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EVOLUTION OF DAMAGE IN COMPOSITES: THE ONE-DIMENSIONAL CASE

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EVOLUTION OF DAMAGE IN COMPOSITES: THE ONE-DIMENSIONAL CASE

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1. Introduction

The definition of quasi-static evolution has been recently widely analyzed in the framework of energetic solutions (see e.g. [2], [7], [8], [11] and [15] and the references therein). The existence of such solutions in general is obtained through the approximation with a family of implicit-time schemes. Namely, given an internal parameter-depending energy $\mathcal{E}$ and a dissipation $\mathcal{D}$, an initial datum U_0 and a time step τ, a discrete trajectory $\{U_j^\tau\}_j$ is defined recursively by setting $U_0^\tau = U_0$ and taking U_j^τ as a minimizer of

$$\min\Big\{\mathcal{E}(j\tau, U) + \mathcal{D}(U - U_{j-1}^\tau)\Big\}. \tag{1.1}$$

A piecewise-constant trajectory $U^\tau(t)$ is then defined by

$$U^\tau(t) = U^\tau_{\lfloor t/\tau \rfloor},$$

and its limit (which exists up to subsequences) is an energetic solution of the evolution inclusion

$$\partial\mathcal{D}(\dot U) + \partial_U \mathcal{E}(t, U) \ni 0.$$

In case of energies and dissipations depending on some parameter ε (e.g., in the case of homogenization, on a small space scale) the same scheme above can be followed. For fixed ε it can be defined a quasi-static evolution U_ε and the limit as $\varepsilon \to 0$ can be studied. Conversely, with fixed τ and ε, we may consider discrete trajectories $U_j^{\tau,\varepsilon}$ defined iteratively as solutions of

$$\min\Big\{\mathcal{E}^\varepsilon(j\tau, U) + \mathcal{D}^\varepsilon(U - U_{j-1}^{\tau,\varepsilon})\Big\} \tag{1.2}$$

and take the limit as $\varepsilon \to 0$ first. Under some coerciveness and continuity assumptions these trajectories converge as $\varepsilon \to 0$ to $U_j^{\tau,0}$, which solve an effective limit problem

$$\min\Big\{\mathcal{F}(j\tau, U, U_{j-1}^{\tau,0})\Big\}, \tag{1.3}$$

where $\mathcal{F}(t, \cdot, V)$ is the Γ-limit of

$$U \mapsto \mathcal{E}^\varepsilon(t, U) + \mathcal{D}^\varepsilon(U - V).$$

We can then define the piecewise constant functions $U^{\tau,0}(t) = U^{\tau,0}_{\lfloor t/\tau \rfloor}$ and take the limit as $\tau \to 0$ to obtain a continuos in time limit $U^0(t)$. In general it is not clear whether the trajectory $U^0(t)$ agrees with the *effective energetic solution* given by the limit of $U^\varepsilon(t)$. Mielke et al. [14] proved that this is the case when $\mathcal{E}^\varepsilon$ and $\mathcal{D}^\varepsilon$ separately Γ-converge to some $\mathcal{E}$ and $\mathcal{D}$ and suitable additional assumptions are satisfied, which in particular imply that

$$\mathcal{F}(t, U, V) = \mathcal{E}(t, U) + \mathcal{D}(U - V)\,, \tag{1.4}$$

and hence the limiting trajectory can be again regarded as an energetic solution. In general the form of the limit $\mathcal{F}$ may not be immediately interpreted as a sum of an internal energy and a dissipation (note that the Γ-convergence of $\mathcal{E}^\varepsilon$ and $\mathcal{D}^\varepsilon$ does not imply the convergence of their sum). In particular, a generalization of the scheme above must be envisaged.

In this paper we illustrate a case when we do have separate Γ-convergence of $\mathcal{E}^\varepsilon$ and $\mathcal{D}^\varepsilon$, but the Γ-limit of the sum does not agree with the sum of the Γ-limits. Nevertheless, the limit $\mathcal{F}$ can be viewed as the sum of an internal energy and a dissipation and the corresponding quasi-static evolution is the limit of the quasi-static evolutions for $\mathcal{E}^\varepsilon$ and $\mathcal{D}^\varepsilon$. This result contributes to the analysis of the interaction between Γ-convergence and variational evolution which has recently attracted much interest both in the framework of energetic solutions and in the theory of gradient flows (see [4, 14, 12, 1, 13, 5]).

We consider the model of damage by Francfort and Marigo [8], where $U = (u, A)$, u is a displacement and A is the damage set,

$$\mathcal{E}(t, U) = E(t, u, A) = \alpha \int_A |\nabla u|^2\, dx + \beta \int_{\Omega\setminus A} |\nabla u|^2\, dx \qquad (\alpha < \beta)$$

with the constraint that $u = g_t$ on the boundary of Ω (given set in $\mathbb{R}^n$), with g_t a continuously parameterized family of boundary data with $g_0 = 0$, and

$$\mathcal{D}(U) = D(A) = \gamma |A|\,.$$

In the formulation (1.1) the dependence of $\mathcal{D}$ on $U - U^\tau_{j-1}$, where $U = (u, A)$ and $U^\tau_{j-1} = (u^\tau_{j-1}, A^\tau_{j-1})$ must be understood as the requirement that $\chi_A - \chi_{A^\tau_{j-1}}$ be a characteristic function; i.e., that $A^\tau_{j-1} \subset A$, so that $\{A^\tau_j\}_j$ is a non-decreasing family of sets. The evolution according to such E and D describes a damage process with a non-decreasing damage zone, driven by the varying boundary value and the competition between the internal energy, which is characterized by the elastic coefficient

$$\sigma_A(x) = \alpha\chi_A(x) + \beta(1 - \chi_A(x))$$

and is lower in the damaged region A, and the dissipation, which accounts for the amount of damaged material. This model is intrinsically non-convex and in general it is not possible to determine a solution of the form $(u(t), A(t))$, with $A(t)$ a family of sets parametrized by t and the problem must be relaxed. In the multi-dimensional case the right framework for the relaxation is that of the G-convergence for the coefficients σ_A (see [8], [6] and [9]). In the one-dimensional case such *weak evolution* can be easily expressed in terms of the weak limits of characteristic functions χ_A. Moreover it can be seen that in this case it is always possible to construct *strong evolutions* of the form $(u(t), A(t))$.

We treat a heterogeneous one-dimensional case, where the functionals take the form

$$E^\varepsilon(t, u, A) = \int_A \alpha\Big(\frac{x}{\varepsilon}\Big)|u'|^2\, dx + \int_{\Omega\setminus A} \beta\Big(\frac{x}{\varepsilon}\Big)|u'|^2\, dx \tag{1.5}$$

and α and β are 1-periodic functions. It is well known that in this case for fixed A these energies Γ-converge to

$$\underline{E}(t, u, A) = \underline{\alpha} \int_A |u'|^2\, dx + \underline{\beta} \int_{\Omega\setminus A} |u'|^2\, dx, \tag{1.6}$$

where $\underline{\alpha}$ and $\overline{\beta}$ are the harmonic means of α and β, respectively. For the sake of simplicity we will consider the case when α and β take only two values, so that E^ε can be interpreted as describing a mixture of two materials with coefficients β_1 and β_2 when undamaged, and α_1 and α_2 when damaged. We also consider dissipations

$$D^\varepsilon(A) = \int_A \gamma\Big(\frac{x}{\varepsilon}\Big)\, dx\,, \tag{1.7}$$

where γ takes the values γ_1 and γ_2. Note that even the case $\gamma_1 = \gamma_2$ (and hence D^ε independent of ε) possesses the same features of the effective evolution as for oscillating γ.

First of all, we note that the Γ-limit of $E^\varepsilon(t,u,A)+D^\varepsilon(A)$ always requires a relaxation process. In fact, minimizing sequences of A will never be compact as sets, and their limit (precisely, the weak limit of their characteristic functions) must be described by a density function $\theta \in [0,1]$. Hence, the limit evolution must be expressed in terms of the relaxed variable (u,θ). In these variables the Γ-limit of $E^\varepsilon(t,u,A) + D^\varepsilon(A)$ takes the form (see Theorem 3.1)

$$\int_{(0,1)} f^{\rm hom}(\theta)|u'|^2 dx + \int_{(0,1)} \gamma^{\rm hom}(\theta)\, dx\,,$$

so that a weak quasi-static evolution can be constructed for this energy. We show that this agrees with the limit of the corresponding strong ε-quasi-static evolutions (see Theorem 3.9).

We show that an equivalent formulation can be given in terms of a *three-phase material* model: the effective evolution can itself be seen as a relaxed evolution of a homogenized energy of the form

$$E(t,u,A_1,A_2) = \underline{\alpha}\int_{A_2} |u'|^2\, dx + C(\alpha,\beta)\int_{A_1} |u'|^2\, dx + \underline{\beta}\int_{\Omega\setminus(A_1\cup A_2)} |u'|^2\, dx$$

with $A_1 \cap A_2 = \emptyset$. The sets A_2 and A_1 can be interpreted, respectively, as the zone where either both materials are damaged, or one of the two (which is uniquely determined by the values of α_i and β_i) is damaged. $C(\alpha,\beta)$ is the corresponding harmonic mean in the latter case.

The effective evolution of a mixture of two homogeneous two-phase materials can therefore be interpreted as the relaxed evolution of a homogeneous three-phase material.

2. Quasi-static evolution for composite materials

In this section we give the definition of quasi-static evolution related to the elastic energy and dissipation in (1.5) and (1.7) for fixed ε, and show explicitly the existence of such evolution.

For fixed $\varepsilon > 0$ we consider the functional

$$E^\varepsilon_{\rm Tot}(u,A) = E^\varepsilon(u,A) + D^\varepsilon(A)\,, \tag{2.1}$$

where

$$E^\varepsilon(u,A) = \int_{(0,1)} \sigma^\varepsilon_A(x)|u'(x)|^2 dx \qquad \text{and} \qquad D^\varepsilon(A) = \int_A \gamma\Big(\frac{x}{\varepsilon}\Big) dx, \tag{2.2}$$

with $u \in H^1(0,1)$, $A \subset (0,1)$,

$$\sigma^\varepsilon_A(x) = \alpha\Big(\frac{x}{\varepsilon}\Big)\chi_A(x) + \beta\Big(\frac{x}{\varepsilon}\Big)(1-\chi_A(x)), \tag{2.3}$$

and

$$\alpha(y) = \begin{cases} \alpha_1 & \text{if } y \in [0, \frac{1}{2}) \\ \alpha_2 & \text{if } y \in [\frac{1}{2}, 1) \end{cases} \qquad \beta(y) = \begin{cases} \beta_1 & \text{if } y \in [0, \frac{1}{2}) \\ \beta_2 & \text{if } y \in [\frac{1}{2}, 1) \end{cases} \tag{2.4}$$

$$\gamma(y) = \begin{cases} \gamma_1 & \text{if } y \in [0, \frac{1}{2}) \\ \gamma_2 & \text{if } y \in [\frac{1}{2}, 1) \end{cases} \tag{2.5}$$

with

$$0 < \alpha_i < \beta_i, \qquad 0 < \gamma_i, \qquad \text{for } i = 1, 2. \tag{2.6}$$

Moreover, we will denote in the following

$$\underline{\alpha} := \Big(\frac{1}{2\alpha_1} + \frac{1}{2\alpha_2}\Big)^{-1} \qquad \text{and} \qquad \underline{\beta} := \Big(\frac{1}{2\beta_1} + \frac{1}{2\beta_2}\Big)^{-1}, \tag{2.7}$$

the *harmonic means* of α_i and β_i, respectively.

