Report no. OxPDE-13/09

Oxford Centre for Nonlinear PDE

ON THE MORSE-SARD THEOREM FOR THE SHARP CASE OF SOBOLEV MAPPINGS

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July 2013

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Version: 29/07/2013

Abstract

We establish Luzin N- and Morse–Sard properties for mappings $v\colon \mathbb{R}^n \to \mathbb{R}^m$ of the Sobolev–Lorentz class $\mathrm{W}^k_{p,1}$ with $k = n - m + 1$ and $p = \frac{n}{k}$ (this is the sharp case that guaranties the continuity of mappings). Using these results we prove that almost all level sets are finite disjoint unions of C^1–smooth compact manifolds of dimension $n - m$.

Key words: *Sobolev–Lorentz space, Luzin N–property, Morse–Sard property, level sets.*

Introduction

The Morse–Sard theorem is a fundamental result with many applications. In its classical form it states that the image of the set of critical points of a C^{n-m+1} smooth mapping $v\colon \mathbb{R}^n \to \mathbb{R}^m$ has zero Lebesgue measure in $\mathbb{R}^m$. More precisely, assuming that $n \geq m$ the set of critical points for v is $Z_v = \{x \in \mathbb{R}^n \,:\, \mathrm{rank}\nabla v(x) < m\}$ and the conclusion is that

$$\mathscr{L}^m(v(Z_v)) = 0. \tag{1}$$

The theorem was proved by Morse [23] in the case $m = 1$ and subsequently by Sard [27] in the general case. It is well–known since the work of Whitney [33] that the C^{n-m+1} smoothness assumption on the mapping v cannot be weakened to C^j smoothness with j less than $n-m+1$. While this is so Dubovitskiĭ [12] obtained results on the structure of level sets for C^j mappings v including the cases where j is smaller than $n - m + 1$ (also see [4]).

An important generalization of the Morse–Sard theorem is the following result that we display as it, together with the classical result, forms the starting point for our investigations here.

*The author was supported by the Russian Foundation for Basic Research (project No. 12-01-00390-a) and by the Grant of the President of Russia for support of young doctors of sciences (grant No. MD-5146.2013.1).

†Work supported by the EPSRC Science and Innovation award to the Oxford Centre for Nonlinear PDE (EP/E035027/1).

Theorem (Federer [14, Theorem 3.4.3]). *Let $m \in \{1, \dots, n\}$, $d, k \in \mathbb{N}$, and let $v\colon \mathbb{R}^n \to \mathbb{R}^d$ be a C^k–smooth mapping. Denote $q_\circ = m - 1 + \frac{n-m+1}{k}$. Then*

$$\mathcal{H}^{q_\circ}(v(Z_{v,m})) = 0, \tag{2}$$

where $\mathcal{H}^\beta$ denotes the β–dimensional Hausdorff measure and $Z_{v,m}$ denotes the set of m–critical points of v: $Z_{v,m} = \{x \in \mathbb{R}^n \,:\, \mathrm{rank}\nabla v(x) < m\}$.

The Morse–Sard–Federer results have subsequently been generalized to mappings in more refined scales of spaces, including Hölder and Sobolev spaces. For Hölder spaces we mention in particular [3, 4, 22, 24, 34] where essentially sharp results were obtained, including examples showing that the smoothness assumption on v in Federer's theorem cannot be weakened within the scale of C^j spaces. However, it follows from [3] that the conclusion (2) remains valid for $\mathrm{C}^{k-1,1}$ mappings v, and according to [22] it fails in general for $\mathrm{C}^{k-1,\alpha}$ mappings whenever $\alpha < 1$. (For $k \in \mathbb{N}_0$ and $\alpha \in (0,1]$ we say that the mapping v is of class $\mathrm{C}^{k,\alpha}$ when v is C^k and the k–th order derivative of v is locally α–Hölder continuous.) One interpretation of these results is that for the validity of (2) one must assume existence of k derivatives of v in a suitably strong sense. At a heuristic level the general problem is then to prove analogs of the Morse–Sard–Federer results where we replace the assumption that the mapping is k times continuously differentiable by a corresponding Sobolev assumption: v has weak derivatives up to and including order k and these weak derivatives must satisfy a suitable integrability condition. The aforementioned examples show that we cannot in general reduce the degree k of differentiability. The question we wish to address here concerns the optimal local integrability condition that the k–th order weak derivative must satisfy for the validity of (2). Previous works on the Morse–Sard property in the context of Sobolev spaces include [4, 9, 10, 15, 17, 25, 31, 32, 7, 8]. The first Morse–Sard result in the Sobolev context that we are aware of is [10]. It states that (1) holds for mappings $v \in \mathrm{W}^k_{p,\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^m)$ when $k \geq \max(n-m+1, 2)$ and $p > n$. Note that by the Sobolev embedding theorem any mapping on $\mathbb{R}^n$ which is locally of Sobolev class W^k_p for some $p > n$ is in particular C^{k-1}, so the critical set Z_v can be defined as usual.

When in the scalar case $m = 1$ we consider functions in $\mathrm{W}^n_{p,\mathrm{loc}}(\mathbb{R}^n)$ with $p \in [1, n]$ we are in general only assured everywhere continuity whereas the differentiability can fail at some points. Hence for such functions one must adapt the definition of critical set accordingly. We define the sets $A_v := \{x \in \mathbb{R}^n \,:\, v \text{ is not differentiable at } x\}$ and $Z_v := \{x \in \mathbb{R}^n \setminus A_v \,:\, \nabla v(x) = 0\}$. In these terms the results of [7, 8] imply that (1) holds with $m = 1$ for all $v \in \mathrm{W}^n_{1,\mathrm{loc}}(\mathbb{R}^n)$ and that also $\mathscr{L}^1(v(A_v)) = 0$. The latter is a consequence of a more general Luzin N property with respect to one–dimensional Hausdorff content that $\mathrm{W}^n_{1,\mathrm{loc}}$ functions are shown to enjoy. In fact the results of [7, 8] even yield (1) with $m = 1$ and an appropriate definition of the critical set, and the Luzin N property within the more general framework of functions of bounded variation $\mathrm{BV}_{n,\mathrm{loc}}(\mathbb{R}^n)$.

In this paper we shall be concerned with the vectorial case $m > 1$. Of course, it is very natural to assume, that the inclusion $v \in \mathrm{W}^k_p(\mathbb{R}^n, \mathbb{R}^d)$ should guarantee at least the continuity of v. For values $k \in \{1, \dots, n-1\}$ it is well–known that $v \in \mathrm{W}^k_p(\mathbb{R}^n, \mathbb{R}^d)$ is continuous for $p > \frac{n}{k}$ and could be discontinuous for $p \leq \frac{n}{k}$. So **the borderline case** is $p = p_\circ = \frac{n}{k}$. It is well–known (see for instance [16]) that really $v \in \mathrm{W}^k_{p_\circ}(\mathbb{R}^n, \mathbb{R}^d)$ is continuous if the derivatives of k-th order belong to the Lorentz space $\mathrm{L}_{p_\circ,1}$, we will denote the space of such mappings by $\mathrm{W}^k_{p_\circ,1}(\mathbb{R}^n,\mathbb{R}^d)$. We refer to section 2 for relevant definitions and notation.

In this paper we prove the precise analog of the above Federer's theorem for mappings $v\colon \mathbb{R}^n \to \mathbb{R}^d$ locally of class $\mathrm{W}^k_{p_\circ,1}$, $k \in \{2,\dots,n\}$, $m \in \{2,\dots,n\}$ (the case $k=1$, and, consequently, $q_\circ = n$, was considered in [16], so we omit it). It is easy to see (using well–known results such as [11]) that such a function is (Fréchet–)differentiable $\mathcal{H}^{q_\circ}$–almost everywhere, where $q_\circ = m-1+\frac{n-m+1}{k}$ is the same as in above Federer's theorem. The critical set $Z_{v,m}$ is defined as the set of points x, where v is differentiable and $\operatorname{rank}\nabla v(x) < m$. As our main result we prove that $\mathcal{H}^{q_\circ}(v(Z_{v,m})) = 0$. In fact, the result in Theorem 3.1 is slightly more general and concerns mappings locally of Sobolev class $\mathrm{W}^k_{p_\circ}$.

