nu) = (\operatorname{curl}E) \cdot \nu = -\mathbf{i}\omega\mu H \cdot \nu + J_m \cdot \nu \text{ in } H^{-\frac{1}{2}}(\partial\Omega), \tag{2.15}$$

see e.g. [18] . In other words, $\mu H \cdot \nu = 0$ in $H^{-\frac{1}{2}}(\partial\Omega)$. Then, by (2.4), for every $\varphi \in W^{2,2}(\Omega;\mathbb{C})$ and $k = 1,2,3$ there holds

$$\int_\Omega H_k \mathrm{div}(\mu^T \nabla\overline{\varphi})\, dx = \int_\Omega G_k \cdot \nabla\overline{\varphi}\, dx - \int_{\partial\Omega} (\mathbf{e}_k \times (H \times \nu)) \cdot (\mu^T \nabla\overline{\varphi})\, d\sigma, \quad (2.16)$$

where

$$G_k = (\partial_k \mu) H - \mu\, (\mathbf{e}_k \times (J_e + \mathbf{i}\omega\varepsilon E)) + \mathbf{i}\omega^{-1} \mathbf{e}_k \mathrm{div} J_m \in L^r(\Omega;\mathbb{C}^3),$$

since $(\partial_k\mu)H \in L^{q(3+\delta)(q+3+\delta)^{-1}}(\Omega;\mathbb{C}^3)$.

Consider now the change of coordinates $A \in C^{1,1}$ in a ball B centred in $x_0 \in \partial\Omega$ already introduced in the proof of Proposition 2.2, and again let us focus on the tangential components first. Take $\chi \in \mathcal{D}(B;\mathbb{R})$ such that $\chi = 1$ in a neighborhood $\tilde{B}$ of x_0 and $j \in \{1,2\}$.

We choose a test function satisfying a Neumann type boundary condition, that is $\varphi \in W^{2,2}(\Omega;\mathbb{C})$ such that $\mu^T \nabla\overline{\varphi} \cdot \nu = 0$ on $\partial\Omega$. Defining $\tilde{H} = \chi A H$ in $B \cap \Omega$ we have

$$\int_\Omega \tilde{H}_j \mathrm{div}(\mu^T \nabla\overline{\varphi})\, dx = \sum_{k=1}^3 \int_\Omega H_k \mathrm{div}(\mu^T \nabla(\chi A_{jk} \overline{\varphi}))\, dx + R(\varphi),$$

where $R(\varphi) = -\sum_{k=1}^3 \int_\Omega H_k \big(\mathrm{div}(\mu^T \overline{\varphi} \nabla(\chi A_{jk})) + \mu \nabla(\chi A_{jk}) \cdot \nabla\overline{\varphi}\big)\, dx$. From the identity (2.16) we obtain

$$\int_\Omega \tilde{H}_j \mathrm{div}(\mu^T \nabla\overline{\varphi})\, dx = \sum_{k=1}^3 \int_\Omega G_k \cdot \nabla(\chi A_{jk}\overline{\varphi})\, dx + R(\varphi) + S(\varphi), \quad (2.17)$$

where

$$S(\varphi) = -\sum_{k=1}^3 \int_{\partial\Omega} (\mathbf{e}_k \times (H \times \nu)) \cdot (\mu^T \nabla(\chi A_{jk}\overline{\varphi}))\, d\sigma.$$

As before, the functional $\varphi \mapsto \sum_{k=1}^3 \int_\Omega G_k \cdot \nabla(\chi A_{jk}\overline{\varphi})\, dx + R(\varphi)$ is in $(W^{1,r'}(\Omega;\mathbb{C}))'$. We shall now prove that $S \in (W^{1,r'}(\Omega;\mathbb{C}))'$. Since $\mu^T \nabla\overline{\varphi} \cdot \nu = 0$ on $\partial\Omega$ we have

$$\sum_{k=1}^3 \chi\, (\mathbf{e}_k \times (H \times \nu)) \cdot (A_{jk} \mu^T \nabla\overline{\varphi}) = -\chi\, (H \cdot \mathbf{t}_j)\, \nu \cdot \big(\mu^T \nabla\overline{\varphi}\big) = 0,$$

as $\sum_{k=1}^3 A_{jk}\mathbf{e}_k = \mathbf{t}_j$, thus

$$S(\varphi) = -\sum_{k=1}^3 \int_{\partial\Omega} (\mathbf{e}_k \times (H \times \nu)) \cdot (\mu^T \nabla(\chi A_{jk}))\, \overline{\varphi}\, d\sigma.$$

By hypothesis we have $H \in W^{1,r}(\mathrm{curl},\Omega)^3$, whence $H \times \nu \in W^{-1/r,r}(\partial\Omega;\mathbb{C}^3)$. It follows that $(\mathbf{e}_k \times (H \times \nu)) \cdot (\mu^T \nabla(\chi A_{jk})) \in W^{-1/r,r}(\partial\Omega;\mathbb{C}^3)$. As a result (see [15]),

$$|S(\varphi)| \le C\, \|\varphi\|_{W^{1-\frac{1}{r'},r'}(\partial\Omega;\mathbb{C})} \le C\, \|\varphi\|_{W^{1,r'}(\Omega;\mathbb{C})},$$

for some $C > 0$ independent of φ; in other words $S \in (W^{1,r'}(\Omega;\mathbb{C}))'$. We can now apply Lemma 2.1 to (2.17) and obtain $\tilde{H}_j \in W^{1,r}(\Omega;\mathbb{C})$, namely $\tilde{H}_1, \tilde{H}_2 \in W^{1,r}(\Omega;\mathbb{C})$. The rest of the proof follows faithfully that of Proposition 2.2. $\square$

To conclude this section, we point out that higher interior regularity results follow naturally under appropriate assumptions.