We suppose that

$$\varepsilon^{-1} \in \mathbb{N}; \tag{2.8}$$

the general case can be always reduced to this assumption up to a negligible error in the energy (2.1) (as $\varepsilon \to 0$).

2.1. **Minimum problems for the ε-energy.** In the following lemma we characterize the minimizers of (2.1) with prescribed boundary data.

Lemma 2.1. *Let $t \in \mathbb{R}$; then there exists a minimizer $(u^\varepsilon, A^\varepsilon)$ of*

$$m(t) := \min\{E^\varepsilon_{\rm Tot}(u, A) : u(0) = 0, u(1) = t, A \subset (0,1)\}. \tag{2.9}$$

Moreover, $m(t)$ can be computed explicitly and it is independent of ε. If

$$p_1 := \sqrt{\frac{\alpha_1\beta_1\gamma_1}{\beta_1 - \alpha_1}} < \sqrt{\frac{\alpha_2\beta_2\gamma_2}{\beta_2 - \alpha_2}} =: p_2, \tag{2.10}$$

(which we may suppose without loss of generality) then

$$m(t) = \begin{cases} \underline{\beta} t^2 & \text{if } |t| \le \dfrac{p_1}{\underline{\beta}} \\ 2p_1 t - \dfrac{p_1^2}{\underline{\beta}} & \text{if } \dfrac{p_1}{\underline{\beta}} < |t| \le \dfrac{p_1(\beta_2+\alpha_1)}{2\beta_2\alpha_1} \\ \dfrac{2\beta_2\alpha_1}{\beta_2+\alpha_1} t^2 + \dfrac{\gamma_1}{2} & \text{if } \dfrac{p_1(\beta_2+\alpha_1)}{2\beta_2\alpha_1} < |t| \le \dfrac{p_2(\beta_2+\alpha_1)}{2\beta_2\alpha_1} \\ 2p_2 t + \dfrac{\gamma_1+\gamma_2}{2} - \dfrac{p_2^2}{\underline{\alpha}} & \text{if } \dfrac{p_2(\beta_2+\alpha_1)}{2\beta_2\alpha_1} < |t| \le \dfrac{p_2}{\underline{\alpha}} \\ \underline{\alpha} t^2 + \dfrac{\gamma_1+\gamma_2}{2} & \text{if } t \ge \dfrac{p_2}{\underline{\alpha}}. \end{cases} \tag{2.11}$$

The function $m(t)$ is plotted in Fig. 1.

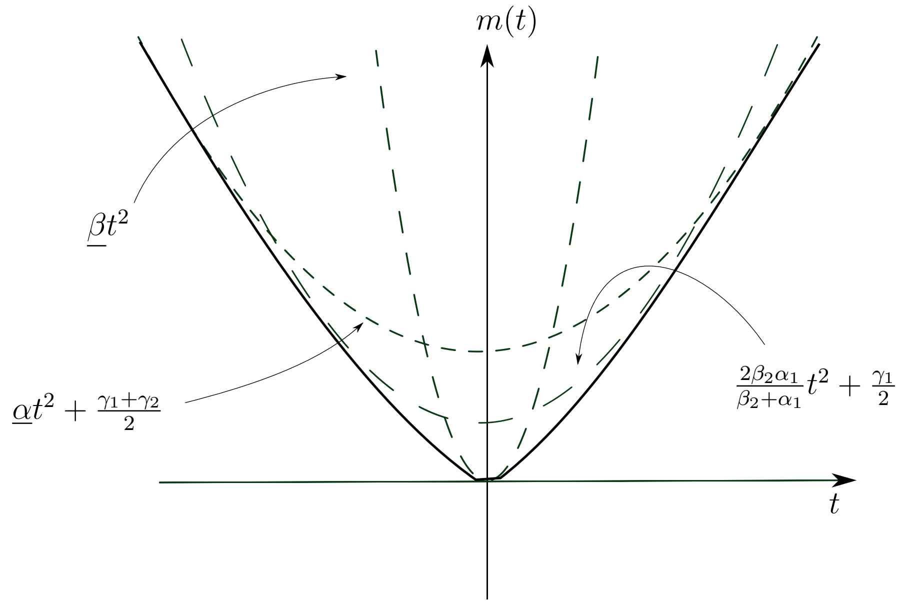

FIGURE 1. The minimal value m

Proof. For $A \subset (0,1)$, we set

$$\begin{aligned} &A_1^\varepsilon := A \cap \Big(\Big[0, \frac{\varepsilon}{2}\Big) + \varepsilon\mathbb{N}\Big), \quad A_2^\varepsilon := A \cap \Big(\Big[\frac{\varepsilon}{2}, \varepsilon\Big) + \varepsilon\mathbb{N}\Big), \\ &B_1^\varepsilon := ((0,1)\setminus A) \cap \Big(\Big[0, \frac{\varepsilon}{2}\Big) + \varepsilon\mathbb{N}\Big), \quad B_2^\varepsilon := ((0,1)\setminus A) \cap \Big(\Big[\frac{\varepsilon}{2}, \varepsilon\Big) + \varepsilon\mathbb{N}\Big). \end{aligned} \tag{2.12}$$

Note that $(0,1) = A_1^\varepsilon \cup A_2^\varepsilon \cup B_1^\varepsilon \cup B_2^\varepsilon$, $\alpha\big(\frac{x}{\varepsilon}\big) = \alpha_i$ for $x \in A_i^\varepsilon$, and $\beta\big(\frac{x}{\varepsilon}\big) = \beta_i$ for $x \in B_i^\varepsilon$ $(i = 1,2)$.

We observe that the value

$$m_A(t) := \min\Big\{E^\varepsilon_{\mathrm{Tot}}(u, A) : u(0) = 0, u(1) = t\Big\} \tag{2.13}$$

depends on A only through the measures $|A_1^\varepsilon|$ and $|A_2^\varepsilon|$. Indeed, by Jensen's inequality and (2.12), for all test functions u we have

$$\begin{aligned} &\int_A \alpha\Big(\frac{x}{\varepsilon}\Big)|u'|^2\,dx + \int_{(0,1)\setminus A} \beta\Big(\frac{x}{\varepsilon}\Big)|u'|^2 dx \\ &\qquad \geq \alpha_1|A_1^\varepsilon||z_{11}|^2 + \beta_1|B_1^\varepsilon||z_{12}|^2 + \alpha_2|A_2^\varepsilon||z_{21}|^2 + \beta_2|B_2^\varepsilon||z_{22}|^2, \end{aligned}$$

where

$$z_{i1} := \frac{1}{|A_i^\varepsilon|}\int_{A_i^\varepsilon} u'\,dx, \quad z_{i2} := \frac{1}{|B_i^\varepsilon|}\int_{B_i^\varepsilon} u'\,dx, \quad i = 1,2, \tag{2.14}$$

with a strict inequality unless u' is constant on A_i^ε and B_i^ε. Hence, each minimizer must have a constant value of the derivative on each of the four sets A_i^ε and B_i^ε. This observation allows to reduce the computation of $m(t)$ to a finite-dimensional minimization. To that end, denote

$$\lambda_i := 2\big|A_i^\varepsilon\big|, \quad i = 1,2. \tag{2.15}$$

Observing that $|B_i^\varepsilon| = \frac{1}{2} - |A_i^\varepsilon| = \frac{1}{2}(1-\lambda_i)$, we have that

(2.16)
$$m(t) = \min_{z_{ij},\lambda_k} \Big\{ \frac{1}{2}(\lambda_1\alpha_1 z_{11}^2 + (1-\lambda_1)\beta_1 z_{12}^2) + \frac{1}{2}(\lambda_2\alpha_2 z_{21}^2 + (1-\lambda_2)\beta_2 z_{22}^2) \\ + \frac{1}{2}\gamma_1\lambda_1 + \frac{1}{2}\gamma_2\lambda_2 : \frac{1}{2}(\lambda_1 z_{11} + (1-\lambda_1)z_{12}) + \frac{1}{2}(\lambda_2 z_{21} + (1-\lambda_2)z_{22}) = t \Big\}.$$

A solution λ_i, z_{ij} $(i,j = 1,2)$ provides a description of all minimizers of problem (2.9) as follows: the set A^ε is any set A such that $2|A_i^\varepsilon| = \lambda_i$, and u^ε is the unique solution of (2.14), which gives

$$u' = z_{i1} \text{ on } A_i^\varepsilon \qquad \text{and} \qquad u' = z_{i2} \text{ on } B_i^\varepsilon, \quad i = 1,2. \tag{2.17}$$

We can explicitly compute the minimum in (2.16). We conclude that $m(t)$ is independent on ε and satisfies

$$m(t) = \frac{1}{2}\min\Big\{ m_1(t_1) + m_2(t_2) : \frac{t_1+t_2}{2} = t \Big\}, \tag{2.18}$$

where

$$m_i(t) := \min_{z_{i1},z_{i2},\lambda_i} \Big\{ \lambda_i\alpha_i z_{i1}^2 + (1-\lambda_i)\beta_i z_{i2}^2 + \gamma_i\lambda_i : \lambda_i z_{i1} + (1-\lambda_i)z_{i2} = t \Big\},$$

whose explicit form is given by

$$m_i(t) = \begin{cases} \beta_i t^2 & \text{if } |t| \le \sqrt{\frac{\alpha_i\gamma_i}{\beta_i(\beta_i-\alpha_i)}} = \frac{p_i}{\beta_i} \\ \alpha_i t^2 + \gamma_i & \text{if } |t| \ge \sqrt{\frac{\beta_i\gamma_i}{\alpha_i(\beta_i-\alpha_i)}} = \frac{p_i}{\alpha_i} \\ 2t\sqrt{\dfrac{\alpha_i\beta_i\gamma_i}{\beta_i-\alpha_i}} - \dfrac{\gamma_i\alpha_i}{\beta_i-\alpha_i} = 2tp_i - \dfrac{p_i^2}{\beta_i} & \text{otherwise.} \end{cases} \tag{2.19}$$