We also establish a related Luzin N property with respect to Hausdorff content in Theorem 2.1. More precisely, when the mapping $v\colon \mathbb{R}^n \to \mathbb{R}^d$ is of class $\mathrm{W}^k_{p_\circ,1}$ we find for any $\varepsilon > 0$ a $\delta > 0$ such that for all subsets E of $\mathbb{R}^n$ with $\mathcal{H}^{q_\circ}_\infty(E) < \delta$ we have $\mathcal{H}^{q_\circ}_\infty(v(E)) < \varepsilon$. Here $\mathcal{H}^{q_\circ}_\infty$ is the $q_\circ$–dimensional Hausdorff content. In particular, it follows that $\mathcal{H}^{q_\circ}(v(E)) = 0$ whenever $\mathcal{H}^{q_\circ}(E) = 0$. So the image of the exceptional "bad" set, where the differential is not defined, has zero $q_\circ$–dimensional Hausdorff measure. This ties nicely with our definition of the critical set and our version of the Federer result.

Finally, using these results we prove that if $v \in \mathrm{W}^k_{p_\circ,1}(\mathbb{R}^n,\mathbb{R}^m)$ with $k = n-m+1$ then for $\mathscr{L}^m$–almost all $y \in \mathbb{R}^m$ the preimage $v^{-1}(y)$ is a finite disjoint union of C^1–smooth compact manifolds of dimension $n-m$ without boundary (see Theorem 5.2).

Of course, the results are in particular valid for functions v from the classical Sobolev spaces $\mathrm{W}^k_p(\mathbb{R}^n,\mathbb{R}^d)$ with $p > p_\circ = \frac{n}{k}$ (see Remark 5.4).

We emphasize again that the similar results were proved for $k=1$ (i.e., $q_\circ = n$ for any $m \in \{1,\dots,n\}$) in [16] and for $m=1$, $k=n$ in [7, 8]. We do not prove the analogs of Federer's theorem for the cases $k > n$ or $m = 1, k < n$. In fact, these cases remain open.

While we have formulated all our results in the context of euclidean spaces it is clear that the results are local and hence could, with the appropriate modifications, be formulated for Sobolev mappings between smooth Riemannian manifolds instead.

Our proofs rely on the results of [21] on advanced versions of Sobolev imbedding theorems (see Theorem 1.6), of [1] on Choquet integrals of Hardy-Littlewood maximal functions with respect to Hausdorff content (see Theorem 1.8), and of [34] on the entropy estimate of near–critical values of differentiable functions (see Theorem 1.9). The key step in the proof of the Morse–Sard–Federer Theorem 3.1 is contained in Lemma 3.2, and it expands on a similar argument used in [8].

Acknowledgment. We are grateful to Professor Jean Bourgain for very useful interaction on the subject of this paper. The main part of the paper was written during a visit of MVK to the Oxford Centre for Nonlinear PDE in June 2013.

1 Preliminaries

By an *n–dimensional interval* we mean a closed cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes. If I is an n–dimensional interval then we write $\ell(I)$ for its sidelength.

For a subset S of $\mathbb{R}^n$ we write $\mathscr{L}^n(S)$ for its outer Lebesgue measure. The m–dimensional Hausdorff measure is denoted by $\mathcal{H}^m$ and the m–dimensional Hausdorff content by $\mathcal{H}^m_\infty$. Recall that for any subset S of $\mathbb{R}^n$ we have by definition

$$\mathcal{H}^m(S) = \lim_{\alpha \searrow 0} \mathcal{H}^m_\alpha(S) = \sup_{\alpha > 0} \mathcal{H}^m_\alpha(S),$$

where for each $0 < \alpha \leq \infty$,

$$\mathcal{H}^m_\alpha(S) = \inf\Big\{\sum_{i=1}^{\infty} (\operatorname{diam} S_i)^m \ : \ \operatorname{diam} S_i \leq \alpha, \ \ S \subset \bigcup_{i=1}^{\infty} S_i\Big\}.$$

It is well known that $\mathcal{H}^n(S) \sim \mathcal{H}^n_\infty(S) \sim \mathscr{L}^n(S)$ for sets $S \subset \mathbb{R}^n$.

To simplify the notation, we write $\|f\|_{\mathrm{L}_p}$ instead of $\|f\|_{\mathrm{L}_p(\mathbb{R}^n)}$, etc.

The Sobolev space $\mathrm{W}^k_p(\mathbb{R}^n, \mathbb{R}^d)$ is as usual defined as consisting of those $\mathbb{R}^d$-valued functions $f \in \mathrm{L}_p(\mathbb{R}^n)$ whose distributional partial derivatives of orders $l \leq k$ belong to $\mathrm{L}_p(\mathbb{R}^n)$ (for detailed definitions and differentiability properties of such functions see, e.g., [13], [35], [11]). Denote by $\nabla^k f$ the vector-valued function consisting of all k-th order partial derivatives of f arranged in some fixed order. However for the case of first order derivatives $k = 1$ we shall often think of $\nabla f(x)$ as the Jacobi matrix of f at x, i.e., the $d \times n$ matrix whose r-th row is the vector of partial derivatives of the r-th coordinate function.

We use the norm

$$\|f\|_{\mathrm{W}^k_p} = \|f\|_{\mathrm{L}_p} + \|\nabla f\|_{\mathrm{L}_p} + \cdots + \|\nabla^k f\|_{\mathrm{L}_p},$$

and unless otherwise specified all norms on the spaces $\mathbb{R}^s$ ($s \in \mathbb{N}$) will be the usual euclidean norms. We state the following result for later references, and only remark that it is well–known and follows from the definition of Sobolev spaces. In its statement we denote by $\mathrm{C}^\infty_c(\mathbb{R}^n)$ the space of C^∞ smooth and compactly supported functions on $\mathbb{R}^n$.

Lemma 1.1. *Let $f \in \mathrm{W}^k_p(\mathbb{R}^n)$. Then for any $\varepsilon > 0$ there exist functions $f_0 \in C^\infty_c(\mathbb{R}^n)$ and $f_1 \in \mathrm{W}^k_p(\mathbb{R}^n)$ such that $f = f_0 + f_1$ and $\|f_1\|_{\mathrm{W}^k_p} < \varepsilon$.*

Working with locally integrable functions, we always assume that the precise representatives are chosen. If $w \in L_{1,\mathrm{loc}}(\Omega)$, then the precise representative w^* is defined by

$$w^*(x) = \begin{cases} \lim\limits_{r \searrow 0} \fint_{B(x,r)} w(z)\,\mathrm{d}z, & \text{if the limit exists and is finite,} \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

where the dashed integral as usual denotes the integral mean,

$$\fint_{B(x,r)} w(z)\,\mathrm{d}z = \frac{1}{\mathscr{L}^n(B(x,r))} \int_{B(x,r)} w(z)\,\mathrm{d}z,$$

and $B(x,r) = \{y : |y - x| < r\}$ is the open ball of radius r centered at x. Henceforth we omit special notation for the precise representative writing simply $w^* = w$.

We will say that x is an L_p Lebesgue point of w (and simply a Lebesgue point when $p = 1$), if

$$\fint_{B(x,r)} |w(z) - w(x)|^p \, \mathrm{d}z \to 0 \quad \text{as} \quad r \searrow 0.$$

If $k < n$, then it is well-known that functions from Sobolev spaces $\mathrm{W}^k_p(\mathbb{R}^n)$ are continuous for $p > \frac{n}{k}$ and could be discontinuous for $p \le p_\circ = \frac{n}{k}$ (see, e.g., [21, 35]). The Sobolev–Lorentz space $\mathrm{W}^k_{p_\circ,1}(\mathbb{R}^n) \subset \mathrm{W}^k_{p_\circ}(\mathbb{R}^n)$ is a refinement of the corresponding Sobolev space that for our purposes turns out to be convenient. Among other things functions that are locally in $\mathrm{W}^k_{p_\circ,1}$ on $\mathbb{R}^n$ are in particular continuous.