Proposition 2.4. *Suppose that* (1.1) *holds and take* $m \in \mathbb{N}^*$. *Assume additionally that*

$$\varepsilon, \mu \in W^{m,3+\delta}\left(\Omega;\mathbb{C}^{3\times 3}\right), \ J_e, J_m \in W^{m,3+\delta}\left(\mathrm{div}, \Omega\right) \ \textit{for some } \delta > 0.$$

If E and H in $H\left(\mathrm{curl}, \Omega\right)$ are weak solutions of (1.2), *then* $E, H \in \mathcal{C}_{loc}^{m-1,\frac{\delta}{3+\delta}}\left(\Omega;\mathbb{C}^3\right)$ *and if* $\Omega_0 \Subset \Omega$ *then*

$$\begin{aligned}\|E\|_{\mathcal{C}^{m-1,\frac{\delta}{3+\delta}}(\Omega_0;\mathbb{C}^3)} + \|H\|_{\mathcal{C}^{m-1,\frac{\delta}{3+\delta}}(\Omega_0;\mathbb{C}^3)} \leq C\big(&\|E\|_{L^2(\Omega)} + \|H\|_{L^2(\Omega)} \\ &+ \|J_e\|_{W^{m,3+\delta}(\mathrm{div},\Omega)} + \|J_m\|_{W^{m,3+\delta}(\mathrm{div},\Omega)}\big),\end{aligned}$$

for some constant C depending on Ω, Ω_0, Λ given in (1.1), *ω, and the* $W^{m,3+\delta}\left(\Omega;\mathbb{C}^{3\times 3}\right)$ *norms of ε and μ only.*

Proof. The case $m = 1$ follows by Theorem 1.3. Then the result follows by an induction argument, applying classical Schauder estimates (see [13]) to (2.1) and (2.3). $\square$

3. Bi-anisotropic materials

In this section, we investigate the interior regularity of the solutions of the following problem

$$\begin{cases} \mathrm{curl} H = \mathbf{i}\omega\left(\varepsilon E + \xi H\right) + J_e & \text{in } \Omega, \\ \mathrm{curl} E = -\mathbf{i}\omega\left(\zeta E + \mu H\right) + J_m & \text{in } \Omega. \end{cases} \tag{1.8}$$

As far as the authors are aware, this question was previously studied only recently in [12], where the parameters are assumed to be at least Lipschitz continuous. In this more general context, hypothesis (1.1) is not sufficient to ensure ellipticity. As we will see in Proposition Appendix A.1, the leading order parameter for the coupled elliptic system is the tensor

$$A = A_{ij}^{\alpha\beta} = \begin{bmatrix} \Re\varepsilon & -\Im\varepsilon & \Re\xi & -\Im\xi \\ \Im\varepsilon & \Re\varepsilon & \Im\xi & \Re\xi \\ \Re\zeta & -\Im\zeta & \Re\mu & -\Im\mu \\ \Im\zeta & \Re\zeta & \Im\mu & \Re\mu \end{bmatrix}, \tag{3.1}$$

where the Latin indices $i, j = 1, \ldots, 4$ identify the different 3×3 block sub-matrices, whereas the Greek letters $\alpha, \beta = 1, 2, 3$ span each of these 3×3 block sub-matrices.

We assume that A is in $L^\infty(\Omega;\mathbb{R})^{12\times 12}$ and satisfies a strong Legendre condition (as in [8,13]), that is, there exists $\Lambda > 0$ such that

$$A_{ij}^{\alpha\beta}\eta_\alpha^i\eta_\beta^j \geq \Lambda\left|\eta\right|^2, \ \eta \in \mathbb{R}^{12} \quad \text{and} \quad \left|A_{ij}^{\alpha\beta}\right| \leq \Lambda^{-1} \qquad \text{a.e. in } \Omega. \tag{3.2}$$

The following result gives a sufficient condition for (3.2) to hold true.

Lemma 3.1. *Assume that $\varepsilon_0, \mu_0, \kappa, \chi$ are real constants, with $\varepsilon_0 > 0$ and $\mu_0 > 0$. Let*

$$\varepsilon = \varepsilon_0 I_3, \quad \mu = \mu_0 I_3, \quad \xi = (\chi - \mathbf{i}\kappa)I_3, \quad \zeta = (\chi + \mathbf{i}\kappa)I_3, \tag{3.3}$$

where I_3 is the 3×3 identity matrix, and construct the matrix A as in (3.1)*. If*

$$\chi^2 + \kappa^2 < \varepsilon_0\mu_0, \tag{3.4}$$

then A satisfies (3.2)*.*

Remark 3.1. This result shows that a wide class of materials satisfy the strong Legendre condition (3.2). Considering for simplicity the case of constant and isotropic parameters, the constitutive relations given in (3.3) describe the so-called chiral materials. It turns out that (3.4) is satisfied for natural materials [22].

Proof. A direct calculation shows that the smallest eigenvalue of A is $(\varepsilon_0 + \mu_0 - (\varepsilon_0^2 - 2\varepsilon_0\mu_0 + \mu_0^2 + 4\chi^2 + 4\kappa^2)^{1/2})/2$, which is strictly positive since $\chi^2 + \kappa^2 < \varepsilon_0\mu_0$. $\square$

We now give the regularity assumptions on the parameters. In contrast to the previous situation, here the mixing coefficients ξ and ζ fully couple electric and magnetic properties. We are thus led to assume that

$$\varepsilon, \xi, \mu, \zeta \in W^{1,3+\delta}\left(\Omega;\mathbb{C}^{3\times 3}\right) \text{ for some } \delta > 0. \tag{3.5}$$

The theorem below shows that at least as far as interior regularity is concerned, Theorem 1.3 also applies in this more general setting.