Using (2.19) and solving (2.18) we obtain the expression of $m(t)$ as in (2.11). □

Remark 2.2. 1) We can explicitly compute the minimum values $\lambda_{i,\min}$ in (2.16) which are given by (assuming (2.10), i.e., $p_1 < p_2$)

$$\lambda_{1,\min}(t) = \begin{cases} 0 & \text{if } 0 \le |t| \le \dfrac{p_1}{\underline{\beta}} \\ \dfrac{2p_1}{\gamma_1}\Big(|t| - \dfrac{p_1}{\underline{\beta}}\Big) & \text{if } \dfrac{p_1}{\underline{\beta}} \le |t| \le \dfrac{p_1(\beta_2+\alpha_1)}{2\beta_2\alpha_1} \\ 1 & \text{if } |t| \ge \dfrac{p_1(\beta_2+\alpha_1)}{2\beta_2\alpha_1}, \end{cases} \tag{2.20}$$

and

$$\lambda_{2,\min}(t) = \begin{cases} 0 & \text{if } 0 \le |t| \le \dfrac{p_2(\beta_2+\alpha_1)}{2\beta_2\alpha_1} \\ \dfrac{2p_2}{\gamma_2}\Big(|t| - \dfrac{p_2(\beta_2+\alpha_1)}{2\beta_2\alpha_1}\Big) & \text{if } \dfrac{p_2(\beta_2+\alpha_1)}{2\beta_2\alpha_1} \le |t| \le \dfrac{p_2}{\underline{\alpha}} \\ 1 & \text{if } |t| \ge \dfrac{p_2}{\underline{\alpha}}. \end{cases} \tag{2.21}$$

We get that

$$\lambda_{\min}(t) := |A^\varepsilon| = \begin{cases} \dfrac{\lambda_{1,\min}(t)}{2} & \text{if } |t| < \dfrac{p_2(\beta_2+\alpha_1)}{2\beta_2\alpha_1} \\ \dfrac{1}{2} + \dfrac{\lambda_{2,\min}(t)}{2} & \text{if } |t| \geq \dfrac{p_2(\beta_2+\alpha_1)}{2\beta_2\alpha_1}. \end{cases} \tag{2.22}$$

The value of $\lambda_{\min}$ is plotted in Fig. 2.

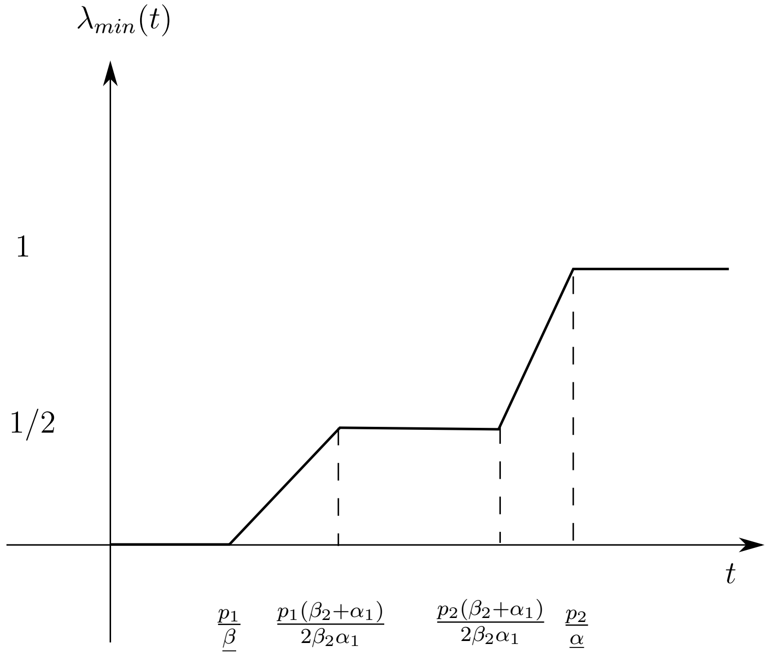

FIGURE 2. The value of $\lambda_{\min}$.

2) The characterization of the minimizers $(u^\varepsilon, A^\varepsilon)$ given by (2.15) and (2.17) gives the existence of infinitely-many minimizers, except in the cases when both $\lambda_{i,\min} \in \{0,1\}$, for which the minimizing pair is unique. Under condition (2.10), i.e., $p_1 < p_2$, this corresponds to $A^\varepsilon = \emptyset$, $A^\varepsilon = (0,1) \cap ([0, \frac{\varepsilon}{2}) + \varepsilon\mathbb{N})$ or $A^\varepsilon = (0,1)$.

Note that the minimality conditions for (2.16) give the relations

$$\alpha_1 z_{11} = \beta_1 z_{21} = \alpha_2 z_{12} = \beta_2 z_{22}. \tag{2.23}$$

3) Among all the minimizers $(u^\varepsilon, A^\varepsilon)$ we have those with

$$A_1^\varepsilon := (0,1) \cap \Big(\Big[0, \lambda_{1,\min}\frac{\varepsilon}{2}\Big) + \varepsilon\mathbb{N}\Big), \quad A_2^\varepsilon = (0,1) \cap \Big(\Big[\frac{\varepsilon}{2}, (1+\lambda_{2,\min})\frac{\varepsilon}{2}\Big) + \varepsilon\mathbb{N}\Big),$$

for which the damage is "uniformly distributed" in $(0,1)$. In this case the weak limit of the characteristic functions of the sets A_i^ε is the constant $\frac{1}{2}\lambda_{i,\min}$.

Another family of minimizers are those with

$$A_1^\varepsilon := (0, \lambda_{1,\min}^\varepsilon) \cap \Big(\Big[0, \frac{\varepsilon}{2}\Big) + \varepsilon\mathbb{N}\Big), \quad A_2^\varepsilon = (0, \lambda_{2,\min}^\varepsilon) \cap \Big(\Big[\frac{\varepsilon}{2}, \varepsilon\Big) + \varepsilon\mathbb{N}\Big),$$

where $\lambda_{i,\min}^\varepsilon$ are such that $2|A_i^\varepsilon| = \lambda_{i,\min}$, for which the damage is "concentrated towards 0". Note that in this case we have $|\lambda_{i,\min}^\varepsilon - \lambda_{i,\min}| \leq \varepsilon$ and hence the weak limit of the characteristic functions of the sets A_i^ε is the function $\frac{1}{2}\chi_{[0,\lambda_{i,\min}]}$.

Remark 2.3. If we introduce the *homogenized coefficient* related to η_1 and η_2 as

$$f^{\mathrm{hom}}(\eta_1,\eta_2) := \Big[\frac{1}{\alpha_1}\eta_1 + \frac{1}{\beta_1}\Big(\frac{1}{2}-\eta_1\Big) + \frac{1}{\alpha_2}\eta_2 + \frac{1}{\beta_2}\Big(\frac{1}{2}-\eta_2\Big)\Big]^{-1}, \tag{2.24}$$

then, remarking that

$$\begin{aligned}\min_{z_{ij}} \Big\{ &\frac{1}{2}(\lambda_1\alpha_1 z_{11}^2 + (1-\lambda_1)\beta_1 z_{12}^2) + \frac{1}{2}(\lambda_2\alpha_2 z_{21}^2 + (1-\lambda_2)\beta_2 z_{22}^2) : \\ &\frac{1}{2}(\lambda_1 z_{11} + (1-\lambda_1)z_{12}) + \frac{1}{2}(\lambda_2 z_{21} + (1-\lambda_2)z_{22}) = 1\Big\} = f^{\mathrm{hom}}\Big(\frac{\lambda_1}{2},\frac{\lambda_2}{2}\Big),\end{aligned} \tag{2.25}$$

we can rewrite

$$m(t) = \min_{\lambda_1,\lambda_2}\Big\{ f^{\mathrm{hom}}\Big(\frac{\lambda_1}{2},\frac{\lambda_2}{2}\Big)t^2 + \frac{1}{2}\gamma_1\lambda_1 + \frac{1}{2}\gamma_2\lambda_2\Big\}. \tag{2.26}$$

Remark 2.4. Let

$$G(\lambda_1,\lambda_2,t) = f^{\mathrm{hom}}\Big(\frac{\lambda_1}{2},\frac{\lambda_2}{2}\Big)t^2 + \frac{\gamma_1}{2}\lambda_1 + \frac{\gamma_2}{2}\lambda_2\,, \tag{2.27}$$

where f^{hom} is defined by (2.24). Then for fixed s and t with $0 \le s \le t$, the unique minimizer of the function $G(\cdot,\cdot,s)$ on $[\lambda_{1,\min}(t),1]\times[\lambda_{2,\min}(t),1]$ is $(\lambda_{1,\min}(t),\lambda_{2,\min}(t))$. This follows from a straightforward, even though somewhat lengthy, calculation.

2.2. **Quasi-static evolution for the ε-energy.** We state now the definition of quasi-static evolution for the energy functional (2.1) and describe explicitly the behaviour of such motions in Theorem 2.8.

From now on we will consider $u = u(t,x)$, with $u(t,\cdot) \in H^1(0,1)$ parametrized by $t \in [0,T]$. As a shorthand we will write $u(t) = u(t,\cdot)$.

Definition 2.5. Given $g \in AC([0,T])$, with $g(0) = 0$, and $\varepsilon > 0$, we say that $(u(t),A(t))$ is a *(strong) quasi-static evolution* (for the energy (2.1) subjected to the boundary condition g) if for all $t \in [0,T]$ we have $u(t) \in H^1(0,1)$, $u(t,0) = 0$, $u(t,1) = g(t)$, $A(t) \subset (0,1)$, and the following properties hold:

- **Damage Irreversibility**: $A(t_1) \subset A(t_2)$ if $t_1 < t_2$;
- **Energy Balance**: for all $t \in [0,T]$ we have

$$E^{\varepsilon}_{\mathrm{Tot}}(u(t),A(t)) = E^{\varepsilon}_{\mathrm{Tot}}(u(0),A(0)) + 2\int_0^t \dot g(s)\int_{(0,1)} \sigma^{\varepsilon}_{A(s)}(x)u'(s,x)\,dxds; \tag{2.28}$$

- **Minimality Condition**: for all $t \in [0,T]$

$$E^{\varepsilon}_{\mathrm{Tot}}(u(t),A(t)) \le E^{\varepsilon}_{\mathrm{Tot}}(v,B) \tag{2.29}$$

for all $v \in H^1(0,1)$ with $v(0) = 0$, $v(1) = g(t)$, and $A(t) \subset B \subset (0,1)$.

