



# ON LOCAL NON-ZERO CONSTRAINTS IN PDE WITH ANALYTIC COEFFICIENTS

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# ON LOCAL NON-ZERO CONSTRAINTS IN PDE WITH ANALYTIC COEFFICIENTS

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**ABSTRACT.** We consider the Helmholtz equation with real analytic coefficients on a bounded domain  $\Omega \subset \mathbb{R}^d$ . We take  $d+1$  prescribed boundary conditions  $f^i$  and frequencies  $\omega$  in a fixed interval  $[a, b]$ . We consider a constraint on the solutions  $u_\omega^i$  of the form  $\zeta(u_\omega^1, \dots, u_\omega^{d+1}, \nabla u_\omega^1, \dots, \nabla u_\omega^{d+1}) \neq 0$ , where  $\zeta$  is analytic, which is satisfied in  $\Omega$  when  $\omega = 0$ . We show that for any  $\Omega' \Subset \Omega$  and almost any  $d+1$  frequencies  $\omega_k$  in  $[a, b]$ , there exist  $d+1$  subdomains  $\Omega_k$  such that  $\Omega' \subset \cup_k \Omega_k$  and  $\zeta(u_{\omega_k}^1, \dots, u_{\omega_k}^{d+1}, \nabla u_{\omega_k}^1, \dots, \nabla u_{\omega_k}^{d+1}) \neq 0$  in  $\Omega_k$ . This question comes from hybrid imaging inverse problems. The method used is not specific to the Helmholtz model and can be applied to other frequency dependent problems.

## 1. INTRODUCTION

The motivation for this work comes from hybrid, or multi-physics, parameter identification problem in boundary value problems for partial differential equations [11, 6]. In such imaging modalities, one part of the inverse problem can be described in general terms as follows. Suppose that  $u_\omega^1, u_\omega^2, \dots, u_\omega^N$  are the solutions of a partial differential equation of the form

$$\begin{cases} P(x, \omega, u_\omega^i) = 0 & \text{in } \Omega, \\ u_\omega^i = f^i & \text{on } \partial\Omega, \end{cases}$$

for  $i = 1, \dots, N$ . Suppose further that the parameter  $\omega$  is known,  $f^i$  is known, and some pointwise information is known about a functional of the solutions, e.g.  $H(u_\omega^1, \dots, u_\omega^N, \nabla u_\omega^1, \dots, \nabla u_\omega^N)$ , in  $\Omega'$ , a subdomain of  $\Omega$ , or possibly in all of  $\Omega$ . The problem is to reconstruct spatial dependence of the operator  $P$  from this information. In this article we focus on a particular model, a problem of Helmholtz type, in a smooth bounded domain  $\Omega \subseteq \mathbb{R}^d$ ,  $d \geq 2$ , given by

$$(1) \quad \begin{cases} -\operatorname{div}(a \nabla u) - (\omega^2 \varepsilon + \mathbf{i} \omega \sigma) u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

We assume that  $a \in L^\infty(\Omega; \mathbb{R}^{d \times d})$  and that  $a$  is symmetric and uniformly positive definite and bounded, that is, for all  $\xi \in \mathbb{R}^d$  there holds

$$(2) \quad \lambda^{-1} |\xi|^2 \leq \xi \cdot a \xi \leq \lambda |\xi|^2 \quad \text{a.e. in } \Omega$$

for some positive constant  $\lambda$ , whereas  $\varepsilon, \sigma \in L^\infty(\Omega; \mathbb{R})$  satisfy

$$(3) \quad \lambda^{-1} \leq \varepsilon \leq \lambda, \quad 0 \leq \sigma \leq \lambda \quad \text{a.e. in } \Omega.$$

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Assumptions (2) and (3) guarantee that problem (1) has a unique solution in  $H^1(\Omega; \mathbb{C})$  for every  $f \in H^{\frac{1}{2}}(\partial\Omega; \mathbb{R})$  and  $\omega \in D = \mathbb{C} \setminus \Sigma$ , where  $\Sigma$  denotes the set of the discrete Dirichlet eigenvalues of the problem. Let  $\mathcal{A} = [A_{\min}, A_{\max}]$  represent the frequencies we have access to, for some  $0 < A_{\min} < A_{\max}$ . For simplicity, we suppose that  $\mathcal{A} \subseteq D$ .