Given a measurable function $f\colon \mathbb{R}^n \to \mathbb{R}$, denote by $f_*\colon (0,\infty) \to \mathbb{R}$ its distribution function

$$f_*(s) := \mathscr{L}^n\{x \in \mathbb{R}^n : |f(x)| > s\},$$

and by f^* the nonincreasing rearrangement of f, defined for $t > 0$ by

$$f^*(t) = \inf\{s \ge 0 : f_*(s) \le t\}.$$

Since f and f^* are equimeasurable we have for every $1 \le p < \infty$,

$$\left(\int_{\mathbb{R}^n} |f(x)|^p \, \mathrm{d}x\right)^{1/p} = \left(\int_0^{+\infty} f^*(t)^p \, \mathrm{d}t\right)^{1/p}.$$

The Lorentz space $\mathrm{L}_{p,q}(\mathbb{R}^n)$ for $1 \le p < \infty$, $1 \le q < \infty$ can be defined as the set of all measurable functions $f\colon \mathbb{R}^n \to \mathbb{R}$ for which the expresssion

$$\|f\|_{\mathrm{L}_{p,q}} = \begin{cases} \left(\dfrac{q}{p}\displaystyle\int_0^{+\infty} (t^{1/p} f^*(t))^q \frac{\mathrm{d}t}{t}\right)^{1/q} & \text{if } 1 \le q < \infty \\[2ex] \sup\limits_{t>0} t^{1/p} f^*(t) & \text{if } q = \infty \end{cases}$$

is finite. We refer the reader to [19], [29] or [35] for information about Lorentz spaces. However, let us remark that in view of the definition of $\|\cdot\|_{\mathrm{L}_{p,q}}$ and the equimeasurability of f and f^* we have an identity $\|f\|_{\mathrm{L}_p} = \|f\|_{\mathrm{L}_{p,p}}$ so that in particular $\mathrm{L}_{p,p}(\mathbb{R}^n) = \mathrm{L}_p(\mathbb{R}^n)$. Further, for a fixed exponent p and $q_1 < q_2$ we have an estimate $\|f\|_{\mathrm{L}_{p,q_2}} \le \|f\|_{\mathrm{L}_{p,q_1}}$, and, consequently, an embedding $\mathrm{L}_{p,q_1}(\mathbb{R}^n) \subset \mathrm{L}_{p,q_2}(\mathbb{R}^n)$ (see [19, Theorem 3.8(a)]). Finally we recall that $\|\cdot\|_{\mathrm{L}_{p,q}}$ is a norm on $\mathrm{L}_{p,q}(\mathbb{R}^n)$ for all $q \in [1,p]$ (see [19, Proposition 3.3]).

Here we shall mainly be concerned with the Lorentz space $\mathrm{L}_{p,1}$, and in this case one may rewrite the norm as (see for instance [19, Proposition 3.6])

$$\|f\|_{p,1} = \int_0^{+\infty} \left[\mathscr{L}^n(\{x \in \mathbb{R}^n : |f(x)| > t\})\right]^{\frac{1}{p}} \mathrm{d}t. \tag{4}$$

We need the following subadditivity property of the Lorentz norm.

Lemma 1.2 (see, e.g., [26] or [19]). *Suppose that* $1 \le p < \infty$ *and* $E = \bigcup_{j\in\mathbb{N}} E_j$, *where* E_j *are measurable and mutually disjoint subsets of* $\mathbb{R}^n$. *Then for all* $f \in \mathrm{L}_{p,1}$ *we have*

$$\sum_j \|f \cdot 1_{E_j}\|^p_{\mathrm{L}_{p,1}} \le \|f \cdot 1_E\|^p_{\mathrm{L}_{p,1}},$$

where 1_E *denotes the indicator function of* E.

Denote by $\mathrm{W}^k_{p,1}(\mathbb{R}^n)$ the space of all functions $v \in \mathrm{W}^k_p(\mathbb{R}^n)$ such that in addition the Lorentz norm $\|\nabla v^k\|_{\mathrm{L}_{p,1}}$ is finite. For given dimensions $n, m, \in \mathbb{N}$, $1 \le m \le n$, and $k \in \{1, \dots, n\}$, we denote the corresponding critical exponents by

$$p_\circ = \frac{n}{k} \quad \text{and} \quad q_\circ = m - 1 + \frac{n-m+1}{k} = p_\circ + (m-1)\big(1 - k^{-1}\big). \tag{5}$$

By direct calculation, from $m \ge 1$, $k \ge 1$ we find

$$p_\circ \le q_\circ \le n. \tag{6}$$

Note that in the double inequality (6) we have equality in the first inequality iff $m = 1$ or $k = 1$, while in the second inequality equality holds iff $k = 1$. In particular,

$$p_\circ < q_\circ < n \quad \text{for } k, m \in \{2, \dots, n\}. \tag{7}$$

For a mapping $u \in \mathrm{L}_1(I, \mathbb{R}^d)$, $I \subset \mathbb{R}^n$, define the polynomial $P_I[u] = P_{I,k-1}[u]$ of degree at most $k-1$ by the following rule:

$$\int_I y^\alpha \left(u(y) - P_I[u](y)\right) \mathrm{d}y = 0 \tag{8}$$

for any multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ of length $|\alpha| = \alpha_1 + \dots + \alpha_n \le k-1$.

The following well–known bound will be used on several occasions.

Lemma 1.3. *Suppose* $v \in \mathrm{W}^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^d)$. *Then* v *is a continuous mapping and for any* n-*dimensional interval* $I \subset \mathbb{R}^n$ *the estimate*

$$\sup_{y\in I}\big|v(y) - P_I[v](y)\big| \le C\|1_I \cdot \nabla^k v\|_{\mathrm{L}_{p_\circ,1}} \tag{9}$$

holds, where C *is a constant depending on* n, d *only. Moreover, the mapping* $v_I(y) = v(y) - P_I[v](y)$, $y \in I$, *can be extended from* I *to the whole of* $\mathbb{R}^n$ *such that the extension (denoted again)* $v_I \in \mathrm{W}^k_{p_\circ}(\mathbb{R}^n, \mathbb{R}^d)$ *and*

$$\|\nabla^k v_I\|_{\mathrm{L}_{p_\circ}(\mathbb{R}^n)} \le C_0\|\nabla^k v\|_{\mathrm{L}_{p_\circ}(I)}, \tag{10}$$

where C_0 *also depends on* n, d *only.*

Proof. By well–known estimates (see for instance [11, Lemma 2] or [19, Proposition 3.7]) we have for any Lebesgue point $y \in I$ of v,

$$
\begin{aligned}
|v(y) - P_I[v](y)| &\leq C \int_I \frac{|\nabla^k v(x)|}{|y-x|^{n-k}} \,\mathrm{d}x \\
&\leq C \|1_I \cdot \nabla^k v\|_{\mathrm{L}_{p_\circ,1}} \cdot \left\| \frac{1_I}{|y - \cdot|^{n-k}} \right\|_{\mathrm{L}_{\frac{n}{n-k},\infty}} \\
&\leq C' \|1_I \cdot \nabla^k v\|_{\mathrm{L}_{p_\circ,1}}.
\end{aligned}
$$

From this estimate the continuity of v follows in a routine manner, and thus (9) holds. Because of coordinate invariance of estimate (10), it is sufficient to prove the assertions about extension for the case when I is a unit cube: $I = [0,1]^n$. By results of [21, §1.1.15] for any $u \in \mathrm{W}^{k,p_\circ}(I)$ the estimate

$$
\|u\|_{\mathrm{W}^k_{p_\circ}(I)} \leq c\big(\|P_I[u]\|_{\mathrm{L}_1(I)} + \|\nabla^k u\|_{\mathrm{L}_{p_\circ}(I)}\big), \tag{11}
$$

holds, where $c = c(n,k)$ is a constant. Taking $u(y) = v_I(y) = v(y) - P_I[v](y)$, the first term on the right hand side of (11) vanishes and so we have

$$
\|v_I\|_{\mathrm{W}^k_{p_\circ}(I)} \leq c\|\nabla^k v\|_{\mathrm{L}_{p_\circ}(I)}. \tag{12}
$$

By the Sobolev Extension Theorem, every function $u \in \mathrm{W}^k_{p_\circ}(I)$ on the unit cube $I = [0,1]^n$ can be extended to a function $U \in \mathrm{W}^k_{p_\circ}(\mathbb{R}^n)$ such that the estimate $\|\nabla^k U\|_{\mathrm{L}_{p_\circ}(\mathbb{R}^n)} \leq c\|u\|_{\mathrm{W}^k_{p_\circ}(I)}$ holds, see [21, §1.1.15]). Applying this result coordinatewise to $u = v_I$ and taking into account (12), we obtain the required estimate (10). □

Remark 1.4. The above proof can easily be adapted to give that $v \in \mathrm{C}_0(\mathbb{R}^n)$, the space of continuous functions on $\mathbb{R}^n$ that vanish at infinity (see for instance [19, Theorem 5.5]).

From Lemma 1.3 we deduce the following oscillation estimate.

Corollary 1.5. *Suppose $v \in \mathrm{W}^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^d)$. Then for any n-dimensional interval $I \subset \mathbb{R}^n$ the estimate*

$$
\operatorname{diam} v(I) \leq C\left(\frac{\|\nabla v\|_{\mathrm{L}_1(I)}}{\ell(I)^{n-1}} + \|1_I \cdot \nabla^k v\|_{\mathrm{L}_{p_\circ,1}}\right) \leq C\left(\frac{\|\nabla v\|_{\mathrm{L}_q(I)}}{\ell(I)^{\frac{n}{q}-1}} + \|1_I \cdot \nabla^k v\|_{\mathrm{L}_{p_\circ,1}}\right) \tag{13}
$$

holds for every $q \in [1,n]$, where C depends on n,k only.