Moreover, we say that $(u(t),A(t))$ is an *approximable (strong) quasi-static evolution* (for the energy (2.1) subjected to the boundary condition g) if it satisfies the conditions above, and can be obtained as the limit of a time-discrete approximation scheme; i.e., up to a subsequence, for all t it is the limit of $(u_\tau(t),A_\tau(t))$ constructed as follows: $u_\tau(t) = u^\tau_{\lfloor t/\tau\rfloor}$, $A_\tau(t) = A^\tau_{\lfloor t/\tau\rfloor}$, where $u^\tau_0 = 0$, $A^\tau_0 = \emptyset$, and (u^τ_k, A^τ_k) is a minimizing pair for the problem

$$\min\Big\{E^{\varepsilon}_{\mathrm{Tot}}(v,A) : v(0) = 0,\ v(1) = g(k\tau),\ A^\tau_{k-1} \subset A\Big\} \tag{2.30}$$

for $k \ge 1$.

Remark 2.6. Note that in general not all quasi-static evolutions are approximable (see [12]). We do not address this issue here.

Remark 2.7. By the minimality condition (2.29), with $B = A(t)$, we deduce that $u(t)$ is the unique minimizer of the quadratic energy $E^\varepsilon(v, A(t))$ satisfying the boundary condition $u(t,0) = 0$ and $u(t,1) = g(t)$. Testing the Euler-Lagrange equation with $u(t,x) - g(t)x$ we deduce the identity

$$\int_{(0,1)} \sigma^\varepsilon_{A(t)}(x) u'(t,x)\, dx = f^\varepsilon(A(t))g(t)\,, \tag{2.31}$$

where

$$f^\varepsilon(A) := \min\Big\{ E^\varepsilon(v, A) \,:\, v \in H^1(0,1)\,,\ v(0) = 0\,,\ v(1) = 1 \Big\}\,. \tag{2.32}$$

Theorem 2.8. *Let $g \in AC([0,T])$, with $g(0) = 0$. Assume (without loss of generality) that* (2.10) *holds. Then each approximable strong quasi-static evolution $(u^\varepsilon(t), A^\varepsilon(t))$ (in the sense of Definition* 2.5*) for the energy in* (2.1)*, subjected to the boundary condition g, is characterized by*

(i) *$A^\varepsilon(t)$ is increasing in t;*

(ii) *if $A^\varepsilon_1(t) := A^\varepsilon(t) \cap \big([0, \frac{\varepsilon}{2}) + \varepsilon\mathbb{N}\big)$ and $A^\varepsilon_2(t) := A^\varepsilon(t) \setminus A^\varepsilon_1(t)$, then*

$$2|A^\varepsilon_1(t)| = \lambda_{1,\min}(\overline{g}(t)) \qquad \textit{and} \qquad 2|A^\varepsilon_2(t)| = \lambda_{2,\min}(\overline{g}(t)),$$

where $\overline{g}$ is the non-decreasing envelope *of the function g, defined by*

$$\overline{g}(t) := \inf_h \{h(t) : h \geq g \text{ on } [0,T], h \text{ non decreasing}\}; \tag{2.33}$$

(iii) *the function $u^\varepsilon(t)$ is the unique minimizer of $E^\varepsilon(\cdot, A^\varepsilon(t))$ under the boundary condition $u^\varepsilon(t,0) = 0$ and $u^\varepsilon(t,1) = g(t)$.*

Proof. Note that the approximability condition in general implies the minimality and the energy balance. This can be derived from [10], upon a relaxation argument in order to fulfil the abstract framework therein. Here we give a direct proof that highlights the homogenization process through the explicit description of the solutions, using $\lambda_{1,\min}$ and $\lambda_{2,\min}$. We consider the case of g non-decreasing first, and then the general case.

If g is non-decreasing, we can assume without loss of generality that $g(t) = t$ for all $t \in \mathbb{R}$. Let $(A^\varepsilon(t), u^\varepsilon(t))$ satisfy (i)–(iii). By the characterization of minimizers in Lemma 2.1 such a pair is a solution to

$$\min\{E^\varepsilon_{\mathrm{Tot}}(v, B) : v(0) = 0, v(1) = t, B \subset (0,1)\}. \tag{2.34}$$

and hence satisfies the minimality condition in Definition 2.5. Damage irreversibility is property (i).

It remains to prove the energy balance. To that end, we first note that by (2.11) the function $s \mapsto E^\varepsilon_{\mathrm{Tot}}(u^\varepsilon(s), A^\varepsilon(s))$ is absolutely continuous and its a.e. derivative is given by

$$\partial_s E^\varepsilon_{\rm Tot}(u^\varepsilon(s),A^\varepsilon(s)) = m'(s) = \begin{cases} 2\underline{\beta}t & \text{if } 0<t<\dfrac{p_1}{\underline{\beta}} \\ 2p_1 & \text{if } \dfrac{p_1}{\underline{\beta}}<t<\dfrac{p_1(\beta_2+\alpha_1)}{2\beta_2\alpha_1} \\ \dfrac{4\beta_2\alpha_1}{\beta_2+\alpha_1}t & \text{if } \dfrac{p_1(\beta_2+\alpha_1)}{2\beta_2\alpha_1}<t<\dfrac{p_2(\beta_2+\alpha_1)}{2\beta_2\alpha_1} \\ 2p_2 & \text{if } \dfrac{p_2(\beta_2+\alpha_1)}{2\beta_2\alpha_1}<t<\dfrac{p_2}{\underline{\alpha}} \\ 2\underline{\alpha}t & \text{if } t>\dfrac{p_2}{\underline{\alpha}}. \end{cases} \tag{2.35}$$

Using this equality we now prove (2.28), rewritten as

$$\int_0^t \partial_s E^\varepsilon_{\rm Tot}(u^\varepsilon(s),A^\varepsilon(s))\,ds = 2\int_0^t\int_{(0,1)} \sigma^\varepsilon_{A^\varepsilon(s)}(x)(u^\varepsilon(x,s))'\,dxds.$$

In order to conclude we show that for all $s\in\mathbb{R}$

$$m'(s) = 2\int_{(0,1)} \sigma^\varepsilon_{A^\varepsilon(s)}(x)(u^\varepsilon(x,s))'\,dx. \tag{2.36}$$

Note that, by Remark 2.2(2) we have (in the notation of Lemma 2.1)

$$(u^\varepsilon)' = z_{i1} \text{ on } A^\varepsilon_i, \qquad (u^\varepsilon)' = z_{i2} \text{ on } B^\varepsilon_i\,.$$

Taking into account conditions (2.23) and the boundary condition

$$\frac12(\lambda_{1,\min}(s)z_{11}+(1-\lambda_{1,\min}(s))z_{12}) + \frac12(\lambda_{2,\min}(s)z_{21}+(1-\lambda_{2,\min}(s))z_{22}) = s\,,$$

this allows us to conclude that the right-hand side of (2.36) equals

$$\frac{4\alpha_1\alpha_2\beta_1\beta_2}{\alpha_2\beta_2(\beta_1-\alpha_1)\lambda_{1,\min}(s)+\alpha_1\beta_1(\beta_2-\alpha_2)\lambda_{2,\min}(s)+\alpha_1\alpha_2(\beta_2+\beta_1)}s.$$

Using (2.20), (2.21), (2.35) we check that this expression is equal to the one for $m'(s)$ above.

By (i)–(iii) and recalling the minimality properties of $\lambda_{1,\min}$ and $\lambda_{2,\min}$, we have that, for every τ and k, $(u^\tau_k, A^\tau_k) = (u^\varepsilon(k\tau), A^\varepsilon(k\tau))$ is a minimizer for (2.30), which implies the approximability of $(u^\varepsilon(t), A^\varepsilon(t))$. This concludes the proof of the energy balance property and the proof of the theorem in the case of g increasing.

In the general case, we define $\overline{g}$ by (2.33) and consider $(u^\varepsilon(t), A^\varepsilon(t))$ satisfying (i)–(iii). If we denote by $\overline{u}^\varepsilon$ the minimizer of

$$\min\{E^\varepsilon_{\rm Tot}(v,A^\varepsilon(t)) : v\in H^1(0,1), v(0)=0, v(1)=\overline{g}(t)\}, \tag{2.37}$$

then, by the previous step, the pair $(\overline{u}^\varepsilon(t), A^\varepsilon(t))$ is an approximable quasi-static evolution for the boundary condition $\overline{g}$. In order to show that $(u^\varepsilon(t), A^\varepsilon(t))$ is an approximable quasi-static evolution for the boundary condition g we first examine the minimality condition. It is enough to consider t such that $g(t) < \overline{g}(t)$. Suppose by contradiction that there exists $B\supset A^\varepsilon(t)$ such that

$$E^\varepsilon_{\rm Tot}(u^\varepsilon(t),A^\varepsilon(t)) > \min\big\{E^\varepsilon_{\rm Tot}(v,B)\ :\ v(0)=0\,,\ v(1)=g(t)\big\}\,.$$

Then, noting that $f^\varepsilon(A)$, as defined in (2.32), is decreasing by inclusion, we have

$$\begin{aligned} E^\varepsilon_{\rm Tot}(\overline{u}^\varepsilon(t), A^\varepsilon(t)) &= E^\varepsilon_{\rm Tot}(u^\varepsilon(t), A^\varepsilon(t)) + f^\varepsilon(A^\varepsilon(t))\big(\overline{g}^2(t) - g^2(t)\big) \\ &> \min\big\{E^\varepsilon_{\rm Tot}(v, B)\ :\ v(0) = 0\,,\ v(1) = g(t)\big\} \\ &\qquad +f^\varepsilon(B)\big(\overline{g}^2(t) - g^2(t)\big) \\ &= \min\big\{E^\varepsilon_{\rm Tot}(v, B)\ :\ v(0) = 0\,,\ v(1) = \overline{g}(t)\big\}\,, \end{aligned}$$

contradicting the minimality condition for $(\overline{u}^\varepsilon(t), A^\varepsilon(t))$. As for the energy balance, it is enough to check it between two points s and t such that $\overline{g}(\tau) = \overline{g}(s) = \overline{g}(t)$ for all $\tau \in (s, t)$; i.e.,

$$E^\varepsilon_{\rm Tot}(u(t), A) - E^\varepsilon_{\rm Tot}(u(s), A) = 2\int_s^t \dot{g}(\tau)\int_{(0,1)} \sigma^\varepsilon_A(x)u'(\tau, x)\,dxd\tau, \tag{2.38}$$

where $A = A(t) = A(s)$. This is easily verified by noting that, in view of Remark 2.7, we can rewrite (2.38) as

$$f^\varepsilon(A)\big(g^2(t) - g^2(s)\big) = 2\int_s^t \dot{g}(\tau)g(\tau)f^\varepsilon(A)\,d\tau\,.$$

The approximability is obtained as in the non-decreasing case above, after recalling the constrained minimality properties of $\lambda_{i,\min}(t)$ in Remark 2.4 which allow to characterize the minimum values of the energy as in Remark 2.3.