Given several boundary conditions  $f^i \in H^{\frac{1}{2}}(\partial\Omega; \mathbb{R})$  and frequencies  $\omega_k \in \mathcal{A}$ , the pointwise information available (or observable) in this case could be

$$u_{\omega_k}^i, a \nabla u_{\omega_k}^i, \text{ or } q u_{\omega_k}^i u_{\omega_k}^j \text{ or } a \nabla u_{\omega_k}^i \cdot \nabla u_{\omega_k}^j,$$

for some  $i, j$  and  $k$ . As all these data come from measurements, it is important to know a priori that the numbers obtained are not mostly measurement error or background noise: we want to ensure that the modulus of these data is non-zero. For instance, given  $\omega$  and a boundary condition  $f^1$ , we want to ensure that

$$(4) \quad u_{\omega}^1 \neq 0.$$

Alternatively, combining multiple data and constraints into one functional, given  $\omega$  and  $d+1$  boundary conditions  $f^1, \dots, f^{d+1}$  we write

$$(5) \quad F(\omega, f^1, \dots, f^{d+1}) = u_{\omega}^1 \det \begin{bmatrix} u_{\omega}^1 & \dots & u_{\omega}^{d+1} \\ \nabla u_{\omega}^1 & \dots & \nabla u_{\omega}^{d+1} \end{bmatrix},$$

and we want to ensure that

$$(6) \quad F(\omega, f^1, \dots, f^{d+1}) \neq 0 \quad \text{in } \Omega'.$$

The constraints (4) and (6), or related quantities, appear in [9, 10, 14, 21, 6]. If  $\omega$  is large, that is, greater than a constant depending on  $\lambda$  and  $\Omega$  only, (4) (and a fortiori (6)) cannot be satisfied in the whole domain, as for any boundary condition in  $H^{\frac{1}{2}}(\partial\Omega; \mathbb{R})$ , the field  $u_{\omega}^1$  must cancel (at least when  $\sigma = 0$ ). Thus multiple boundary conditions or frequencies must be considered.

**Definition 1.** Given a finite set of frequencies  $\{\omega_1, \dots, \omega_K\} \in \mathcal{A}^K$  and a finite set of boundary conditions  $f^1, \dots, f^N \in H^{\frac{1}{2}}(\partial\Omega; \mathbb{R})$ , we say that  $\{\omega_1, \dots, \omega_K\} \times \{f^1, \dots, f^N\}$  is a *set of measurements*.

The first concern is whether there exists a set of measurements such that (6) is satisfied everywhere by a subset of this set. The precise meaning of this statement is given by the following definition.

**Definition 2.** Take  $\Omega' \subseteq \Omega$ . Given  $K, N \in \mathbb{N}^*$ , a set of measurements  $\{\omega_1, \dots, \omega_K\} \times \{f^1, \dots, f^N\}$  is *F-complete in  $\Omega'$*  if there exists an open cover of  $\Omega'$

$$\Omega' = \bigcup_{p=1}^P \Omega'_p,$$

such that for each  $p$  there exist  $k \in \{1, \dots, K\}$  and  $i_1, \dots, i_{d+1} \in \{1, \dots, N\}$  such that

$$(7) \quad |F(\omega_k, f^{i_1}, \dots, f^{i_{d+1}})(x)| > 0, \quad x \in \Omega'_p.$$

In other words, a *F-complete* set of measurements gives a cover of the domain  $\Omega'$  into a finite collection of subdomains, such that the constraints (7) are satisfied in each subdomain for different frequencies and boundary conditions.