Proof. Because of coordinate invariance of estimate (13) it is sufficient to prove the estimates for the case when I is a unit cube: $I = [0,1]^n$. But for a such fixed interval I the estimate follows from (9) and from the fact that the coefficients of the polynomial $P_I[u]$ depend continuously on u with respect to L_1-norm. □

We need a version of the Sobolev Embedding Theorem that gives inclusions in Lebesgue spaces with respect to suitably general positive measures. Very general and precise statements are known, but here we restrict attention to the following class of measures. For $\beta \in (0,n)$ denote by $\mathscr{M}^\beta$ the space of all nonnegative Radon measures μ on $\mathbb{R}^n$ such that

$$\|\mu\|_\beta = \sup_{I \subset \mathbb{R}^n} \ell(I)^{-\beta} \mu(I) < \infty,$$

where the supremum is taken over all n-dimensional intervals $I \subset \mathbb{R}^n$.

Theorem 1.6 (see [21], §1.4.4)**.** *Let μ be a positive Radon measure on $\mathbb{R}^n$ and $p(k-1) < n$, $1 \le p < q < \infty$. Then for any function $v \in \mathrm{W}^k_p(\mathbb{R}^n)$ the estimate*

$$\int |\nabla v|^q \,\mathrm{d}\mu \le C \|\mu\|_\beta \cdot \|\nabla^k v\|^q_{\mathrm{L}_p}, \tag{14}$$

holds with $\beta = \bigl(\frac{n}{p} - k + 1\bigr) q$, where C depends on $n,\ p,\ q,\ k$.

We use also the following important strong-type estimate for maximal functions.

Theorem 1.7 (see Theorem A, Proposition 1 and its Corollary in [1])**.** *Let $\beta \in (0,n)$. Then for nonnegative functions $f \in C_0(\mathbb{R}^n)$ the estimates*

$$\begin{aligned}\int_0^\infty \mathcal{H}^\beta_\infty(\{x \in \mathbb{R}^n : \mathcal{M}f(x) \ge t\})\,\mathrm{d}t &\le C_1 \int_0^\infty \mathcal{H}^\beta_\infty(\{x \in \mathbb{R}^n : f(x) \ge t\})\,\mathrm{d}t \\ &\le C_2 \sup\Bigl\{\int f\,\mathrm{d}\mu : \mu \in \mathscr{M}^\beta,\ \|\mu\|_\beta \le 1\Bigr\},\end{aligned}$$

hold, where the constants C_1, C_2 depend on β, n only and

$$\mathcal{M}f(x) = \sup_{r>0} r^{-n} \int\limits_{B(x,r)} |f(y)|\,\mathrm{d}y$$

is the usual Hardy-Littlewood maximal function of f.

Applying the two foregoing theorems for $p = p_\circ = \frac{n}{k}$, $q = \beta = q_\circ = m - 1 + \frac{n-m+1}{k}$, we obtain the first key ingredient of our proof.

Corollary 1.8. *Let $m, k \in \{2, \dots, n\}$. Then for any function $v \in \mathrm{W}^k_{p_\circ}(\mathbb{R}^n)$ the estimates*

$$\|\nabla v\|^{q_\circ}_{L_{q_\circ}(\mu)} \le C \|\mu\|_{q_\circ} \|\nabla^k v\|^{q_\circ}_{\mathrm{L}_{p_\circ}} \quad \forall \mu \in \mathscr{M}^{q_\circ}, \tag{15}$$

$$\int_0^\infty \mathcal{H}^{q_\circ}_\infty(\{x \in \mathbb{R}^n : \mathcal{M}\bigl(|\nabla v|^{q_\circ}\bigr)(x) \ge t\})\,\mathrm{d}t \le C \|\nabla^k v\|^{q_\circ}_{\mathrm{L}_{p_\circ}} \tag{16}$$

hold, where the exponents $p_\circ, q_\circ$ are defined by (5) *and the constant C depends on n, k, m only.*

For a subset A of $\mathbb{R}^m$ and $\varepsilon > 0$ the ε–entropy of A, denoted by $\mathrm{Ent}(\varepsilon, A)$, is the minimal number of balls of radius ε covering A. Further, for a linear map $L\colon \mathbb{R}^n \to \mathbb{R}^d$ denote by $\lambda_j(L)$, $j = 1, \dots, d$, the lengths of the semiaxes of the ellipsoid $L(B(0,1))$ ordered by the rule $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_d$. Obviously the numbers λ_j are exactly the eigenvalues repeated according to multiplicity of the symmetric nonnegative linear map $\sqrt{LL^*}\colon \mathbb{R}^d \to \mathbb{R}^d$. Also for a differentiable mapping $f\colon \mathbb{R}^n \to \mathbb{R}^d$ put $\lambda_j(f,x) = \lambda_j(d_x f)$, where by $d_x f$ we denote the differential of f at x. The next result is the second basic ingredient of our proof.

Theorem 1.9 ([34])**.** *For any polynomial $P\colon \mathbb{R}^n \to \mathbb{R}^d$ of degree at most k, for each ball $B \subset \mathbb{R}^n$ of radius $r > 0$, and any number $\varepsilon > 0$ the estimate*

$$\operatorname{Ent}\big(\varepsilon r, \{P(x) : x \in B,\ \lambda_1 \le 1+\varepsilon, \dots, \lambda_{m-1} \le 1+\varepsilon, \lambda_m \le \varepsilon, \dots, \lambda_d \le \varepsilon\}\big) \le C_Y\big(1+\varepsilon^{1-m}\big),$$

holds, where the constant C_Y depends on n, d, k only and for brevity we wrote $\lambda_j = \lambda_j(P, x)$.

The application of Corollary 1.8 is facilitated through the following simple estimate (see for instance Lemma 2 in [11]).

Lemma 1.10. *Let $u \in \mathrm{W}_1^1(\mathbb{R}^n)$. Then for any ball $B(z,r) \subset \mathbb{R}^n$, $B(z,r) \ni x$, the estimate*

$$\left| u(x) - \fint_{B(z,r)} u(y)\, dy \right| \le C r (\mathcal{M}\nabla u)(x)$$

holds, where C depends on n only and $\mathcal{M}\nabla u$ is the Hardy-Littlewood maximal function of $|\nabla u|$.

By use of the triangle inequality we then deduce the following oscillation estimate (cf. [6]).

Corollary 1.11. *Let $u \in \mathrm{W}_1^1(\mathbb{R}^n, \mathbb{R}^d)$. Then for any ball $B \subset \mathbb{R}^n$ of radius $r > 0$ and for any number $\varepsilon > 0$ the estimate*

$$\operatorname{diam}(\{u(x) : x \in B,\ (\mathcal{M}\nabla u)(x) \le \varepsilon\}) \le C_M \varepsilon r$$

holds, where C_M is a constant depending on n, d only.

Finally, recall the following approximation properties of Sobolev functions.

Theorem 1.12 (see, e.g., Chapter 3 in [35] or [5])**.** *Let $p \in (1, \infty)$, $k, l \in \{1, \dots, n\}$, $l \le k$, $(k-l)p < n$. Then for any $f \in \mathrm{W}_p^k(\mathbb{R}^n)$ and for each $\varepsilon > 0$ there exist an open set $U \subset \mathbb{R}^n$ and a function $g \in \mathrm{C}^l(\mathbb{R}^n)$ such that*

(i) each point $x \in \mathbb{R}^n \setminus U$ is a Lebesgue point for f and for $\nabla^j f$, $j = 1, \dots, l$;

(ii) $f \equiv g$, $\nabla^j f \equiv \nabla^j g$ on $\mathbb{R}^n \setminus U$ for $j = 1, \dots, l$;

(iii) $\mathscr{L}^n(U) < \varepsilon$ if $l = k$;

(iv) $\mathcal{B}_{k-l,p}(U) < \varepsilon$ if $l < k$, where $\mathcal{B}_{\alpha,p}(U)$ denotes the Bessel capacity of the set U.