Theorem 3.1. *Assume that* (3.2) *and* (3.5) *hold. Suppose that the current sources J_e and J_m are in $W^{1,p}(\mathrm{div},\Omega)$ for some $p \geq 2$.*

If E and H in $H(\mathrm{curl},\Omega)$ are weak solutions of (1.8)*, then $E, H \in W^{1,q}_{loc}(\Omega;\mathbb{C}^3)$ with $q = \min(p, 3+\delta)$. Furthermore, for any open set Ω_0 such that $\overline{\Omega_0} \subset \Omega$ there holds*

$$\|E\|_{W^{1,q}(\Omega_0;\mathbb{C}^3)} + \|H\|_{W^{1,q}(\Omega_0;\mathbb{C}^3)} \leq C\big(\|E\|_{L^2(\Omega)} + \|H\|_{L^2(\Omega)} + \|(J_e, J_m)\|_{W^{1,p}(\mathrm{div},\Omega)^2}\big),$$

where C is a constant depending on $\Omega, \Omega_0, q, \Lambda, \omega$, and the $W^{1,3+\delta}\left(\Omega;\mathbb{C}^{3\times 3}\right)$ norms of ε, μ, ξ and *ζ. In particular, if $p > 3$ then $E, H \in \mathcal{C}^{0,1-\frac{3}{q}}_{loc}(\Omega;\mathbb{C}^3)$.*

We did not investigate the regularity up to the boundary in this problem for two reasons. From a modelling point of view, the natural (or pertinent) boundary conditions to be considered in this case are not completely clear. There is also a technical reason: the boundary terms are a rather intricate mix of Neumann and

Dirichlet type terms on both E and H, and the correct space of test functions to consider is not readily apparent (see Proposition Appendix A.1).

The proof of this result is a variant of the proof of Theorem 1.3. In this case, the system is written in $\mathbb{R}^{12}$ (instead of a weakly coupled system of 6 complex unknowns) and the proof is detailed in the appendix.

4. Proof of Theorem 1.4 using Campanato estimates

The purpose of this section is to prove Theorem 1.4. We shall apply classical Campanato estimates for elliptic equations to (2.1), namely the elliptic equations satisfied by E. We first state the properties of Campanato spaces that we shall use, and then proceed to the proof of Theorem 1.4.

For $\lambda \geq 0$ and $p \geq 1$ we denote the Campanato space by $L^{p,\lambda}(\Omega;\mathbb{C})$ [7], namely the Banach space of functions $u \in L^p(\Omega;\mathbb{C})$ such that

$$[u]_{p,\lambda;\Omega}^p := \sup_{x\in\Omega, 0<\rho<\operatorname{diam}\Omega} \rho^{-\lambda} \int_{\Omega(x,\rho)} \left| u(y) - \fint_{\Omega(x,\rho)} u(z)\,dz \right|^p dy < \infty,$$

where $\Omega(x,\rho) = \Omega \cap \{y \in \mathbb{R}^3 : |y-x| < \rho\}$, equipped with the norm

$$\|u\|_{L^{p,\lambda}(\Omega;\mathbb{C})} = \|u\|_{L^p(\Omega;\mathbb{C})} + [u]_{p,\lambda;\Omega}.$$

Lemma 4.1. *Take $\lambda \geq 0$.*

(1) Suppose $\lambda > 3$. If $u \in L^{2,\lambda}(\Omega;\mathbb{C})$ then $u \in \mathcal{C}^{0,\frac{\lambda-3}{2}}\left(\overline{\Omega};\mathbb{C}\right)$, and the embedding is continuous.

(2) Suppose $\lambda < 3$. If $u \in L^2(\Omega;\mathbb{C})$ and $\nabla u \in L^{2,\lambda}\left(\Omega;\mathbb{C}^3\right)$ then $u \in L^{2,2+\lambda}(\Omega;\mathbb{C})$, and the embedding is continuous.

(3) Suppose $\delta > 0$ and $\lambda \neq 1$. If $f \in L^{3+\delta}(\Omega;\mathbb{C})$ and $u \in L^2(\Omega;\mathbb{C})$ with $\nabla u \in L^{2,\lambda}\left(\Omega;\mathbb{C}^3\right)$ then $fu \in L^{2,\lambda'}(\Omega;\mathbb{C})$ with $\lambda' = \min(\lambda + 2\delta(3+\delta)^{-1}, 3(1+\delta)(3+\delta)^{-1})$, and the embedding is continuous.

Proof. Statements (1) and (2) are classical, see e.g. [21]. For (3), note that Hölder's inequality implies that $f \in L^{2,3(1+\delta)(3+\delta)^{-1}}(\Omega;\mathbb{C})$. When $\lambda < 1$, the result follows from [11]. When $\lambda > 1$, (3) follows from (1) and (2). □

We now state the regularity result regarding Campanato estimates we will use. It can be found in [21] (Theorems 2.19 and 3.16).