It now remains to prove that every approximable quasi-static evolution $(u^\varepsilon(t), A^\varepsilon(t))$ satisfies properties (i)–(iii). Properties (i) and (iii) are immediately implied by the definition. Let (u^τ_k, A^τ_k) be as in Definition 2.5. We define the piecewise-constant function g_τ by

$$g_\tau(t) = g\Big(\tau\Big\lfloor\frac{t}{\tau}\Big\rfloor\Big),$$

and $\overline{g}_\tau$ as its non-decreasing envelope in the notation (2.33).

The sets A^τ_k satisfy

$$\begin{aligned} 2\Big|A^\tau_k \cap \Big(\Big[0, \frac{\varepsilon}{2}\Big) + \varepsilon\mathbb{N}\Big)\Big| &= \lambda_{1,\min}(\overline{g}_\tau(k\tau)) \quad \text{and} \\ 2\Big|A^\tau_k \setminus \Big(\Big[0, \frac{\varepsilon}{2}\Big) + \varepsilon\mathbb{N}\Big)\Big| &= \lambda_{2,\min}(\overline{g}_\tau(k\tau)). \end{aligned} \tag{2.39}$$

This can be proved by induction. Indeed, (2.39) is satisfied for $k = 0$, since $g(0) = 0$ and $A^\tau_0 = \emptyset$. Assume it holds true with $k-1$ in the place of k. We have two cases: if $\overline{g}_\tau(k\tau) > \overline{g}_\tau((k-1)\tau)$ then $\overline{g}(k\tau) = g_\tau(k\tau)$ and the validity of (2.39) follows by the minimality properties of $\lambda_{1,\min}$ and $\lambda_{2,\min}$; if otherwise $\overline{g}_\tau(k\tau) = \overline{g}_\tau((k-1)\tau)$ then the conclusion follows by noting that $A^\tau_k = A^\tau_{k-1}$ as a consequence of Remark 2.4.

Passing to the limit as $\tau \to 0$ we then obtain property (ii), after noting the uniform convergence of $\overline{g}_\tau$ to $\overline{g}$. □

Remark 2.9. For any quasi-static evolution $(u^\varepsilon(t), A^\varepsilon(t))$ in the sense of Definition 2.5, for fixed t the sets $A^\varepsilon(t)$ do not converge to sets as $\varepsilon \to 0$, except for the trivial cases $\emptyset$ and $(0, 1)$. Indeed, for example for $p_1 < p_2$ and $t \in \Big[\frac{p_1(\beta_2+\alpha_1)}{2\beta_2\alpha_1}, \frac{p_2(\beta_2+\alpha_1)}{2\beta_2\alpha_1}\Big]$, we have that $A^\varepsilon(t) = \varepsilon\big(\mathbb{N} + \big[0, \frac{1}{2}\big)\big)$, whose characteristic functions weakly converge to the constant $\frac{1}{2}$.

3. QUASI-STATIC EVOLUTION FOR NON-HOMOGENEOUS MATERIALS

In this section we show that the approximable quasi-static evolutions related to the energy functionals $E^\varepsilon_{\rm Tot}$ converge, up to subsequences, to the approximable quasi-static evolutions related to the Γ-limit of such energy functionals. Vice versa, any approximable quasi-static evolution for the Γ-limit of the functionals (2.1) is the limit of the corresponding approximable quasi-static evolutions.

3.1. **Relaxed homogenization.** First we compute the Γ-limit of the family of functionals $E^\varepsilon_{\rm Tot}$. We tacitly identify sets with the characteristic functions as elements of $L^1(0,1)$.

Theorem 3.1 (relaxed homogenization)**.** *Let* (2.10) *hold. Then the family* $E^\varepsilon_{\rm Tot}$ *in* (2.1) Γ*-converges, in the* $L^2\times L^1$*-weak topology, to the functional*

$$E^{\rm hom}_{\rm Tot}(u,\theta)=E^{\rm hom}(u,\theta)+D^{\rm hom}(\theta)\,, \tag{3.1}$$

where

$$E^{\rm hom}(u,\theta)=\int_{(0,1)} f_{\rm hom}(\theta)|u'|^2dx, \tag{3.2}$$

with

$$f_{\rm hom}(\theta)=\begin{cases}\Big[\frac{\beta_1+\beta_2}{2\beta_1\beta_2}+\frac{(\beta_1-\alpha_1)}{\beta_1\alpha_1}\theta\Big]^{-1} & \text{if}\quad \theta\in[0,\frac12)\\ \Big[\frac{\beta_2+\alpha_1}{2\beta_2\alpha_1}+\frac{(\beta_2-\alpha_2)}{2\beta_2\alpha_2}(2\theta-1)\Big]^{-1} & \text{if}\quad \theta\in[\frac12,1)\end{cases} \tag{3.3}$$

and

$$D^{\rm hom}(\theta)=\int_{(0,1)}\gamma_{\rm hom}(\theta)\,dx,\quad \gamma_{\rm hom}(\theta)=\begin{cases}\gamma_1\theta & \text{if } \theta\in[0,\frac12)\\ \frac{\gamma_1}{2}+\gamma_2(\theta-\frac12) & \text{if } \theta\in[\frac12,1).\end{cases} \tag{3.4}$$

Proof. This is a particular case of homogenization in L^p spaces, where the cell-problem formula rewrites as

$$\begin{aligned}\phi(\theta,z)\ &:=\ \min\Big\{\int_A\alpha(y)|v|^2\,dy+\int_{(0,1)\setminus A}\beta(y)|v|^2\,dy+\int_A\gamma(y)\,dy\ :\\ &\qquad A\subset(0,1),\ |A|=\theta,\int_0^1 v\,dx=z\Big\}.\end{aligned}$$

Note that, minimizing first in v, and denoting by $\eta_1=|A\cap[0,\frac12]|$ and $\eta_2=|A\setminus[0,\frac12]|$, we obtain

$$\phi(\theta,z)=\min\Big\{f^{\rm hom}(\eta_1,\eta_2)z^2+\gamma_1\eta_1+\gamma_2\eta_2\ :\eta_1+\eta_2=\theta\Big\}, \tag{3.5}$$

with $f^{\rm hom}(\eta_1,\eta_2)$ defined in (2.24). By a direct computation we get the unique minimal

$$\eta_1=\theta\wedge\frac12,\qquad \eta_2=\Big(\theta-\frac12\Big)\vee 0 \tag{3.6}$$

and

$$\phi(\theta,z)=f_{\rm hom}(\theta)z^2+\gamma_{\rm hom}(\theta),$$

and the desired characterization. □

Corollary 3.2. *For all $t \geq 0$ we have*

$$\min\Big\{E^{\rm hom}_{\rm Tot}(v,\theta) : v \in H^1(0,1),\ v(0)=0,\ v(1)=t,\ 0\leq\theta\leq 1\Big\} = m(t), \tag{3.7}$$

where m is given by (2.11)*. Furthermore, the minimizers (u,θ) for this problem are characterized by the following properties:*

(i) *either $\theta \geq \frac{1}{2}$ a.e. or $\theta \leq \frac{1}{2}$ a.e.;*

(ii) *we have*

$$\int_{(0,1)} \theta\, dx = \lambda_{\rm min}(t), \tag{3.8}$$

where $\lambda_{\rm min}(t)$ is given by (2.22)*;*

(iii) *u is the unique minimizer of*

$$\min\Big\{E^{\rm hom}(v,\theta) : v \in H^1(0,1),\ v(0)=0,\ v(1)=t\Big\}. \tag{3.9}$$

Proof. The corollary follows from a direct computation, or from the previous theorem, Lemma 2.1 and the Fundamental Theorem of Γ-convergence. To that end, note that the characterization of m in the proof of Lemma 2.1 guarantees that sequences $(u_\varepsilon, A_\varepsilon)$ such that $u_\varepsilon(0)=0$, $u_\varepsilon(1)=t$ and $E^\varepsilon_{\rm Tot}(u_\varepsilon,A_\varepsilon) = m(t)+o(1)$ as $\varepsilon\to 0$ have the same cluster points as the sequences of minimizers of (3.7).

If $(u(t),\theta(t))$ satisfy (i)–(iii) then we can define $A_\varepsilon(t)$ such that $\chi_{A_\varepsilon(t)}$ weakly converges to $\theta(t)$, $|A_\varepsilon(t)| = \lambda_{\rm min}(t)$, $A_\varepsilon(t) \supset [0,\frac{\varepsilon}{2})+\varepsilon\mathbb{N}$ or $A_\varepsilon(t) \subset [0,\frac{\varepsilon}{2})+\varepsilon\mathbb{N}$, and u_ε is the corresponding solution of $\min E^\varepsilon_{\rm Tot}(u, A_\varepsilon(t))$ with $u(0)=0$ and $u(1)=t$. By Lemma 2.1 $(u_\varepsilon(t), A_\varepsilon(t))$ is a minimizer of $E^\varepsilon_{\rm Tot}(u,A)$ with $u(0)=0$ and $u(1)=t$ and then converges to a minimizer of $E^{\rm hom}_{\rm Tot}(u,\theta)$ subject to the same boundary conditions. □

Remark 3.3. Note that we do not have the separate Γ-convergence of E^ε and D^ε to $E^{\rm hom}$ and $D^{\rm hom}$. This is evident from the dependence of the form of the limit functionals on inequality (2.10).