Several results show the existence of such  $F$ -complete sets. In the single-frequency case, namely with  $K = 1$ , the existence of such  $F$ -complete sets can be proved by using Complex Geometric Optics (CGO) solutions [16, 20, 13] or the Runge approximation property [14, 12, 18] under appropriate regularity hypotheses on  $a, \varepsilon$  and  $\sigma$ . When using CGO, only  $P = 2$  subdomains are needed, while with the Runge approximation approach  $P$  is larger than 2. These approaches do not indicate how suitable boundary conditions should be chosen in practice, as the proof of their existence relies on the unknown coefficients: without additional a priori information, the search for these boundary conditions requires many trials. Moreover, since CGO are very oscillatory, they may require a very sophisticated practical apparatus to be implemented.

An alternative method consists in fixing the boundary conditions and varying the frequency.

**Theorem 3.** *Take  $d = 2$  and suppose that  $\Omega$  is convex and  $a \in C^{0,\alpha}$  for some  $\alpha > 0$ . There exist  $K \in \mathbb{N}^*$  and  $C > 0$  depending only on  $\Omega, \Omega', \lambda, \alpha, A_{\min}$  and  $A_{\max}$  such that*

$$\{\omega_k = A_{\min} + (A_{\max} - A_{\min}) \frac{k-1}{K-1} : k = 1, \dots, K\} \times \{1, x_1, x_2\}$$

*is  $F$ -complete in  $\Omega'$ . More precisely, there exists an open cover  $\Omega' = \bigcup_{k=1}^K \Omega'_k$  such that*

$$|F(\omega_k, 1, x_1, x_2)| \geq C \text{ in } \Omega'_k.$$

*This result also holds when  $d = 3$  with the boundary conditions  $1, x_1, x_2, x_3$  provided that  $\|a - I_d\|_{C^{0,\alpha}} \leq \delta$  where  $\delta$  depends only on  $\Omega, \lambda$  and  $\alpha$ .*

This result is proved in [2] (see also [1, 4]). In this result,  $K$  is bounded a priori and possibly large, but the boundary conditions are fixed a priori and non-oscillatory. The proof of Theorem 3 relies on an analytic continuation argument with respect to the frequency, immersed in the complex plane. When  $\omega = 0$ , problem (1) becomes

$$\begin{cases} -\operatorname{div}(a \nabla u_0) = 0 & \text{in } \Omega, \\ u_0 = f & \text{on } \partial\Omega. \end{cases}$$

For the boundary condition  $f^1 = 1$ , the solution is simply  $u_0^1 \equiv 1$ . In two dimensions, for the boundary conditions  $f^2 = x_1$  and  $f^3 = x_2$ , since  $\Omega$  is convex, it is known that  $\det(\nabla u_0^2, \nabla u_0^3) > 0$  in  $\Omega$  [7, 15, 8]. Therefore

$$(8) \quad F(0, 1, x_1, x_2) = 1 \times \det \begin{bmatrix} 1 & u_0^2 & u_0^3 \\ 0 & \nabla u_0^2 & \nabla u_0^3 \end{bmatrix} > 0 \text{ in } \Omega.$$

In higher dimension such a result is not available, however Schauder elliptic regularity theory shows that if

$$(9) \quad \|a - I_d\|_{C^{0,\alpha}} \leq \delta$$

for some  $\delta > 0$  small enough, then

$$(10) \quad F(0, 1, x_1, \dots, x_d) = 1 \times \det \begin{bmatrix} 1 & u_0^2 & \dots & u_0^{d+1} \\ 0 & \nabla u_0^2 & & \nabla u_0^{d+1} \end{bmatrix} > 0 \text{ in } \Omega.$$

The principle of the proof of Theorem 3 is then to show that this positivity property can be transported to any interval  $\mathcal{A}$  in a predictable manner [2].

The goal of this article is to investigate what is the minimal number of required frequencies  $K$  (or, equivalently, the number  $P$  of subdomains in Definition 2) in dimension  $d \geq 2$ , for the fixed boundary conditions  $1, x_1, \dots, x_d$ . We consider a technically very convenient particular case, namely we assume that

$$(11) \quad a, \varepsilon, \text{ and } \sigma \text{ are real analytic in } \Omega.$$

The main result of the paper reads as follows.