Proposition 4.1. *Assume (1.1) and $\Im\mu = 0$. There exists $\lambda_\mu \in (1,2]$ depending only on Ω and Λ given in (1.1), such that if $F \in L^{2,\lambda}\left(\Omega;\mathbb{C}^3\right)$ for some $\lambda \in [0,\lambda_\mu)$, and $u \in W^{1,2}(\Omega;\mathbb{C})$ satisfies*

$$\begin{cases} -\operatorname{div}(\mu\nabla u) = \operatorname{div}(F) & \text{in } \Omega, \\ \mu\nabla u \cdot \nu = F \cdot \nu & \text{on } \partial\Omega, \end{cases}$$

then $\nabla u \in L^{2,\lambda}\left(\Omega;\mathbb{C}^3\right)$ and

$$\|\nabla u\|_{L^{2,\lambda}(\Omega;\mathbb{C}^3)} \leq C\,\|F\|_{L^{2,\lambda}(\Omega;\mathbb{C}^3)}, \tag{4.1}$$

where the constant C depends only on Λ, λ and Ω.

Alternatively, assume (1.1) *and* (1.6). *For all $\lambda \in [0,2]$, if $F \in L^{2,\lambda}\left(\Omega;\mathbb{C}^3\right)$, $f \in L^2\left(\Omega;\mathbb{C}\right)$, and $u \in W^{1,2}\left(\Omega;\mathbb{C}\right)$ satisfies*

$$\begin{cases} -\mathrm{div}(\mu\nabla u) = \mathrm{div}(F) + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

then $\nabla u \in L^{2,\lambda}\left(\Omega;\mathbb{C}^3\right)$ and

$$\|\nabla u\|_{L^{2,\lambda}(\Omega;\mathbb{C}^3)} \leq C\left(\|F\|_{L^{2,\lambda}(\Omega;\mathbb{C}^3)} + \|f\|_{L^2(\Omega;\mathbb{C})}\right), \tag{4.2}$$

where the constant C depends on Λ, Ω, and $\|\mu\|_{W^{1,3+\delta}(\Omega;\mathbb{C}^{3\times 3})}$ only.

We first study the regularity of H following a variant of an argument given in [24].

Proposition 4.2. *Assume that Ω is simply connected, that* (1.1) *holds with $\Im\mu = 0$ and $J_m \in L^{2,\lambda}(\Omega)$ with $1 < \lambda < \lambda_\mu$, where λ_μ is given by Proposition 4.1. Let E and H in $H(\mathrm{curl},\Omega)$ be weak solutions of* (1.2) *with $G = 0$. Then $H \in L^{2,\lambda}\left(\Omega;\mathbb{C}^3\right)$ and*

$$\|H\|_{L^{2,\lambda}(\Omega;\mathbb{C}^3)} \leq C\big(\|E\|_{L^2(\Omega)} + \|J_e\|_{L^2(\Omega)} + \|J_m\|_{L^{2,\lambda}(\Omega;\mathbb{C}^3)}\big), \tag{4.3}$$

where the constant C depends only on Λ, λ, ω and Ω.

Proof. Since $\mathbf{i}\omega\varepsilon E + J_e$ is divergence free in Ω, and Ω is $C^{2,1}$ and simply connected, it is well known that there exists $T \in H^1(\Omega)$ such that $\mathbf{i}\omega\varepsilon E + J_e = \mathrm{curl}T$, satisfying

$$\|T\|_{H^1(\Omega)} \leq C\big(\|J_e\|_{L^2(\Omega)} + \|E\|_{L^2(\Omega)}\big) \tag{4.4}$$

where C depends on Ω, Λ given in (1.1) and ω only, see e.g. [14]. Thanks to Lemma 4.1, this implies $\mu T \in L^{2,2}(\Omega;\mathbb{C}^3) \subset L^{2,\lambda}(\Omega;\mathbb{C}^3)$, and therefore $\mu T + i\omega^{-1}J_m \in L^{2,\lambda}(\Omega;\mathbb{C}^3)$.

As $H - T$ is curl free in Ω, in view of [14] there exists $p \in H^1(\Omega;\mathbb{C})$ such that $H - T = \nabla p$. The potential p is defined up to a constant by

$$\begin{cases} \mathrm{div}(\mu\nabla p) = \mathrm{div}(-\mu T - i\omega^{-1}J_m) & \text{in } \Omega, \\ \mu\nabla p\cdot\nu = (-\mu T - i\omega^{-1}J_m)\cdot\nu & \text{on } \partial\Omega. \end{cases}$$

Note that the boundary condition follows from that of E and (2.15). Thanks to estimate (4.1) in Proposition 4.1, we have

$$\|\nabla p\|_{L^{2,\lambda}(\Omega;\mathbb{C}^3)} \leq C\left\|\mu T + i\omega^{-1}J_m\right\|_{L^{2,\lambda}(\Omega;\mathbb{C}^3)} \leq \tilde{C}\big(\|T\|_{H^1(\Omega)} + \|J_m\|_{L^{2,\lambda}(\Omega;\mathbb{C}^3)}\big). \tag{4.5}$$

The conclusion follows from the identity $H = T + \nabla p$ and the estimates (4.4) and (4.5). □

We now adapt Proposition 2.2 to be able to use Campanato estimates in the bootstrap argument.