Since for $1 < p < \infty$ and $0 < n - \alpha p < \beta \le n$ the smallness of $\mathcal{B}_{\alpha,p}(U)$ implies the smallness of $\mathcal{H}^{\beta}_{\infty}(U)$ (see, e.g., [2]), we have

Corollary 1.13. *Let $k \in \{2, \dots, n\}$ and $v \in \mathrm{W}_{p_\circ}^k(\mathbb{R}^n)$. Then there exists a Borel set $A_v \subset \mathbb{R}^n$ such that $\mathcal{H}^q(A_v) = 0$ for every $q \in (p_\circ, n]$ and all points of $\mathbb{R}^n \setminus A_v$ are Lebesgue points for ∇v. Further, for every $\varepsilon > 0$ and $q \in (p_\circ, n]$ there exist an open set $U \supset A_v$ and a function $g \in \mathrm{C}^1(\mathbb{R}^n)$ such that $\mathcal{H}^q_{\infty}(U) < \varepsilon$ and $v \equiv g$, $\nabla v \equiv \nabla g$ on $\mathbb{R}^n \setminus U$.*

2 On images of sets of small Hausdorff contents

The main result of this section is the following Luzin N–property with respect to Hausdorff content for $\mathrm{W}^k_{p_\circ,1}$–mappings:

Theorem 2.1. *Let $k \in \{2,\dots,n\}$, $q \in (p_\circ, n]$, and $v \in \mathrm{W}^k_{p_\circ,1}(\mathbb{R}^n,\mathbb{R}^d)$. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any set $E \subset \mathbb{R}^n$ if $\mathcal{H}^q_\infty(E) < \delta$, then $\mathcal{H}^q_\infty(v(E)) < \varepsilon$. In particular, $\mathcal{H}^q(v(E)) = 0$ whenever $\mathcal{H}^q(E) = 0$.*

For the case $d = 1$, $k = n$, and $q = p_\circ = 1$ the assertion of Theorem 2.1 was obtained in the paper [8], and the argument given there easily adapts to cover also the cases $k = n$, $q = 1$, and $d > 1$. Our proof here for the remaining cases follows and expands on the ideas from [8].
For the remainder of this section we fix $k \in \{2,\dots,n\}$, $q \in (p_\circ, n]$, and a mapping v in $\mathrm{W}^k_{p_\circ,1}(\mathbb{R}^n,\mathbb{R}^d)$. To prove Theorem 2.1, we need some preliminary lemmas that we turn to next.
By *a dyadic interval* we understand an interval of the form $[\frac{k_1}{2^l}, \frac{k_1+1}{2^l}] \times \cdots \times [\frac{k_n}{2^l}, \frac{k_n+1}{2^l}]$, where k_i, l are integers. The following assertion is straightforward, and hence we omit its proof here.

Lemma 2.2. *For any n-dimensional interval $I \subset \mathbb{R}^n$ there exist dyadic intervals $Q_1,\dots,Q_{2^n}$ such that $I \subset Q_1 \cup \cdots \cup Q_{2^n}$ and $\ell(Q_1) = \cdots = \ell(Q_{2^n}) \le 2\ell(I)$.*

Let $\{I_\alpha\}_{\alpha\in A}$ be a family of n-dimensional dyadic intervals. We say that the family $\{I_\alpha\}$ is *regular*, if for any n-dimensional dyadic interval Q the estimate

$$\ell(Q)^q \ge \sum_{\alpha: I_\alpha \subset Q} \ell(I_\alpha)^q \tag{17}$$

holds. Since dyadic intervals are either disjoint or contained in one another, (17) implies that any regular family $\{I_\alpha\}$ must in particular consist of mutually disjoint[1] intervals.

Lemma 2.3 (see Lemma 2.3 in [8]). *Let $\{I_\alpha\}$ be a family of n–dimensional dyadic intervals. Then there exists a regular family $\{J_\beta\}$ of n–dimensional dyadic intervals such that $\bigcup_\alpha I_\alpha \subset \bigcup_\beta J_\beta$ and*

$$\sum_\beta \ell(J_\beta)^q \le \sum_\alpha \ell(I_\alpha)^q.$$

Lemma 2.4. *For each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, v) > 0$ such that for any regular family $\{I_\alpha\}$ of n–dimensional dyadic intervals we have if*

$$\sum_\alpha \ell(I_\alpha)^q < \delta, \tag{18}$$

then

$$\sum_\alpha \|1_{I_\alpha} \cdot \nabla^k v\|^q_{\mathrm{L}_{p_\circ,1}} < \varepsilon \tag{19}$$

and

$$\sum_\alpha \frac{1}{\ell(I_\alpha)^{n-q}} \int_{I_\alpha} |\nabla v|^q < \varepsilon. \tag{20}$$

[1]By *disjoint dyadic intervals* we mean intervals with disjoint interior.

Proof. Fix $\varepsilon \in (0,1)$ and let $\{I_\alpha\}$ be a regular family of n–dimensional dyadic intervals satisfying (18), where $\delta > 0$ will be specified below.

We start by checking (19). Of course, for sufficiently small δ we can achieve that $\|1_{I_\alpha} \cdot \nabla^k v\|_{\mathrm{L}_{p_\circ,1}}$ is strictly less than say 1 for every α. Then in view of the inequalities $q > p_\circ$ and Lemma 1.2 we have

$$\sum_\alpha \|1_{I_\alpha} \cdot \nabla^k v\|^q_{\mathrm{L}_{p_\circ,1}} \le \sum_\alpha \|1_{I_\alpha} \cdot \nabla^k v\|^{p_\circ}_{\mathrm{L}_{p_\circ,1}} \le \|1_{\bigcup_\alpha I_\alpha} \cdot \nabla^k v\|^{p_\circ}_{\mathrm{L}_{p_\circ,1}}.$$

Using (4), we can rewrite the last estimate as

$$\sum_\alpha \|1_{I_\alpha} \cdot \nabla^k v\|^q_{\mathrm{L}_{p_\circ,1}} \le \left(\int_0^{+\infty} \left[\mathscr{L}^n(\{x \in \bigcup_\alpha I_\alpha : |\nabla^k v(x)| > t\})\right]^{\frac{1}{p_\circ}} \mathrm{d}t \right)^{p_\circ}. \tag{21}$$

Since

$$\int_0^{+\infty} \left[\mathscr{L}^n(\{x \in \mathbb{R}^n : |\nabla^k v(x)| > t\})\right]^{\frac{1}{p_\circ}} \mathrm{d}t < \infty,$$

it follows that the integral on the right–hand side of (21) tends to zero as $\mathscr{L}^n(\bigcup_\alpha I_\alpha) \to 0$. In particular, it will be less than ε if the condition (18) is fulfilled with a sufficiently small δ. Thus (19) is established for all $\delta \in (0, \delta_1]$, where $\delta_1 = \delta_1(\varepsilon, v) > 0$.

Next we check (20). By virtue of Lemma 1.1, applied coordinate–wise, we can find a decomposition $v = v_0 + v_1$, where $\|\nabla v_0\|_{\mathrm{L}^\infty} \le K = K(\varepsilon, v)$ and

$$\|\nabla^k v_1\|_{\mathrm{L}_{p_\circ}} < \varepsilon. \tag{22}$$

Assume that $\delta \in (0, \delta_1]$ and

$$\sum_\alpha \ell(I_\alpha)^q < \delta < \tfrac{1}{K^q+1}\varepsilon. \tag{23}$$

Define the measure μ by

$$\mu = \left(\sum_\alpha \frac{1}{\ell(I_\alpha)^{n-q}} 1_{I_\alpha} \right) \mathscr{L}^n, \tag{24}$$

where 1_{I_α} denotes the indicator function of the set I_α.

Claim. The estimate

$$\sup_I \{\ell(I)^{-q} \mu(I)\} \le 2^{n+q} \tag{25}$$

holds, where the supremum is taken over all n–dimensional intervals.

Indeed, write for a dyadic interval Q

$$\mu(Q) = \sum_{\alpha : I_\alpha \subset Q} \ell(I_\alpha)^q + \sum_{\alpha : I_\alpha \not\subseteq Q} \frac{\ell(Q \cap I_\alpha)^n}{\ell(I_\alpha)^{n-q}}.$$

By regularity of $\{I_\alpha\}$ the first sum is bounded above by $\ell(Q)^q$. If the second sum is nonzero then there must exist an index α such that $I_\alpha \nsubseteq Q$ and I_α, Q overlap. But as the intervals $\{I_\alpha\}$ are disjoint and dyadic we must then precisely have one such interval I_α and $I_\alpha \supset Q$. But then the first sum is empty and the second sum has only the one term $\ell(Q)^n/\ell(I_\alpha)^{n-q}$, hence is at most $\ell(Q)^q$. Thus the estimate $\mu(Q) \leq \ell(Q)^q$ holds for dyadic Q. The inequality (25) in the case of a general interval I follows from the above dyadic case and Lemma 2.2. The proof of the claim is complete.