Proposition 3.4 (compatibility of constraints)**.** Let B_ε be a family of subsets of $(0,1)$ and $\varphi \in L^1(0,1)$, such that $\chi_{B_\varepsilon} \overset{*}{\rightharpoonup} \varphi$ and

$$\Gamma\text{-}\lim_{\varepsilon\to 0} E^\varepsilon_{\rm Tot}(\cdot, B_\varepsilon) = E^{\rm hom}_{\rm Tot}(\cdot,\varphi) \tag{3.10}$$

with respect to the L^2-convergence, then the Γ-limit of

$$E^\varepsilon_{\rm Tot}(u,A;B_\varepsilon) := \begin{cases} E^\varepsilon(u,A) + D^\varepsilon(A) & \text{if} \quad A \supset B_\varepsilon \\ +\infty & \text{otherwise} \end{cases} \tag{3.11}$$

with respect to the $L^2\times L^1$-weak convergence is

$$E^{\rm hom}_{\rm Tot}(u,\theta;\varphi) := \begin{cases} E^{\rm hom}(u,\theta) + D^{\rm hom}(\theta) & \text{if} \quad \theta \geq \varphi \\ +\infty & \text{otherwise.} \end{cases} \tag{3.12}$$

Remark 3.5. Condition (3.10) is equivalent to requiring that

$$\chi_{B_\varepsilon\cap([0,\frac{\varepsilon}{2})+\varepsilon\mathbb{N})} \rightharpoonup \varphi \wedge \frac{1}{2} \tag{3.13}$$

or, equivalently, that

$$\chi_{B_\varepsilon\cap([\frac{\varepsilon}{2},\varepsilon)+\varepsilon\mathbb{N})} \rightharpoonup \Big(\varphi-\frac{1}{2}\Big)\vee 0. \tag{3.14}$$

In order to check (3.13), denote with φ_1 the weak limit of the sequence on the left-hand side of (3.13), which exists up to subsequences, and $\varphi_2=\varphi-\varphi_1$, which is the weak limit of the sequence on the left-hand side of (3.14). Note that (we do not relabel the subsequence)

$$\Gamma\text{-}\lim_{\varepsilon\to 0} E^\varepsilon_{\rm Tot}(u,B_\varepsilon)=F(u,\varphi_1,\varphi_2), \tag{3.15}$$

where

$$F(u,\varphi_1,\varphi_2):=\int_{(0,1)} f^{\rm hom}(\varphi_1,\varphi_2)|u'|^2dx+\gamma_1\int_{(0,1)}\varphi_1\,dx+\gamma_2\int_{(0,1)}\varphi_2\,dx \tag{3.16}$$

and $f^{\rm hom}$ is defined in (2.24). This immediately follows from the convergence of the dissipation term, and the characterization of one-dimensional Γ-convergence (see [3] Appendix B).

It follows that (3.13) is equivalent to (3.10), since, by (3.5) and (3.6), $F(u,\varphi_1,\varphi_2)=E^{\rm hom}_{\rm Tot}(u,\varphi)$ if and only if φ_1 and φ_2 are as in (3.13) and (3.14).

Proof of Proposition 3.4. The lower bound inequality is trivial since the constraint is closed. As for the upper bound, with fixed $\theta\geq\varphi$, we use a diagonal argument, first constructing a recovery sequence of sets for a sequence of θ^σ converging to θ.

With fixed $\sigma>0$, for all x Lebesgue point of φ, φ_1 (as defined in Remark 3.5) and θ, we consider the family

$$\begin{aligned}\mathcal{I}^\sigma_x \;=\;& \Big\{I=(x-\delta,x+\delta)\subset(0,1):\delta<\sigma,\\ &\int_I|\varphi(x)-\varphi|\,dy+\int_I|\varphi_1(x)-\varphi_1|\,dy+\int_I|\theta(x)-\theta|\,dy<\sigma|I|\Big\}.\end{aligned}$$

Since $\mathcal{I}^\sigma=\bigcup_x\mathcal{I}^\sigma_x$ is a fine cover of the set of Lebesgue points of $(0,1)$ we can find a finite family of disjoint intervals $\{I^\sigma_k\}$ of $\mathcal{I}^\sigma$ such that

$$\Big|(0,1)\setminus\bigcup_k I^\sigma_k\Big|<\sigma.$$

We construct subsets A^σ_ε of $(0,1)$ defining them on each such interval

$$I^\sigma_k=(x^\sigma_k-\delta^\sigma_k,x^\sigma_k+\delta^\sigma_k)$$

as follows:

(i) $A^\sigma_\varepsilon\cap I^\sigma_k\supset B_\varepsilon\cap I^\sigma_k$;
(ii) $|A^\sigma_\varepsilon\cap I^\sigma_k|=\int_{I^\sigma_k}\theta\,dy$.

If $\varphi(x^\sigma_k)>\frac{1}{2}$ conditions (i) and (ii) are the only ones required in our construction; otherwise, if $\varphi(x^\sigma_k)\leq\frac{1}{2}$, we have to require some additional conditions. In order to specify such conditions we introduce the notation

$$A^\sigma_{\varepsilon,1}=A^\sigma_\varepsilon\cap\Big(\Big[0,\frac{\varepsilon}{2}\Big)+\varepsilon\mathbb{N}\Big),\qquad A^\sigma_{\varepsilon,2}=A^\sigma_\varepsilon\setminus A^\sigma_{\varepsilon,1}$$

and

$$B_{\varepsilon,1}=B_\varepsilon\cap\Big(\Big[0,\frac{\varepsilon}{2}\Big)+\varepsilon\mathbb{N}\Big),\qquad B_{\varepsilon,2}=B_\varepsilon\setminus B_{\varepsilon,1}.$$

(iiia) if $\theta(x_k^\sigma) \le \frac{1}{2}$ then

$$A^\sigma_{\varepsilon,2} \cap I_k^\sigma = B_{\varepsilon,2} \cap I_k^\sigma; \tag{3.17}$$

(iiib) if $\theta(x_k^\sigma) > \frac{1}{2}$ then

$$|A^\sigma_{\varepsilon,2} \cap I_k^\sigma| = |B_{\varepsilon,2} \cap I_k^\sigma| \vee \Big(\int_{I_k^\sigma} \Big(\theta - \frac{1}{2}\Big) dy\Big). \tag{3.18}$$

We finally include in the sets A^σ_ε the complement of $\bigcup_k I_k^\sigma$.

Up to a subsequence we have that

$$\chi_{A^\sigma_\varepsilon} \rightharpoonup \theta^\sigma, \qquad \text{and} \qquad \chi_{A^\sigma_{\varepsilon,2}} \rightharpoonup \theta_2^\sigma$$

as $\varepsilon \to 0$, for some θ_2^σ and θ^σ.

By the fact that I_k^σ belong to $\mathcal{I}^\sigma_{x_k^\sigma}$, that B_ε satisfy the optimality condition (3.14), and by the properties of A^σ_ε and $A^\sigma_{\varepsilon,2}$, we have: for all intervals $I \subset (0,1)$

$$\Big|\int_I \theta_2^\sigma \, dy - \int_I \widetilde{\theta}^\sigma \, dy\Big| \le 4\sigma,$$

where

$$\widetilde{\theta}^\sigma(x) = \begin{cases} \Big(\theta(x_k^\sigma) - \frac{1}{2}\Big)^+ & \text{if } x \in I_k^\sigma \\ 0 & \text{otherwise.} \end{cases}$$

Since $\widetilde{\theta}^\sigma$ converges in $L^1(0,1)$ to $\Big(\theta - \frac{1}{2}\Big)^+$, we deduce that

$$\theta^\sigma \rightharpoonup \theta \qquad \text{and} \qquad \theta_2^\sigma \rightharpoonup \Big(\theta - \frac{1}{2}\Big)^+$$

as $\sigma \to 0$.

By a diagonal argument, we can construct $A_\varepsilon = A_\varepsilon^{\sigma(\varepsilon)} \subset B_\varepsilon$ which thanks to (3.14) satisfies

$$\Gamma\text{-}\lim_{\varepsilon \to 0} E^\varepsilon_{\rm Tot}(\cdot, A_\varepsilon) = E^{\rm hom}_{\rm Tot}(\cdot, \theta), \tag{3.19}$$

which implies the desired upper bound. □

Corollary 3.6. *Given $s \in [0,T]$. Assume that $\varphi : [0,1] \to [0,1]$ satisfies $\varphi \le \frac{1}{2}$ a.e. or $\varphi \ge \frac{1}{2}$ a.e. and*

$$\int_{(0,1)} \varphi \, dx > \lambda_{\min}(s). \tag{3.20}$$

Then

$$\min\{E^{\rm hom}_{\rm Tot}(u,\varphi) : u(0) = 0,\ u(1) = s\} \le \min\{E^{\rm hom}_{\rm Tot}(u,\theta) : u(0) = 0,\ u(1) = s\}$$

for all $\theta \ge \varphi$.

Proof. This is a direct consequence of the Γ-convergence result above, combined with Remark 2.4. Indeed, denoting by $t > s$ the value such that

$$\int_{(0,1)} \varphi \, dx = \lambda_{\min}(t), \tag{3.21}$$

we have, using Corollary 3.2, that φ can be approximated by a sequence χ_{B_ε}, with B_ε satisfying the assumption of Proposition 3.4 and

$$2|B_\varepsilon \cap ([0,\varepsilon/2)+\varepsilon\mathbb{N}))| = \lambda_{1,\min}(t)$$

and

$$2|B_\varepsilon \setminus ([0,\varepsilon/2)+\varepsilon\mathbb{N}))| = \lambda_{2,\min}(t).$$

Then by Remark 2.4 we get that

$$\min\{E^\varepsilon_{\rm Tot}(u,B_\varepsilon) : u(0)=0,\ u(1)=s\} \le \min\{E^\varepsilon_{\rm Tot}(u,A):\ u(0)=0,\ u(1)=s\}$$

for all $A\supset B_\varepsilon$. We conclude by applying Proposition 3.4. □

3.2. **Quasi-static evolution.** Now we give the definition of a quasi-static evolution related to the energy functional and the dissipation in (3.12).

Definition 3.7. Given $g\in AC([0,T])$, with $g(0)=0$, we say that $(u(t),\theta(t))$ is a *(weak) quasi-static evolution* (for the energy (3.1)) if for all $t\in[0,T]$ we have $u(t)\in H^1(0,1)$, $u(0)=0$, $u(1)=g(t)$, $\theta(t)\in L^\infty(0,1)$, $0\le\theta\le 1$, and the following properties hold:

- **Damage Irreversibility**: $\theta(t)$ is non-decreasing in time;
- **Energy Balance**:

$$E^{\rm hom}_{\rm Tot}(u(t),\theta(t)) = E^{\rm hom}_{\rm Tot}(u(0),\theta(0)) + 2\int_0^t \dot g(s)\int_{(0,1)} f_{\rm hom}(\theta)\,u'(s,x)\,dxds; \tag{3.22}$$

- **Minimality Condition**:

$$E^{\rm hom}_{\rm Tot}(u(t),\theta(t)) \le E^{\rm hom}_{\rm Tot}(v,\psi), \tag{3.23}$$

  for all $v\in H^1(0,1)$ $v(0)=0$, $v(1)=g(t)$ and $\psi\in L^\infty(0,1)$, $\psi\ge\theta(t)$.

Moreover, we say that $(u(t),\theta(t))$ is an *approximable (weak) quasi-static evolution* (for the energy (3.1) subjected to the boundary condition g) if it satisfies the conditions above, and can be obtained as the limit of a time-discrete approximation scheme; i.e., up to a subsequence, for all t it is the limit of $(u_\tau(t),\theta_\tau(t))$ constructed as follows: $u_\tau(t)=u^\tau_{\lfloor t/\tau\rfloor}$, $\theta_\tau(t)=\theta^\tau_{\lfloor t/\tau\rfloor}$, where $u^\tau_0=0$, $\theta^\tau_0=\emptyset$ and (u^τ_k,θ^τ_k) is a minimizing pair for the problem

$$\min\Big\{E^{hom}_{\rm Tot}(v,\theta) : v(0)=0,\ v(1)=g(k\tau),\ \theta^\tau_{k-1}\le\theta\Big\} \tag{3.24}$$

for $k\ge 1$.