Proposition 4.3. *Assume that Ω is simply connected, that* (1.1) *holds with $\Im\mu = 0$ and that* (1.3) *holds. Suppose $J_m \in L^{2,\tilde{\lambda}}(\Omega;\mathbb{C}^3)$ and $\mathrm{div}J_e \in L^{2,\tilde{\lambda}}(\Omega;\mathbb{C})$ for some $1 < \tilde{\lambda} < \lambda_\mu$, where λ_μ is given by Proposition 4.1. Let E and H in $H(\mathrm{curl},\Omega)$ be weak solutions of* (1.2) *with $G = 0$.*

If $\nabla E \in L^{2,\lambda_0}\left(\Omega;\mathbb{C}^{3\times 3}\right)$ for some $\lambda_0 \in [0,\infty)\setminus\{1\}$ then $\nabla E \in L^{2,\lambda_1}(\Omega;\mathbb{C})^9$, with $\lambda_1 = \min\left(\tilde{\lambda}, \lambda_0 + 2\delta(3+\delta)^{-1}, 3(1+\delta)(3+\delta)^{-1}\right)$. Moreover there holds

$$\|\nabla E\|_{L^{2,\lambda_1}(\Omega;\mathbb{C})^9} \leq C\big(\|E\|_{L^2(\Omega)} + \|\nabla E\|_{L^{2,\lambda_0}(\Omega;\mathbb{C})^9} + \|J_e\|_{L^2(\Omega)} + \|J_m\|_{L^{2,\tilde{\lambda}}(\Omega;\mathbb{C}^3)} + \|\mathrm{div}J_e\|_{L^{2,\tilde{\lambda}}(\Omega;\mathbb{C})}\big), \quad (4.6)$$

where the constant C depends only on Ω, Λ, λ_1, ω and $\|\varepsilon\|_{W^{1,3+\delta}(\Omega;\mathbb{C}^{3\times 3})}$.

Proof. In view of Theorem 1.1 and Proposition 2.1, for each $k = 1,2,3$, $E_k \in H^1(\Omega;\mathbb{C})$ is a weak solution of

$$-\mathrm{div}\left(\varepsilon\nabla E_k\right) = \mathrm{div}\left(\partial_k\varepsilon\, E + S_k\right) \text{ in } \Omega, \quad (4.7)$$

with

$$S_k = -\varepsilon\left(\mathbf{e}_k \times (J_m - \mathbf{i}\omega\mu H)\right) - \mathbf{i}\omega^{-1}\mathbf{e}_k\mathrm{div}J_e.$$

Thanks to Proposition 4.2 we have that $S_k \in L^{2,\tilde{\lambda}}(\Omega;\mathbb{C})$. Furthermore $\partial_k\varepsilon E \in L^{2,\tilde{\lambda}_0}\left(\Omega;\mathbb{C}^3\right)$ with $\tilde{\lambda}_0 = \min\left(\lambda_0 + 2\delta(3+\delta)^{-1}, 3(1+\delta)(3+\delta)^{-1}\right)$ thanks to Lemma 4.1. Thus

$$\partial_k\varepsilon\, E + S_k \in L^{2,\lambda_1}\left(\Omega;\mathbb{C}^3\right). \quad (4.8)$$

Interior regularity. Given a smooth subdomain $\Omega_0 \Subset \Omega$, introduce a cut-off function $\chi \in \mathcal{D}(\Omega)$ such that $\chi = 1$ in Ω_0. From (4.7) we deduce

$$-\mathrm{div}\left(\varepsilon\nabla(\chi E_k)\right) = \mathrm{div}\left(\chi\left(\partial_k\varepsilon\, E + S_k\right)\right) + f_k \text{ in } \Omega, \quad (4.9)$$

where

$$f_k = -\nabla\chi\cdot\left(\partial_k\varepsilon\, E + S_k\right) - \varepsilon\nabla E_k\cdot\nabla\chi - \mathrm{div}(\varepsilon E_k\nabla\chi) \in L^2\left(\Omega;\mathbb{C}\right).$$

As $\lambda_1 < 2$ and ε satisfies (1.3), we may apply Proposition 4.1 (withε in lieu of μ) to show that $\nabla(\chi E_k)$ is in $L^{2,\lambda_1}\left(\Omega;\mathbb{C}^3\right)$, which implies $\nabla E \in L^{2,\lambda_1}\left(\Omega_0;\mathbb{C}^3\right)$.

Boundary regularity. Consider now the change of coordinate $A \in C^{1,1}$ in a ball B centred in x_0 already introduced in the proof of Proposition 2.2, and again let us focus on the tangential components first. Take $\chi \in \mathcal{D}(B;\mathbb{R})$ such that $\chi = 1$ in a neighborhood $\tilde{B}$ of x_0 and $j \in \{1,2\}$. Defining $\tilde{E} = \chi AE$, the identity (4.7) yields, for $j = 1,2$

$$-\mathrm{div}\big(\varepsilon\nabla\tilde{E}_j\big) = \mathrm{div}\left(\chi A_{jk}\left(\partial_k\varepsilon\, E + S_k\right)\right) + f_j \text{ in } \Omega,$$

where $f_j = \sum_{k=1}^3 -\nabla(\chi A_{jk})\cdot(\partial_k\varepsilon\, E + S_k) - \varepsilon\nabla E_k\cdot\nabla(\chi A_{jk}) - \mathrm{div}(\varepsilon E_k\nabla(\chi A_{jk})) \in L^2(\Omega;\mathbb{C})$. Note that $E\times\nu = 0$ on $\partial\Omega$ implies $\tilde{E}_1 = \tilde{E}_2 = 0$ on $\partial\Omega$. Proposition 4.2 together with (4.8) then shows that $\nabla\tilde{E}_j$ belongs to $L^{2,\lambda_1}\left(\Omega;\mathbb{C}^3\right)$ for $j = 1,2$.

Arguing as in the proof of Proposition 2.2, we also derive that $\nabla \tilde{E}_3 \in L^{2,\lambda_1}\left(\Omega;\mathbb{C}^3\right)$. Therefore $\nabla \tilde{E} \in L^{2,\lambda_1}\left(\tilde{B};\mathbb{C}^{3\times 3}\right)$, and in turn $\nabla E \in L^{2,\lambda_1}\left(\tilde{B};\mathbb{C}^{3\times 3}\right)$.