Now return to (20). By properties (22), (14) (applied to the mapping v_1 and parameters $p = p_\circ,\ \beta = (\frac{n}{p_\circ} - k + 1)q = q$), we have

$$\begin{aligned}\sum_\alpha \frac{1}{\ell(I_\alpha)^{n-q}} \int_{I_\alpha} |\nabla v|^q &\leq \frac{K^q}{K^q+1}\varepsilon + \sum_\alpha \frac{1}{\ell(I_\alpha)^{n-q}} \int_{I_\alpha} |\nabla v_1|^q \\ &\leq C'\varepsilon + \int |\nabla v_1|^q \, d\mu \leq C''\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the proof of Lemma 2.4 is complete. □

Proof of Theorem 2.1. Fix $\varepsilon > 0$ and take $\delta = \delta(\varepsilon, v)$ from Lemma 2.4. Then by Corollary 1.5 for any regular family $\{I_\alpha\}$ of n–dimensional dyadic intervals we have if $\sum_\alpha \ell(I_\alpha)^q < \delta$, then $\sum_\alpha \big(\operatorname{diam} v(I_\alpha)\big)^q < C\varepsilon$. Now we may conclude the proof of Theorem 2.1 by use of Lemmas 2.2 and 2.3. Indeed they allow us to find a $\delta_0 > 0$ such that if for a subset E of $\mathbb{R}^n$ we have $\mathcal{H}^q_\infty(E) < \delta_0$, then E can be covered by a regular family $\{I_\alpha\}$ of n– dimensional dyadic intervals with $\sum_\alpha \ell(I_\alpha)^q < \delta$. □

Remark 2.5. Recall, that the assertion of Theorem 2.1 is also true for $k = n, q = p_\circ = 1$ by [8] and for $k = 1, q = p_\circ = n$ by results of [16]. Note that the order of integrability $p_\circ$ is sharp: for example, the Luzin N–property fails in general for continuous mappings $v \in \mathrm{W}^1_n(\mathbb{R}^n, \mathbb{R}^n)$ (here $k = 1, q = p_\circ = n$), see, e.g., [18].

3 Morse–Sard–Federer theorem for Sobolev mappings

Let $k, m \in \{2, \dots, n\}$ and $v \in \mathrm{W}^k_{p_\circ,\mathrm{loc}}(\Omega, \mathbb{R}^d)$, where Ω is an open subset of $\mathbb{R}^n$. Then, by Corollary 1.13, there exists a Borel set A_v such that $\mathcal{H}^{q_\circ}(A_v) = 0$ and all points of the complement $\Omega \setminus A_v$ are Lebesgue points for the gradient $\nabla v(x)$. We remark that with the assumed Sobolev regularity the mapping v need not be differentiable at any point of Ω, and that $\nabla v(x)$ simply is the precise representative of the weak gradient of v. There are of course many other ways to give pointwise meaning to $\nabla v(x)$, but as these play no role in our considerations here we omit any further discussion. Denote $Z_{v,m} = \{x \in \Omega \setminus A_v : \operatorname{rank}\nabla v(x) < m\}$. We can now state the main result of the section:

Theorem 3.1. *If $k, m \in \{2, \dots, n\}$, Ω is an open subset of $\mathbb{R}^n$, and $v \in \mathrm{W}^k_{p_\circ,\mathrm{loc}}(\Omega, \mathbb{R}^d)$, then $\mathcal{H}^{q_\circ}(v(Z_{v,m})) = 0$.*

The exponents occuring in the theorem are the critical exponents that were defined in (5):

$$p_\circ = \frac{n}{k} \quad \text{and} \quad q_\circ = m - 1 + \frac{n-m+1}{k}.$$

We emphasize the fact that, in contrast with the Luzin $N-$ property with respect to Hausdorff content of Theorem 2.1, the Morse–Sard–Federer Theorem 3.1 is valid within the wider context of W^k_p–Sobolev spaces (finiteness of the Lorentz norm is **not required**).
Before embarking on the detailed proof let us make some preliminary observations that will enable us to make some convenient additional assumptions. Namely because the result is local we can without loss in generality assume that $\Omega = \mathbb{R}^n$ and that $v \in \mathrm{W}^k_{p_\circ}(\mathbb{R}^n, \mathbb{R}^d)$. Indeed note that it suffices to prove that

$$\mathcal{H}^{q_\circ}(v(Z_{v,m} \cap \Omega')) = 0 \tag{26}$$

for all smooth domains Ω' whose closure $\overline{\Omega'}$ is compact and contained in Ω. Now such domains Ω' are extension domains for $\mathrm{W}^k_{p_\circ}$ and so $v|_{\Omega'}$ can be extended to $V \in \mathrm{W}^k_{p_\circ}(\mathbb{R}^n, \mathbb{R}^d)$, and hence proving the statement for V we deduce (26) and therefore prove the theorem.

For the remainder of the section we fix $k, m \in \{2, \dots n\}$ and a mapping $v \in \mathrm{W}^k_{p_\circ}(\mathbb{R}^n, \mathbb{R}^d)$. In view of the definition of critical set adopted here we have that

$$Z_{v,m} = \bigcup_{j \in \mathbb{N}} \{x \in Z_{v,m} : \, |\nabla v(x)| \le j\}.$$

Consequently we only need to prove that $\mathcal{H}^{q_\circ}(Z'_v) = 0$, where

$$Z'_v = \{x \in Z_{v,m} : |\nabla v(x)| \le 1\}.$$

The following lemma contains the main step in the proof of Theorem 3.1.

Lemma 3.2. *For any n-dimensional dyadic interval $I \subset \mathbb{R}^n$ the estimate*

$$\mathcal{H}^{q_\circ}_\infty(v(Z'_v \cap I)) \le C\big(\|\nabla^k v\|^{q_\circ}_{\mathrm{L}_{p_\circ}(I)} + \ell(I)^{m-1}\|\nabla^k v\|^{1-m+q_\circ}_{\mathrm{L}_{p_\circ}(I)}\big) \tag{27}$$

holds, where the constant C depends on n, m, k, d only.

Proof. By virtue of (10) it suffices to prove that

$$\mathcal{H}^{q_\circ}_\infty(v(Z'_v \cap I)) \le C\big(\|\nabla^k v_I\|^{q_\circ}_{\mathrm{L}_{p_\circ}(\mathbb{R}^n)} + \ell(I)^{m-1}\|\nabla^k v_I\|^{1-m+q_\circ}_{\mathrm{L}_{p_\circ}(\mathbb{R}^n)}\big) \tag{28}$$

for the mapping v_I defined in Lemma 1.3, where $C = C(n, m, k, d)$ is a constant.

Fix an n-dimensional dyadic interval $I \subset \mathbb{R}^n$ and recall that $v_I(x) = v(x) - P_I(x)$ for all $x \in I$. Denote

$$\sigma = \|\nabla^k v_I\|^{q_\circ}_{\mathrm{L}_{p_\circ}}, \quad \sigma_* = \ell(I)^{m-1}\|\nabla^k v_I\|^{1-m+q_\circ}_{\mathrm{L}_{p_\circ}},$$

and for each $j \in \mathbb{Z}$

$$E_j = \big\{x \in \mathbb{R}^n : (\mathcal{M}|\nabla v_I|^{q_\circ})(x) \in (2^{j-1}, 2^j]\big\} \quad \text{and} \quad \delta_j = \mathcal{H}^{q_\circ}_\infty(E_j).$$

Then by Corollary 1.8,

$$\sum_{j=-\infty}^{\infty} \delta_j 2^j \le C\sigma$$

for a constant C depending on n, m, k, d only. By construction, for each $j \in \mathbb{Z}$ there exists a family of balls $B_{ij} \subset \mathbb{R}^n$ of radii r_{ij} such that

$$E_j \subset \bigcup_{i=1}^{\infty} B_{ij} \quad \text{and} \quad \sum_{i=1}^{\infty} r_{ij}^{q_\circ} \le 2^{q_\circ}\delta_j.$$

Denote

$$Z_j = Z'_v \cap I \cap E_j \quad \text{and} \quad Z_{ij} = Z_j \cap B_{ij}.$$

By construction $Z'_v \cap I = \bigcup_j Z_j$ and $Z_j = \bigcup_i Z_{ij}$. Put

$$\varepsilon_* = \frac{1}{\ell(I)} \|\nabla^k v_I\|_{\mathrm{L}_{p_\circ}},$$

and let j_* be the integer satisfying $\varepsilon_*^{q_\circ} \in (2^{j_*-1}, 2^{j_*}]$. Denote $Z_* = \bigcup_{j<j_*} Z_j$, $Z_{**} = \bigcup_{j \ge j_*} Z_j$. Then by construction

$$Z'_v \cap I = Z_* \cup Z_{**}, \qquad Z_* \subset \{x \in Z'_v \cap I : (\mathcal{M}|\nabla v_I|^{q_\circ})(x) < \varepsilon_*^{q_\circ}\}.$$