Theorem 3.8. *Every approximable (weak) quasi-static evolution $(u(t),\theta(t))$ for $E^{\rm hom}_{\rm Tot}$ is characterized by the following properties:*

(i) *$\theta(t)$ is non-decreasing in t;*
(ii) *$\theta(t)\le\frac12$ a.e. or $\theta(t)\ge\frac12$ a.e., and $\int_{(0,1)}\theta(t)\,dx = \lambda_{\min}(\overline{g}(t))$;*
(iii) *u is the unique minimizer of*

$$\min\Big\{E^{\rm hom}_{\rm Tot}(v,\theta(t)) : v(0)=0,\ v(1)=g(t)\Big\}. \tag{3.25}$$

Proof. By [10] Theorem 4.5, all limits of incremental problems (3.24), which exist up to subsequences, are (weak) quasi-static evolutions for the energy $E^{\rm hom}_{\rm Tot}$. Then it is enough to show that, for any pair $(u(t),\theta(t))$ satisfying (i)–(iii), we can construct an incremental problem whose solutions converge to $(u(t),\theta(t))$, and that any limit of solutions of incremental problems satisfy (i)–(iii).

Let $(u(t),\theta(t))$ satisfy (i)–(iii) and for every $\tau>0$, as in the proof of Theorem 2.8, denote by $g_\tau(t)$ the piecewise-constant interpolation of the values $\{g(k\tau)\}_k$ and let $\overline{g}_\tau(t)$ be its non-decreasing envelope as in (2.33). Then we consider a family θ^τ_k, with either $\theta^\tau_k\ge\frac12$ a.e. or $\theta^\tau_k\le\frac12$ a.e.,

$$\int_{(0,1)}\theta^\tau_k\,dx=\max\{\lambda_{\min}(g(j\tau)):j\le k\}=\lambda_{\min}\big(\overline{g}_\tau(k\tau)\big),$$

and

$$\theta^\tau_{k-1}\le\theta^\tau_k\le\theta(k\tau). \tag{3.26}$$

This can be done by induction. We also consider the corresponding u^τ_k minimizing (3.25) with boundary conditions $u^\tau_k(0)=0$ and $u^\tau_k(1)=g(k\tau)$, and θ replaced by θ^τ_k. We can show that, by construction, the family (u^τ_k,θ^τ_k) is a solution of the incremental problem

$$E^{\rm hom}_{\rm Tot}(u^\tau_k,\theta^\tau_k)\le E^{\rm hom}_{\rm Tot}(v,\varphi)$$

for every $v\in H^1(0,1)$, with $v(0)=0$ and $v(1)=g(k\tau)$ and for every $\varphi\ge\theta^\tau_{k-1}$. Indeed if

$$\int_{(0,1)}\theta^\tau_{k-1}\,dx\le\lambda_{\min}(g(k\tau))$$

then by Corollary 3.2 such θ^τ_k minimizes

$$\begin{aligned}\min\Big\{&E^{\rm hom}_{\rm Tot}(v,\varphi)\ :\ v\in H^1(0,1),\ v(0)=0,\ v(1)=g(k\tau)\ \text{ and }\varphi\ge\theta^\tau_{k-1}\Big\}\\&=\min\Big\{E^{\rm hom}_{\rm Tot}(v,\theta^\tau_k)\ :\ v\in H^1(0,1),\ v(0)=0,\ v(1)=g(k\tau)\Big\},\end{aligned} \tag{3.27}$$

while if

$$\int_{(0,1)}\theta^\tau_{k-1}\,dx>\lambda_{\min}(g(k\tau))\,,$$

then, by Corollary 3.6, we deduce that $\theta^\tau_k=\theta^\tau_{k-1}$. By (3.26) we deduce that the piecewise-constant functions $(u^\tau(t),\theta^\tau(t))=(u^\tau_k,\theta^\tau_k)$ if $t\in[k\tau,(k+1)\tau)$ converge to $(u(t),\theta(t))$ for all $t\in[0,T]$, which proves the approximability of $(u(t),\theta(t))$.

On the other hand if $(u(t),\theta(t))$ is an approximable quasi-static evolution, let (u^τ_k,θ^τ_k) be a solution of the incremental problem (3.24) which converges to $(u(t),\theta(t))$. We can prove by induction that

$$\theta^\tau_k\le\frac12\ \text{a.e., or }\theta^\tau_k\ge\frac12\ \text{a.e., and }\int_{(0,1)}\theta^\tau_k\,dx=\lambda_{\min}(\overline{g}_\tau(k\tau)). \tag{3.28}$$

Indeed, if $k=0$ this is trivially true. Assume that (3.28) holds with k replaced by $k-1$. If $\lambda_{\min}(\overline{g}_\tau(k\tau))=\lambda_{\min}(\overline{g}_\tau((k-1)\tau))$ then $\lambda_{\min}(\overline{g}_\tau(k\tau))\ge\lambda_{\min}(g(k\tau))$, and hence, by Corollary 3.6 we have $\theta^\tau_k=\theta^\tau_{k-1}$. Otherwise, if $\lambda_{\min}(\overline{g}_\tau(k\tau))>\lambda_{\min}(\overline{g}_\tau((k-1)\tau))$ then $\lambda_{\min}(\overline{g}_\tau(k\tau))=\lambda_{\min}(g(k\tau))$, and the conclusion follows by Corollary 3.2. Properties (i)–(iii) then follow by (3.28) taking the limit as $\tau\to0$. □

We show now that an approximable quasi-static evolution $(u^\varepsilon(t), A^\varepsilon(t))$ for $E^\varepsilon_{\rm Tot}(u, A)$ converges (up to subsequences) to a pair $(u(t), \theta(t))$, approximable quasi-static evolution for $E^{\rm hom}_{\rm Tot}(u, \theta)$.

Theorem 3.9. *Any approximable quasi-static evolution $(u^\varepsilon(t), A^\varepsilon(t))$ for $E^\varepsilon_{\rm Tot}(u, A)$ converges (up to subsequences) to a pair $(u(t), \theta(t))$ in the $L^2 \times L^1$-weak convergence. Moreover, $(u(t), \theta(t))$ is an approximable quasi-static evolution for $E^{\rm hom}_{\rm Tot}(u, \theta)$.*

Conversely, any approximable quasi-static evolution $(u(t), \theta(t))$ for $E^{\rm hom}_{\rm Tot}(u, \theta)$ is the limit as $\varepsilon \to 0$ of an approximable quasi-static evolution $(u^\varepsilon(t), A^\varepsilon(t))$ for $E^\varepsilon_{\rm Tot}(u, A)$.

Proof. By the monotonicity condition on $A^\varepsilon(t)$, using Helly's theorem, we can find a subsequence such that (up to relabelling the apices)

$$\chi_{A^\varepsilon(t)} \rightharpoonup \theta(t) \qquad \text{and} \qquad \chi_{A^\varepsilon(t)\cap([0,\frac{\varepsilon}{2})+\varepsilon\mathbb{N})} \rightharpoonup \theta_1(t)$$

in $L^1(0, 1)$ for all t.

Since (i)–(iii) of Theorem 2.8 are satisfied for A_ε, then, taking the limit as $\varepsilon \to 0$ we deduce (i)–(iii) of Theorem 3.8 for (u, θ).

On the other hand, let $(u(t), \theta(t))$ be an approximable quasi-static evolution for $E^{\rm hom}_{\rm Tot}(u, \theta)$. By Theorem 3.8 it satisfies (i)–(iii) therein. We then construct for all $t \in [0, T]$ the set $A_\varepsilon(t)$ as follows

$$A_\varepsilon(t) = \bigcup_k \Big(k\varepsilon, k\varepsilon + \int_{(k\varepsilon,(k+1)\varepsilon)} \theta(t)\, dx\Big),$$

and let $u_\varepsilon(t)$ be the corresponding minimizer of $v \mapsto E^\varepsilon_{\rm Tot}(v, A_\varepsilon(t))$ with boundary conditions $v(0) = 0$ and $v(1) = g(t)$. With this definition $(u_\varepsilon(t), A_\varepsilon(t))$ satisfy (i)–(iii) of Theorem 2.8 and hence, it is an approximable quasi-static evolution for $E^\varepsilon_{\rm Tot}(u, A)$, and converge to $(u(t), \theta(t))$. □

4. Quasi-static evolution for a three-phase material

In this final section, we use the characterization in Theorem 3.8 to show that the limit evolution can be interpreted as a weak evolution of a three-phase material. To that end, we introduce a double damage set model that generalizes the one introduced by Francfort and Marigo as follows. We consider positive constants $a < b < c$, k_1 and k_2, the energy

$$E^{\rm 3P}(u, A_1, A_2) = a\int_{A_2} |u'|^2\, dx + b\int_{A_1} |u'|^2\, dx + c\int_{(0,1)\setminus(A_1\cup A_2)} |u'|^2\, dx$$

and the dissipation

$$D^{\rm 3P}(A_1, A_2) = k_1|A_1| + k_2|A_2|\,,$$

with domain pairs of disjoint subsets A_1, A_2 of $(0, 1)$. This can be interpreted as the damage model of a three-phase material, where c is the elastic constant of the undamaged state, b the one of the 'partly damaged' state, and a the one of the 'totally damaged' state. The constant k_1 represents the cost of the partly damaged state and k_2 the one of the totally damaged state. In general, we could consider also an 'intermediate' dissipation $k_{1,2}$ which accounts for the transition from the partly damaged state to the totally damaged state. Our model corresponds to the case

$$k_{1,2} = k_2 - k_1\,.$$

This assumption reflects the fact that the material in order to reach the totally damaged state should pass through the intermediate partly damaged state.

The incremental problem for this model consists in solving iteratively

$$\min_{u,A_1,A_2}\Big\{E^{3\mathrm{P}}(u,A_1,A_2)+D^{3\mathrm{P}}(A_1,A_2):\ A_1\cap A_2=\emptyset,\ A_1\cup A_2\supset A_1^{k-1}\cup A_2^{k-1},\ A_2\supset A_2^{k-1},\ u(0)=0,\ u(1)=g(k\tau)\Big\}. \tag{4.1}$$

The monotonicity conditions on the sets correspond to the assumption that the totally damaged state can only increase, while the partially damaged set can become totally damaged.