Global regularity. Combining the interior and the boundary estimates we obtain that ∇E is in $L^{2,\lambda_1}\left(\Omega;\mathbb{C}^{3\times 3}\right)$, together with (4.6). □

We are now ready to prove the global Hölder regularity result.

Proof. [Proof of Theorem 1.4] Considering the system satisfied by $E-G$ and H, we may assume $G=0$. Choose any $\tilde{\lambda}>1$ such that $\tilde{\lambda}<\lambda_\mu$ and $\tilde{\lambda}\leq 3\frac{p-2}{p}$. Hölder's inequality shows that $J_m \in L^{2,\tilde{\lambda}}(\Omega;\mathbb{C}^3)$ and $\mathrm{div} J_e \in L^{2,\tilde{\lambda}}(\Omega;\mathbb{C})$. We apply Proposition 4.3 a finite number of times, starting with $\nabla E \in L^{2,\lambda_n}\left(\Omega;\mathbb{C}^{3\times 3}\right)$ for some $\lambda_n<1$ (in the initial step we take $\lambda_0=0$, in view of Theorem 1.1), and obtain that $\nabla E \in L^{2,\lambda_{n+1}}\left(\Omega;\mathbb{C}^{3\times 3}\right)$, with $\lambda_{n+1}=\min\left(\tilde{\lambda},(n+1)2\delta'\left(\delta'+3\right)^{-1}\right)$, where $0<\delta'\leq\delta$ is such that $\frac{\delta'+3}{2\delta'}\notin\mathbb{N}$. We stop the iterative procedure as soon as $\lambda_{n+1}>1$ and we infer that $\nabla E\in L^{2,\lambda}\left(\Omega;\mathbb{C}^{3\times 3}\right)$ for some $1<\lambda\leq\tilde{\lambda}$. A final application of Proposition 4.3 gives $\nabla E \in L^{2,\min\left(\tilde{\lambda},3(1+\delta)(3+\delta)^{-1}\right)}\left(\Omega;\mathbb{C}^{3\times 3}\right)$; the result then follows from Lemma 4.1. □

Appendix A. Proof of Theorem 3.1

The first step is to derive an appropriate very weak formulation.

Proposition Appendix A.1. *Under the hypotheses of Theorem 3.1, let $E,H\in H(\mathrm{curl},\Omega)$ be a weak solution of* (1.8).

Then for each $k=1,2,3$, (E_k,H_k) is a very weak solution of the elliptic system

$$\begin{cases} -\mathrm{div}\left(\varepsilon\nabla E_k+\xi\nabla H_k\right)=\mathrm{div}\left((\partial_k\varepsilon)E+(\partial_k\xi)\,H-\varepsilon\left(\mathbf{e}_k\times\left(-\mathbf{i}\omega\zeta E-\mathbf{i}\omega\mu H+J_m\right)\right)\right) \\ \qquad\qquad +\mathrm{div}\left(-\xi\left(\mathbf{e}_k\times\left(\mathbf{i}\omega\varepsilon E+\mathbf{i}\omega\xi H+J_e\right)\right)-\mathbf{i}\omega^{-1}\mathbf{e}_k\mathrm{div}J_e\right)\ \text{in } \Omega. \\ -\mathrm{div}\left(\zeta\nabla E_k+\mu\nabla H_k\right)=\mathrm{div}\left((\partial_k\zeta)E+(\partial_k\mu)\,H-\mu\left(\mathbf{e}_k\times\left(\mathbf{i}\omega\varepsilon E+\mathbf{i}\omega\xi H+J_e\right)\right)\right) \\ \qquad\qquad +\mathrm{div}\left(\zeta\left(\mathbf{e}_k\times\left(\mathbf{i}\omega\zeta E+\mathbf{i}\omega\mu H-J_m\right)\right)+\mathbf{i}\omega^{-1}\mathbf{e}_k\mathrm{div}J_m\right)\ \text{in } \Omega. \end{cases}$$

More precisely, for any $\varphi\in W^{2,2}(\Omega;\mathbb{C})$ there holds

$$\begin{aligned} &\int_\Omega E_k\mathrm{div}\left(\varepsilon^T\nabla\overline{\varphi}\right)\,dx+\int_\Omega H_k\mathrm{div}\left(\xi^T\nabla\overline{\varphi}\right)\,dx=\int_\Omega\left(\left(\partial_k\varepsilon\right)E+\left(\partial_k\xi\right)H\right)\cdot\nabla\overline{\varphi}\,dx \\ &-\int_\Omega\left(\varepsilon\left(\mathbf{e}_k\times\left(-\mathbf{i}\omega\zeta E-\mathbf{i}\omega\mu H+J_m\right)\right)+\xi\left(\mathbf{e}_k\times\left(\mathbf{i}\omega\varepsilon E+\mathbf{i}\omega\xi H+J_e\right)\right)+\mathbf{i}\omega^{-1}\mathrm{div}J_e\mathbf{e}_k\right)\cdot\nabla\overline{\varphi}\,dx \\ &+\int_{\partial\Omega}(\partial_k\overline{\varphi})(\varepsilon E+\xi H)\cdot\nu\,ds-\int_{\partial\Omega}\left(\mathbf{e}_k\times(H\times\nu)\right)\cdot(\xi^T\nabla\overline{\varphi})\,d\sigma-\int_{\partial\Omega}\left(\mathbf{e}_k\times(E\times\nu)\right)\cdot(\varepsilon^T\nabla\overline{\varphi})\,d\sigma, \end{aligned} \tag{A.1}$$