**Theorem 4.** *Assume that (2), (3) and (11) hold true. Suppose that  $\mathcal{A} \subseteq D$  and take  $F$  as in (5),  $\Omega' \Subset \Omega$  and  $f^1, \dots, f^{d+1} \in H^{\frac{1}{2}}(\partial\Omega; \mathbb{R})$ . If*

$$(12) \quad F(0, f^1, \dots, f^{d+1})(x) \neq 0, \quad x \in \Omega,$$

*then*

$$\{(\omega_1, \dots, \omega_{d+1}) \in \mathcal{A}^{d+1} : \{\omega_k\}_k \times \{f^1, \dots, f^{d+1}\} \text{ is } F\text{-complete in } \Omega'\}$$

*is open and dense in  $\mathcal{A}^{d+1}$ .*

*Remark 5.* This argument can be used for many other partial differential equations, see [2] and [1, 4, 3, 5] for variants of this argument. It applies in particular to the anisotropic Maxwell system of equations.

*Remark 6.* There is nothing special about the function  $F$ , which was used as an illustration in this paper. We only use that  $F(\omega, 1, x_1, \dots, x_d)$  expressed in terms of  $u_\omega^1, \dots, u_\omega^{d+1}$ , that is,

$$F(\omega, 1, x_1, \dots, x_d) = \zeta(u_\omega^1, \dots, u_\omega^{d+1}, \nabla u_\omega^1, \dots, \nabla u_\omega^{d+1})$$

where

$$\zeta(z_1, \dots, z_{d+1}, \xi_1, \dots, \xi_{d+1}) = z_1 \det \left( \begin{bmatrix} z_1 & \dots & z_{d+1} \\ \xi_1 & \dots & \xi_{d+1} \end{bmatrix} \right),$$

is holomorphic (complex analytic). Any other constraint that can be written in terms of a holomorphic function  $\zeta$  can be substituted to  $F$ .

In view of (8) and (10), the following conclusion then naturally follows.

**Corollary 7.** *Assume that (2), (3) and (11) hold true. When  $d = 2$  assume that  $\Omega$  is convex and when  $d \geq 3$  assume that (9) holds. Suppose that  $\mathcal{A} \subseteq D$  and take  $\Omega' \Subset \Omega$ . Then*

$$\{(\omega_1, \dots, \omega_{d+1}) \in \mathcal{A}^{d+1} : \{\omega_k\}_k \times \{1, x_1, \dots, x_d\} \text{ is } F\text{-complete in } \Omega'\}$$

*is open and dense in  $\mathcal{A}^{d+1}$ .*

*Remark 8.* This result shows that almost any  $d + 1$  frequencies in  $\mathcal{A}$  give a  $F$ -complete set. An a priori estimate on the lower bound  $C$  for  $F$  (as in Theorem 3) cannot be obtained for arbitrary frequencies in an open and dense set in  $\mathcal{A}^{d+1}$ , as this bound tends to zero when the frequencies are chosen near the residual set.

It is easy to verify that this result is optimal in dimension 1 and 2 numerically. To intuitively see that  $K = d + 1$  is natural in any dimension, consider the level set  $u_\omega^1 = 0$ , for a given  $\omega$ . It is a priori a  $d - 1$  dimensional object. If we consider the intersection of two such level sets for  $\omega_1$  and  $\omega_2$ , we expect the resulting object to be  $d - 2$  dimensional, the intersection of  $d$  such level sets to be zero dimensional, i.e.