Since $\nabla P_I(x) = \nabla v(x) - \nabla v_I(x)$, $|\nabla v_I(x)| \le 2^{j/q_\circ}$, $|\nabla v(x)| \le 1$, and $\lambda_\nu(v, x) = 0$ for $x \in Z_{ij}$ and $\nu \in \{m, \dots, d\}$, we have[2]

$$Z_{ij} \subset \Big\{x \in B_{ij} : \lambda_1(P_I, x) \le 1 + 2^{j/q_\circ}, \dots, \lambda_{m-1}(P_I, x) \le 1 + 2^{j/q_\circ},$$
$$\lambda_m(P_I, x) \le 2^{j/q_\circ}, \dots, \lambda_d(P_I, x) \le 2^{j/q_\circ}\Big\}.$$

Applying Theorem 1.9 and Corollary 1.11 to mappings P_I, v_I, respectively, with $B = B_{ij}$ and $\varepsilon = \varepsilon_j = 2^{j/q_\circ}$, we find a finite family of balls $T_s \subset \mathbb{R}^d$, $s = 1, \dots, s_j$ with $s_j \le C_Y(1+\varepsilon_j^{1-m})$, each of radius $(1 + C_M)\varepsilon_j r_{ij}$, such that

$$\bigcup_{s=1}^{s_j} T_s \supset v(Z_{ij}).$$

Therefore, for $j \ge j_*$ we have

$$\mathcal{H}^{q_\circ}_\infty(v(Z_{ij})) \le C_1 s_j \varepsilon_j^{q_\circ} r_{ij}^{q_\circ} = C_2(1 + \varepsilon_j^{1-m}) 2^j r_{ij}^{q_\circ} \le C_2(1 + \varepsilon_*^{1-m}) 2^j r_{ij}^{q_\circ}, \tag{29}$$

where all the constants C_ν above depend on n, m, d only. By the same reasons, but this time applying Theorem 1.9 and Corollary 1.11 with $\varepsilon = \varepsilon_*$ and instead of the balls B_{ij} we take a ball $B \supset I$ with radius $r = \sqrt{n}\ell(I)$, we have

$$\mathcal{H}^{q_\circ}_\infty(v(Z_*)) \le C_3(1 + \varepsilon_*^{1-m}) \varepsilon_*^{q_\circ} \ell(I)^{q_\circ} = C_3(1 + \varepsilon_*^{1-m})\sigma = C_3(\sigma + \sigma_*). \tag{30}$$

[2]Here we use the following elementary fact: for any linear maps $L_1 : \mathbb{R}^n \to \mathbb{R}^d$ and $L_2 : \mathbb{R}^n \to \mathbb{R}^d$ the estimates $\lambda_k(L_2 + L_2) \le \lambda_k(L_1) + \|L_2\|$ hold for all $k = 1, \dots, d$, see, e.g., [34, Proposition 2.5 (ii)].

From (29) we get immediately

$$\mathcal{H}^{q_\circ}_\infty(v(Z_{**})) \le \sum_{j\ge j_*}\sum_i C_2(1+\varepsilon_*^{1-m})2^j r_{ij}^{q_\circ} \le \sum_{j\ge j_*} C_2(1+\varepsilon_*^{1-m})2^{j+q_\circ}\delta_j$$

$$\le C_4(1+\varepsilon_*^{1-m})\sigma = C_4(\sigma+\sigma_*).$$

The last two estimates combine to give $\mathcal{H}^{q_\circ}_\infty(v(Z'_v\cap I)) = \mathcal{H}^{q_\circ}_\infty(v(Z_*\cup Z_{**})) \le C(\sigma+\sigma_*)$, and hence finish the proof of the lemma. □

Corollary 3.3. *For any $\varepsilon>0$ there exists $\delta>0$ such that for any subset E of $\mathbb{R}^n$ we have $\mathcal{H}^{q_\circ}_\infty(v(Z'_v\cap E)) \le \varepsilon$ provided $\mathscr{L}^n(E)\le\delta$. In particular, $\mathcal{H}^{q_\circ}(v(Z'_v\cap E))=0$ whenever $\mathscr{L}^n(E)=0$.*

Proof. Let $\mathscr{L}^n(E)\le\delta$, then we can find a family of disjoint n-dimensional dyadic intervals I_α such that $E\subset\bigcup_\alpha I_\alpha$ and $\sum\limits_\alpha \ell^n(I_\alpha) < C\delta$. Of course, for sufficiently small δ the estimate $\|\nabla^k v\|_{\mathrm{L}_{p_\circ}(I_\alpha)}<1$ is fulfilled for every α. Then in view of $q_\circ>p_\circ$ and Lemma 1.2 we have

$$\sum_\alpha \|\nabla^k v\|^{q_\circ}_{\mathrm{L}_{p_\circ}(I_\alpha)} \le \|\nabla^k v\|^{p_\circ}_{\mathrm{L}_{p_\circ}(\bigcup I_\alpha)} \tag{31}$$

Analogously, by Hölder inequality and by virtue of the equalities $1-m+q_\circ = \frac{n-m+1}{k}$ and $(1-m+q_\circ)\frac{n}{n-m+1} = \frac{n}{k} = p_\circ$, we have

$$\begin{aligned}\sum_\alpha \ell(I_\alpha)^{m-1}\|\nabla^k v\|^{1-m+q_\circ}_{\mathrm{L}_{p_\circ}(I_\alpha)} &\le \left(\sum_\alpha \ell(I_\alpha)^n\right)^{\frac{m-1}{n}}\left(\sum_\alpha \|\nabla^k v\|^{p_\circ}_{\mathrm{L}_{p_\circ}(I_\alpha)}\right)^{\frac{n-m+1}{n}}\\ &\le \delta^{\frac{m-1}{n}}\|\nabla^k v\|^{\frac{n-m+1}{k}}_{\mathrm{L}_{p_\circ}(\bigcup I_\alpha)}.\end{aligned}$$

The last two estimates together with Lemma 3.2 allow us to conclude the required smallness of

$$\sum_\alpha \mathcal{H}^{q_\circ}_\infty(Z'_v\cap I_\alpha) \ge \mathcal{H}^{q_\circ}_\infty(Z'_v\cap E).$$

□

Invoking Federer's Theorem for the smooth case $g\in \mathrm{C}^k(\mathbb{R}^n)$, Theorem 1.12 (iii) (applied to the case $k=l$) implies

Corollary 3.4 (see, e.g., [10])**.** *There exists a set $\widetilde{Z}_v$ of n-dimensional Lebesgue measure zero such that $\mathcal{H}^{q_\circ}(v(Z'_v\setminus\widetilde{Z}_v))=0$. In particular, $\mathcal{H}^{q_\circ}(v(Z'_v)) = \mathcal{H}^{q_\circ}(v(\widetilde{Z}_v))$.*

From Corollaries 3.3 and 3.4 we conclude that $\mathcal{H}^{q_\circ}(v(Z'_v))=0$, and this ends the proof of Theorem 3.1.

Theorem 3.1 implies the following analog of the classical Morse–Sard Theorem:

Corollary 3.5. *Let Ω be an open subset of $\mathbb{R}^n$. If $m \in \{1, \dots, n\}$ and $v \in \mathrm{W}^{n-m+1}_{\frac{n}{n-m+1},\mathrm{loc}}(\Omega, \mathbb{R}^m)$, then $\mathscr{L}^m(v(Z_{v,m})) = 0$.*

This assertion follows directly from Theorem 3.1 for $m > 1$ and from the results of [8] for $m = 1$.

Remark 3.6. Arguments of [8] allow also to prove the assertion of Theorem 3.1 for the vectorial case $d > 1$, $k = n$, $m = 1$.

4 On differentiability properties of Sobolev-Lorentz functions

We start with the following simple technical observation.

Lemma 4.1. *If $k \in \{2, \dots, n\}$ and $v \in \mathrm{W}^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^d)$, then for $\mathcal{H}^{p_\circ}$–almost all $x \in \mathbb{R}^n$,*

$$\lim_{r \searrow 0} r^{-1} \|1_{B(x,r)} \cdot \nabla^k v\|_{\mathrm{L}_{p_\circ,1}} = 0$$

holds.