and

$$
\begin{aligned}
&\int_\Omega E_k \mathrm{div}\left(\zeta^T \nabla\overline{\varphi}\right)\,dx + \int_\Omega H_k \mathrm{div}\left(\mu^T \nabla\overline{\varphi}\right)\,dx = \int_\Omega \left(\left(\partial_k\zeta\right)E + \left(\partial_k\mu\right)H\right)\cdot\nabla\overline{\varphi}\,dx\\
&-\int_\Omega \left(\mu\left(\mathbf{e}_k\times\left(\mathbf{i}\omega\varepsilon E + \mathbf{i}\omega\xi H + J_e\right)\right) - \zeta\left(\mathbf{e}_k\times\left(\mathbf{i}\omega\zeta E + \mathbf{i}\omega\mu H - J_m\right)\right) - \mathbf{i}\omega^{-1}\mathrm{div}J_m\mathbf{e}_k\right)\cdot\nabla\overline{\varphi}\,dx\\
&+\int_{\partial\Omega}(\partial_k\overline{\varphi})(\zeta E+\mu H)\cdot\nu\,ds - \int_{\partial\Omega}(\mathbf{e}_k\times(E\times\nu))\cdot(\zeta^T\nabla\overline{\varphi})\,d\sigma - \int_{\partial\Omega}(\mathbf{e}_k\times(H\times\nu))\cdot(\mu^T\nabla\overline{\varphi})\,d\sigma.
\end{aligned}
\tag{A.2}
$$

Proof. The proof is similar to that of Proposition 2.1. □

We only study interior regularity for the problem at hand. The boundary regularity does not follow easily from the method used in Section 2. Indeed, mixed boundary terms appear in (A.1) and (A.2), and the technique used in Proposition 2.2 and in Proposition 2.3, with test functions satisfying either Dirichlet or Neumann boundary conditions, does not apply, as both conditions would be required simultaneously.

The "very weak to weak" Lemma 2.1 adapted to this mixed system is given below.

Lemma Appendix A.1. *Assume* (3.2) *and* (3.5) *hold, and let* A *be given by* (3.1). *Given* $r \geq \frac{6}{5}$, $u \in L^2\left(\Omega;\mathbb{R}^4\right)\cap L^r\left(\Omega;\mathbb{R}^4\right)$ *and* $F \in W^{1,r'}\left(\Omega;\mathbb{R}^4\right)'$, *if*

$$
\int_\Omega u^j \partial_\alpha(A_{ij}^{\alpha\beta}\partial_\beta\varphi^i)\,dx = \langle F_i,\varphi^i\rangle, \qquad \varphi\in W^{2,2}\left(\Omega;\mathbb{R}^4\right)\cap W_0^{1,2}\left(\Omega;\mathbb{R}^4\right), \tag{A.3}
$$

then $u \in W^{1,r}\left(\Omega;\mathbb{R}^4\right)$ *and*

$$
\|\nabla u\|_{L^r(\Omega;\mathbb{R}^{4\times 3})} \leq C\,\|F\|_{W^{1,r'}(\Omega;\mathbb{R}^4)'}, \tag{A.4}
$$

for some constant $C = C(r,\Omega,\Lambda,\|\varepsilon,\xi,\mu,\zeta\|_{W^{1,3}(\Omega;\mathbb{C}^{3\times3})^4})$.

Proof. Let $\psi \in \mathcal{D}(\Omega;\mathbb{R})$ be a test function and take $\alpha^* \in \{1,2,3\}$ and $j^* \in \{1,\ldots,4\}$. Since A satisfies the strong Legendre condition (3.2), the system

$$
\begin{cases}
\partial_\alpha(A_{ij}^{\alpha\beta}\partial_\beta\varphi_*^i) = \delta_{jj^*}\partial_{\alpha^*}\psi & \text{in } \Omega,\\
\varphi_* = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{A.5}
$$

has a unique solution $\varphi_* \in H_0^1\left(\Omega;\mathbb{R}^4\right)$ (see e.g. [13,8]). Further, since $A_{ij}^{\alpha\beta} \in W^{1,3}(\Omega;\mathbb{R})$, by [6] for any $q\in(1,\infty)$

$$
\|\varphi_*\|_{W^{1,q}(\Omega;\mathbb{R}^4)} \leq c\,\|\psi\|_{L^q(\Omega;\mathbb{R}^4)}, \tag{A.6}
$$

for some $c = c(q,\Omega,\Lambda,\|\varepsilon,\xi,\mu,\zeta\|_{W^{1,3}(\Omega;\mathbb{C}^{3\times3})^4}) > 0$. Hence, the usual difference quotient argument given in [13] shows that $\varphi_* \in W^{2,2}\left(\Omega;\mathbb{R}^4\right)$. Therefore, by assumption we have

$$
\left|\int_\Omega u^{j^*}\partial_{\alpha^*}\psi\,dx\right| = \left|\int_\Omega u^j\partial_\alpha(A_{ij}^{\alpha\beta}\partial_\beta\varphi_*^i)\,dx\right| = \left|\langle F_i,\varphi_*^i\rangle\right| \leq \|F\|_{W^{1,r'}(\Omega;\mathbb{R}^4)'}\,\|\varphi_*\|_{W^{1,r'}(\Omega;\mathbb{R}^4)},
$$

which in view of (A.6) gives

$$\left| \int_\Omega u^{j^*} \partial_{\alpha^*} \psi \, dx \right| \leq c \, \|F\|_{W^{1,r'}(\Omega;\mathbb{R}^4)'} \, \|\psi\|_{L^{r'}(\Omega;\mathbb{R}^4)} \,,$$

whence the result. □

The following proposition mirrors Propositions 2.2 and 2.3. Theorem 3.1 then follows by the bootstrap argument used in the proof of Theorem 1.3.