We first note that problems (4.1) may undergo relaxation with respect to the weak convergence in H^1 for u and weak convergence in L^1 for the sets, understood as the weak convergence of their characteristic functions. We are then lead to considering the following relaxed functional

$$E^{3\mathrm{P}}_{\mathrm{Tot}}(u,\varphi,\psi):=\int_{(0,1)}H(\varphi,\psi)|u'|^2dx+k_1\int_{(0,1)}\varphi\,dx+k_2\int_{(0,1)}\psi\,dx. \tag{4.2}$$

where

$$H(\eta_1,\eta_2)=\left[\frac{1-(\eta_1+\eta_2)}{c}+\frac{\eta_1}{b}+\frac{\eta_2}{a}\right]^{-1}. \tag{4.3}$$

This is an immediate consequence of the characterization of one-dimensional Γ-convergence, once we observe that

$$E^{3\mathrm{P}}(u,A_1,A_2)=\int_{(0,1)}\Big(c\chi_{(0,1)\setminus(A_1\cup A_2)}+b\chi_{A_1}+a\chi_{A_2}\Big)|u'|^2\,dx$$

and we can write

$$\frac{1}{c\chi_{(0,1)\setminus(A_1\cup A_2)}+b\chi_{A_1}+a\chi_{A_2}}=\frac{1}{c}\chi_{(0,1)\setminus(A_1\cup A_2)}+\frac{1}{b}\chi_{A_1}+\frac{1}{a}\chi_{A_2}.$$

We give a definition of (weak) quasi-static evolution for these energies as follows. Note that in this definition the monotonicity conditions on A_1 and A_2 given in problems (4.1) correspond to conditions on the functions φ and $\varphi+\psi$.

Definition 4.1. Given $g\in AC([0,T])$, with $g(0)=0$, we say that $(u(t),\varphi(t),\psi(t))$ is a (three-phase) quasi-static evolution for the energy (4.2) if for all $t\in[0,T]$ we have $u(t)\in H^1(0,1)$, $u(0)=0$, $u(1)=g(t)$, $\psi(t)\in L^\infty(0,1)$, $0\le\psi(t)\le 1$, $\varphi(t)\in L^\infty(0,1)$, $0\le\varphi(t)\le 1$, $\varphi(t)+\psi(t)\le 1$, and the following properties hold

- **Damage irreversibility** $\psi(t)$ and $\varphi(t)+\psi(t)$ are increasing in time for each $x\in(0,1)$,
- **Energy Balance**

$$E^{3\mathrm{P}}_{\mathrm{Tot}}(u(t),\psi(t),\varphi(t))=E^{3\mathrm{P}}_{\mathrm{Tot}}(u(0),\psi(0),\varphi(0))+2\int_0^t\dot g(s)\int_{(0,1)}H(\varphi,\psi)u'dxds$$

  for all $t\in[0,T]$,
- **Minimality Condition**

$$E^{3\mathrm{P}}_{\mathrm{Tot}}(u(t),\varphi(t),\psi(t))\le E^{3\mathrm{P}}_{\mathrm{Tot}}(v,\tilde\varphi,\tilde\psi)$$

for all $v : v - u(t) \in H^1_0$, and $(\tilde\varphi, \tilde\psi)$ such that $\tilde\psi \geq \psi(t)$ and $\psi(t) + \varphi(t) \leq \tilde\psi + \tilde\varphi \leq 1$.

Now we prove that the limit of the quasi-static evolutions considered in Section 2 can be seen as a quasi-static evolution of a three-phase homogenized material as in Definition 4.1. This will be an immediate consequence of the following proposition.

Proposition 4.2. If $(u(t), \theta(t))$ is a quasi-static evolution according to the Definition 3.7 and we set

$$(\varphi(t), \psi(t)) = \begin{cases} (2\theta(t), 0) & \text{if } \theta(t) \in [0, \frac{1}{2}) \\ (2(1-\theta(t)), 2\theta(t) - 1) & \text{if } \theta(t) \in [\frac{1}{2}, 1), \end{cases} \tag{4.4}$$

then $(u(t), \psi(t), \varphi(t))$ is a quasi-static evolution according to Definition 4.1, with

$$k_1 = \frac{\gamma_1}{2} \qquad \text{and} \qquad k_2 = \frac{\gamma_1 + \gamma_2}{2},$$

and

$$a = \frac{2\alpha_1\alpha_2}{\alpha_1 + \alpha_2}, \qquad b = \frac{2\alpha_1\beta_2}{\alpha_1 + \beta_2}, \qquad c = \frac{2\beta_1\beta_2}{\beta_1 + \beta_2}.$$

Proof. By the definition of $(\psi(t), \varphi(t))$ the irreversibility of damage is preserved. Moreover, from a direct computation, using the definition of (ψ, φ), k_1 and k_2, a, b, c and the following expression for $f_{\text{hom}}(\theta)$

$$f_{\text{hom}}(\theta) = \begin{cases} \left[\dfrac{\beta_1 + \beta_2}{2\beta_1\beta_2}(1 - 2\theta) + \dfrac{(\beta_2 + \alpha_1)}{2\beta_2\alpha_1} 2\theta\right]^{-1} & \text{if} \quad \theta \in [0, \frac{1}{2}) \\ \left[\dfrac{\beta_2 + \alpha_1}{2\beta_2\alpha_1} 2(1 - \theta) + \dfrac{(\alpha_1 + \alpha_2)}{2\alpha_1\alpha_2}(2\theta - 1)\right]^{-1} & \text{if} \quad \theta \in [\frac{1}{2}, 1) \end{cases} \tag{4.5}$$

we obtain that

$$f_{\text{hom}}(\theta) = H(\varphi, \psi) \qquad \text{and} \qquad D^{\text{hom}}(\theta) = k_1 \int_0^1 \varphi \, dx + k_2 \int_0^1 \psi \, dx,$$

which implies immediately the energy balance. It remains to prove the minimality property. To this end we just need to show that for any admissible test pairs $(\tilde\varphi, \tilde\psi)$ for $E^{3P}_{\text{Tot}}(v, \psi, \varphi)$ (i.e. such that $\tilde\psi \geq \psi(t)$ and $\psi(t) + \varphi(t) \leq \tilde\psi + \tilde\varphi \leq 1$) we can construct an admissible test functions $\tilde\theta \geq \theta(t)$ for $E^{\text{hom}}_{\text{Tot}}(v, \theta)$ such that

$$E^{\text{hom}}_{\text{Tot}}(v, \tilde\theta) = E^{3P}_{\text{Tot}}(v, \tilde\psi, \tilde\varphi).$$

It is enough, given $(\tilde\varphi, \tilde\psi)$ such that $\psi(t) + \varphi(t) \leq \tilde\psi + \tilde\varphi \leq 1$, to define $\tilde\theta = \tilde\varphi/2$ if $\tilde\psi = 0$ and $\tilde\theta = (\tilde\psi + 1)/2$ otherwise. This choice allows to conclude. □

Corollary 4.3. *Let $(u^\varepsilon(t), A^\varepsilon(t))$ be a family of approximable quasi-static evolutions for the inhomogeneous two-phase damage energy $E^\varepsilon_{\text{Tot}}(u, A)$. Denoting by $A^\varepsilon_1(t) = A^\varepsilon(t) \cap ([0, \frac{\varepsilon}{2}) + \varepsilon\mathbb{N})$ and $A^\varepsilon_2(t) = A^\varepsilon(t) \setminus ([0, \frac{\varepsilon}{2}) + \varepsilon\mathbb{N})$ the triple $(u^\varepsilon(t), A^\varepsilon_1(t), A^\varepsilon_2(t))$ converges (up to subsequences) to a triple $(u(t), \theta_1(t), \theta_2(t))$ in the $L^2 \times L^1 \times L^1$-weak convergence such that, defining $\varphi(t) = 2(\theta_1(t) - \theta_2(t))$ and $\psi(t) = 2\theta_2(t)$, $(u(t), \psi(t), \varphi(t))$ is a (three-phase) quasi-static evolution in the sense of Definition* 4.1.

Proof. The proof is an immediate consequence of Theorem 3.9 and the characterization of $\theta(t)$ in terms of $\theta_1(t)$ and $\theta_2(t)$ (see Remark 3.5). □

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References

[1] Ambrosio, L., Gigli, N., A user's guide to optimal transport. In: B. Piccoli and M. Rascle (eds.) *Modelling and Optimisation of Flows on Networks*. Lecture Notes in Mathematics 2062, pp. 1–155. Springer, Berlin, 2013.

[2] Bourdin, B., Francfort, G.A., Marigo, J., The variational approach to fracture, *J. Elasticity* 91 (2008), no. 1-3, 5–148.

[3] Braides, A., *Γ-convergence for Beginners*. Oxford University Press, Oxford, 2002.

[4] Braides, A., *Local minimization, variational evolution and Γ-convergence*. Lecture Notes in Math. 2094, Springer, Berlin, 2013.

[5] Braides, A., Larsen, C., Γ-convergence for stable states and local minimizers. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 10 (2011), 193–206

[6] Francfort, G., Garroni, A., A Variational View of Brittle Damage Evolution, *Arch. Rat. Mech. Anal.*, 182 (1) (2006), 125–152.

[7] Francfort, G.A., Larsen, C.J., Existence and convergence for quasi-static evolution in brittle fracture. *Comm. Pure Appl. Math.* 56 (10) (2003), 1465–1500.

[8] Francfort, G., Marigo, J., Stable damage evolution in a brittle continuous media, *Eur. J. Mech. A/Solids* , 12 (2) (1993), 149–189.

[9] Garroni, A., Larsen, C.J., Threshold-based Quasi-static Brittle Damage Evolution. *Arch. Rat. Mech. Anal.*, 194 (2009), 585–609.

[10] Mainik, A., Mielke, A., Existence results for energetic models for rate-independent systems, *Calc. Var. Partial Differential Equations.*, 22 (2005), 73–99.

[11] Mielke, A., Differential, energetic, and metric formulations for rate-independent processes, *Nonlinear PDE's and applications*, Lecture Notes in Math. 2028, Springer, Berlin, 2011, 87–170.

[12] Mielke, A., Rindler, F., Reverse approximation of energetic solutions to rate-independent processes. *NoDEA Nonlinear Differential Equations Appl.* 16 (1) (2009), 7–40.

[13] Mielke, A., Rossi, R., Savaré, G., BV solutions and viscosity approximations of rate-independent systems. ESAIM Control Optim. Calc. Var. 18 (2012), 36–80

[14] Mielke, A., Roubicek, T., Stefanelli, U., Γ-limits and relaxations for rate-independent evolutionary problems, *Calc. Var. Partial Differential Equations* 31 (3) (2008), 387–416.

[15] Mielke, A; Theil, F., On rate-independent hysteresis models. *NoDEA Nonlinear Differential Equations Appl.* 11 (2) (2004), 151–189.

[16] Rindler, F., Approximation of rate-independent optimal control problems. *SIAM J. Numer. Anal.* 47 (5) (2009), 3884–3909.

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