Proof. Fix $\varepsilon > 0$. Let $\{B_\alpha\}$ be a family of disjoint balls $B_\alpha = B(x_\alpha, r_\alpha)$ such that

$$\|1_{B_\alpha} \cdot \nabla^k v\|_{\mathrm{L}_{p_\circ,1}} \geq \varepsilon r_\alpha$$

and $\sup_\alpha r_\alpha < \delta$ for some $\delta > 0$, where δ is choosen small enough to guarantee $\sup_\alpha \|1_{B_\alpha} \cdot \nabla^k v\|_{\mathrm{L}_{p_\circ,1}} < 1$. Then by Lemma 1.2 we have

$$\sum_\alpha r_\alpha^{p_\circ} \leq \varepsilon^{-1} \sum_\alpha \|1_{B_\alpha} \cdot \nabla^k v\|^{p_\circ}_{\mathrm{L}_{p_\circ,1}} \leq \varepsilon^{-1} \|1_{\bigcup_\alpha B_\alpha} \cdot \nabla^k v\|^{p_\circ}_{\mathrm{L}_{p_\circ,1}}. \tag{32}$$

Since the last term tends to 0 as $\mathscr{L}^n(\bigcup_\alpha B_\alpha) \to 0$, and $\mathscr{L}^n(\bigcup_\alpha B_\alpha) \leq \delta^{n-p_\circ} \sum_\alpha r_\alpha^{p_\circ}$, we get easily that $\sum_\alpha r_\alpha^{p_\circ} \to 0$ as $\delta \to 0$. Using this fact and some standard covering lemmas we arrive in a routine manner at the required assertion

$$\mathcal{H}^{p_\circ}\big(\{x \in \mathbb{R}^n : \limsup_{r \searrow 0} r^{-1} \|1_{B(x,r)} \cdot \nabla^k v\|_{\mathrm{L}_{p_\circ,1}} \geq \varepsilon\}\big) = 0.$$

□

From the last lemma, Corollary 1.13 and estimate (13) we obtain the following result that is probably well–known to specialists:

Theorem 4.2. *Let $k \in \{2, \dots, n\}$ and $v \in \mathrm{W}^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^d)$. Then there exists a Borel set $A_v \subset \mathbb{R}^n$ such that $\mathcal{H}^q(A_v) = 0$ for every $q \in (p_\circ, n]$ and for any $x \in \mathbb{R}^n \setminus A_v$ the function v is differentiable (in the classical Fréchet sense) at x, furthermore, the classical derivative coincides with $\nabla v(x)$, where*

$$\lim_{r \searrow 0} \fint_{B(x,r)} |\nabla v(z) - \nabla v(x)| \, \mathrm{d}z = 0.$$

The case $k = 1$, $q = p_\circ = n$ is a classical result due to Stein [28] (see also [16]), and for $m = 1$, $k = n$ the result is also proved in [11].

5 Application to the level sets of Sobolev-Lorentz mappings

Applying Theorems 2.1 and 3.1 in combination with the Corollary 1.13, we obtain

Corollary 5.1. *Let* $k, m \in \{2, \dots, n\}$, $v \in \mathrm{W}^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^d)$, *and* $\mathrm{rank}\nabla v(x) \le m$ *for all* $x \in \mathbb{R}^n \setminus A_v$. *Then for any* $\varepsilon > 0$ *there exist an open set* $V \subset \mathbb{R}^d$ *and a mapping* $g \in \mathrm{C}^1(\mathbb{R}^n, \mathbb{R}^d)$ *such that* $\mathcal{H}^{q_\circ}_\infty(V) < \varepsilon$, $v(A_v) \subset V$ *and* $v|_{v^{-1}(\mathbb{R}^d \setminus V)} = g|_{v^{-1}(\mathbb{R}^d \setminus V)}$, $\nabla v|_{v^{-1}(\mathbb{R}^d \setminus V)} = \nabla g|_{v^{-1}(\mathbb{R}^d \setminus V)}$, *and* $\mathrm{rank}\nabla v|_{v^{-1}(\mathbb{R}^d \setminus V)} \equiv m$.

Here A_v is the Borel set with $\mathcal{H}^{q_\circ}(A_v) = 0$ from Theorem 4.2.

Theorem 5.2. *Let* $k, m \in \{2, \dots, n\}$ *and* $v \in \mathrm{W}^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^m)$. *Then for* $\mathscr{L}^m$*–almost all* $y \in v(\mathbb{R}^n)$ *the preimage* $v^{-1}(y)$ *is a finite disjoint family of* $(n-m)$*–dimensional* C^1*-smooth compact manifolds (without boundary)* S_j, $j = 1, \dots, N(y)$.

Proof. The inclusion $v \in \mathrm{W}^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^m)$ and Lemma 1.3 easily imply the following statement (see also Remark 1.4):

(i) For any $\varepsilon > 0$ there exists $R_\varepsilon \in (0, +\infty)$ such that $|v(x)| < \varepsilon$ for all $x \in \mathbb{R}^n \setminus B(0, R_\varepsilon)$.

Fix an arbitrary $\varepsilon > 0$. Take the corresponding set $V \subset \mathbb{R}^m$ and mapping $g \in \mathrm{C}^1(\mathbb{R}^n, \mathbb{R}^m)$ from Corollary 5.1. Let $0 \ne y \in \mathbb{R}^m \setminus V$. Denote $F_v = v^{-1}(y)$, $F_g = g^{-1}(y)$. We assert the following properties of these sets.

(ii) F_v is a compact set;

(iii) $F_v \subset F_g$;

(iv) $\nabla v = \nabla g$ and $\mathrm{rank}\nabla v = \mathrm{rank}\nabla g = m$ on F_v;

(v) The function v is differentiable (in the classical sense) at each $x \in F_v$, and the classical derivative coincides with $\nabla v(x) = \lim\limits_{r \searrow 0} ⨍_{B(x,r)} \nabla v(z)\,\mathrm{d}z$.

Indeed, (ii) follows by continuity and from (i) since $y \ne 0$, (iii)-(iv) follow from Corollary 5.1, and (v) follows from the condition $v(A_v) \subset V$ of Corollary 5.1 (see also Theorem 4.2).
We require one more property of these sets:

(vi) For any $x_0 \in F_v$ there exists $r > 0$ such that $F_v \cap B(x_0, r) = F_g \cap B(x_0, r)$.

Indeed, take any point $x_0 \in F_v$ and suppose the claim (vi) is false. Then there exists a sequence of points $F_g \setminus F_v \ni x_i \to x_0$. For $r > 0$ we put

$$H_m = (\ker d_{x_0} g)^\perp \cap B(0, r), \quad S_m = (\ker d_{x_0} g)^\perp \cap \partial B(0, r),$$

$$H_m(x) = x + H_m, \quad S_m(x) = x + S_m,$$

where $(\ker d_{x_0} g)^{\perp}$ is the orthogonal complement of the $(n-m)$-dimensional linear subspace $\ker d_{x_0} g$. Evidently, for sufficiently small $r > 0$ we have $H_m(x) \cap F_g = \{x\}$ for any $x \in F_g \cap B(x_0, r)$. Then by construction

$$H_m(x_i) \cap F_v = \emptyset \tag{33}$$

for sufficiently large i. Since v is differentiable (in the classical sense) at x_0 with $\nabla v(x_0) = \nabla g(x_0)$, for sufficiently small $r > 0$ we have $v(x) \neq y$ for all $x \in S_m(x_0)$, and $\deg(v, H_m(x_0), y) = \pm 1$, where we denote by $\deg(v, H_m(x_0), y)$ the topological degree of $v|_{H_m(x_0)}$ at y. Then for sufficiently large i we must have $v^{-1}(y) \cap S_m(x_i) = \emptyset$ and $\deg(v, H_m(x_i), y) = \deg(v, H_m(x_0), y) = \pm 1$. But this contradicts (33) and finishes the proof of (vi).

Obviously, (ii)–(vi) imply that each connected component of the set $F_v = v^{-1}(y)$ is a compact $(n-m)$-dimensional C^1- smooth manifold (without boundary). □

Remark 5.3. The assertion of Theorem 5.2 is also true for $k = 1, m = n$ by results of [16] and for $k = n, m = 1$ by [8].

Remark 5.4. Since for an open set $U \subset \mathbb{R}^n$ of finite measure the estimate $\|1_U \cdot f\|_{\mathrm{L}_{p_\circ,1}} \leq C_U \|f\|_{\mathrm{L}_p(U)}$ holds for $p > p_\circ$ (see, e.g., [19, Theorem 3.8]), the results of the above theorems 2.1, 3.1, 4.2, and 5.2 are in particular valid for mappings $v \in \mathrm{W}^k_p(\mathbb{R}^n, \mathbb{R}^d)$ with $p > p_\circ = \frac{n}{k}$.

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