Proposition Appendix A.2. *Under the hypotheses of Theorem 3.1 and given* $q \in [2, \infty)$*, set* $r = \min((3q + q\delta)(q + 3 + \delta)^{-1}, p)$*. Let* E *and* H *in* $H(\mathrm{curl}, \Omega)$ *be weak solutions of* (1.8).

Suppose $E, H \in L^q(\Omega; \mathbb{C}^3)$*. Then* $E, H \in W^{1,r}_{loc}(\Omega; \mathbb{C}^3)$ *and if* $\Omega_0 \Subset \Omega$*,*

$$\|(E, H)\|_{W^{1,r}(\Omega_0;\mathbb{C}^3)} \leq C(\|(E, H)\|_{L^q(\Omega;\mathbb{C})^6} + \|(J_e, J_m)\|_{W^{1,p}(\mathrm{div},\Omega)^2}), \qquad \text{(A.7)}$$

for some constant $C = C(r, \Omega, \Omega_0, \Lambda, \omega, \|\varepsilon, \xi, \mu, \zeta\|_{W^{1,\delta+3}(\Omega;\mathbb{C}^{3\times 3})^4})$*.*

Proof. From (A.1) we see that for every compactly supported $\varphi^1, \varphi^2 \in W^{2,2}(\Omega; \mathbb{C})$ and $k = 1, 2, 3$ there holds

$$\begin{cases} \int_\Omega E_k \mathrm{div}\left(\varepsilon^T \nabla \overline{\varphi^1}\right) + H_k \mathrm{div}\left(\xi^T \nabla \overline{\varphi^1}\right) \, dx = \int_\Omega F_k \cdot \nabla \overline{\varphi^1} \, dx, \\ \int_\Omega E_k \mathrm{div}\left(\zeta^T \nabla \overline{\varphi^2}\right) + H_k \mathrm{div}\left(\mu^T \nabla \overline{\varphi^2}\right) \, dx = \int_\Omega G_k \cdot \nabla \overline{\varphi^2} \, dx, \end{cases} \qquad \text{(A.8)}$$

with

$$F_k = (\partial_k \varepsilon) \, E + (\partial_k \xi) \, H - \varepsilon \left(\mathbf{e}_k \times (-\mathbf{i}\omega\zeta E - \mathbf{i}\omega\mu H + J_m)\right) \\ - \xi \left(\mathbf{e}_k \times (\mathbf{i}\omega\varepsilon E + \mathbf{i}\omega\xi H + J_e)\right) - \mathbf{i}\omega^{-1} \mathrm{div} J_e \mathbf{e}_k,$$

and

$$G_k = (\partial_k \zeta) \, E + (\partial_k \mu) \, H - \mu \left(\mathbf{e}_k \times (\mathbf{i}\omega\varepsilon E + \mathbf{i}\omega\xi H + J_e)\right) \\ + \zeta \left(\mathbf{e}_k \times (\mathbf{i}\omega\zeta E + \mathbf{i}\omega\mu H - J_m)\right) + \mathbf{i}\omega^{-1} \mathrm{div} J_m \mathbf{e}_k.$$

By construction, $F_k, G_k \in L^r(\Omega; \mathbb{C}^3)$.

Given a smooth subdomain $\Omega_0 \Subset \Omega$, we consider a cut-off function $\chi \in \mathcal{D}(\Omega)$ such that $\chi = 1$ in Ω_0. A straightforward computation shows

$$\begin{cases} \int_\Omega \chi E_k \mathrm{div}(\varepsilon^T \nabla \overline{\varphi^1}) + \chi H_k \mathrm{div}\left(\xi^T \nabla \overline{\varphi^1}\right) \, dx = \int_\Omega F_k \cdot \nabla(\chi \overline{\varphi^1}) \, dx + T_k(\varphi^1), \\ \int_\Omega \chi E_k \mathrm{div}(\zeta^T \nabla \overline{\varphi^2}) + \chi H_k \mathrm{div}\left(\mu^T \nabla \overline{\varphi^2}\right) \, dx = \int_\Omega G_k \cdot \nabla(\chi \overline{\varphi^2}) \, dx + R_k(\varphi^2), \end{cases}$$

where

$$T_k(\varphi^1) = -\int_\Omega E_k \left(\mathrm{div}(\varepsilon^T \overline{\varphi^1} \nabla \chi) + \varepsilon \nabla \chi \cdot \nabla \overline{\varphi^1}\right) + H_k \left(\mathrm{div}(\xi^T \overline{\varphi^1} \nabla \chi) + \xi \nabla \chi \cdot \nabla \overline{\varphi^1}\right) dx,$$

and

$$R_k(\varphi^2) = -\int_\Omega E_k \left(\mathrm{div}(\zeta^T \overline{\varphi^2} \nabla \chi) + \zeta \nabla \chi \cdot \nabla \overline{\varphi^2}\right) + H_k \left(\mathrm{div}(\mu^T \overline{\varphi^2} \nabla \chi) + \mu \nabla \chi \cdot \nabla \overline{\varphi^2}\right) dx.$$

This last system can be reformulated in the form (A.3), with A given by (3.1). We then apply Lemma Appendix A.1 and obtain $\chi E_k, \chi H_k \in W^{1,r}(\Omega; \mathbb{C})$, namely $E, H \in W^{1,r}(\Omega_0; \mathbb{C}^3)$. Finally, (A.7) follows from (A.4). □

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