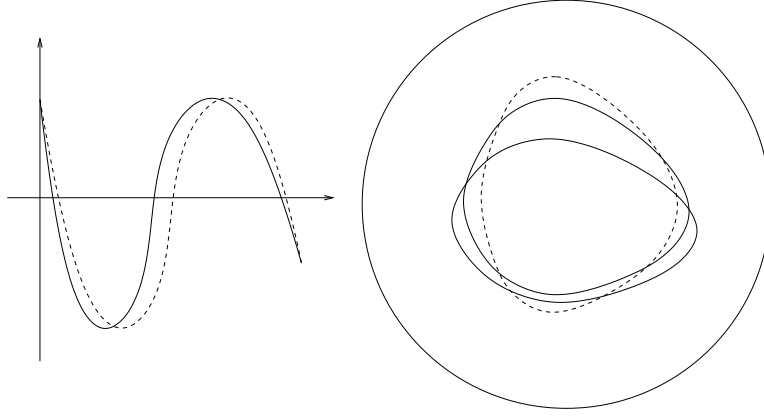


FIGURE 1. A sketch of the  $d + 1$  rationale in dimension 1 and 2. On the left, we represent two solutions with the same boundary data when  $d = 1$ : the locus  $\{u_\omega = 0\}$  moves with the parameter  $\omega$ , and the intersection of two zero sets is empty. On the right, we represent three zero level sets of the function  $F$  in two dimensions. The outer circle represents the boundary of  $\Omega$ . When only two values of  $\omega$  are used, the zero level sets still contain common points, but the intersection of the three level sets is empty.

discrete, and the intersection of  $d + 1$  level sets to be empty. Figure 1 is a graphical illustration of this idea.

The rest of the paper is devoted to the proof of Theorem 4.

## 2. PROOF OF THEOREM 4

First recall that the analyticity of the coefficients implies the analyticity of the solutions. Let  $C^A(\Omega; \mathbb{C})$  denote the space of complex-valued real analytic maps over  $\Omega$ .

**Lemma 9** ([19]). *Assume that (2), (3) and (11) hold true. If  $\omega \in D$  and  $f^i \in H^{\frac{1}{2}}(\partial\Omega; \mathbb{R})$  then  $u_\omega^i \in C^A(\Omega; \mathbb{C})$ .*

Theorem 4 will be a consequence of the following result.

**Proposition 10.** *Let  $\Omega, \Omega' \subseteq \mathbb{R}^d$  be smooth domains such that  $\Omega' \Subset \Omega$ . Let  $D \subseteq \mathbb{C}$  be an open set such that  $0 \in D$  and  $\mathcal{A} \subseteq D$ . Consider a map*

$$\theta: D \rightarrow C^A(\Omega; \mathbb{C}), \quad \omega \mapsto \theta_\omega$$

*such that for all  $x \in \Omega$ ,  $\omega \in D \mapsto \theta_\omega(x) \in \mathbb{C}$  is holomorphic and  $\theta_0(x) \neq 0$ . The set*

$$\left\{ (\omega_1, \dots, \omega_{d+1}) \in \mathcal{A}^{d+1} : \min_{\Omega'} (|\theta_{\omega_1}| + \dots + |\theta_{\omega_{d+1}}|) > 0 \right\}$$

*is open and dense in  $\mathcal{A}^{d+1}$ .*

First, let us see why Theorem 4 follows from this result.

*Proof of Theorem 4.* Consider the map

$$\theta: \omega \mapsto F(\omega, f^1, \dots, f^{d+1}) = u_\omega^1 \det \left( \begin{bmatrix} u_\omega^1 & \dots & u_\omega^{d+1} \\ \nabla u_\omega^1 & \dots & \nabla u_\omega^{d+1} \end{bmatrix} \right).$$

In view of Lemma 9,  $\theta_\omega \in C^A(\Omega; \mathbb{C})$ . By the general fact that for any  $i$  and  $x$  the map  $\omega \in D \mapsto (u_\omega^i(x), \nabla u_\omega^i(x)) \in \mathbb{C}^{d+1}$  is holomorphic [2], the map  $\omega \in D \mapsto \theta_\omega(x) \in \mathbb{C}$  is holomorphic for all  $x \in \Omega$ . Moreover,  $\theta_0(x) \neq 0$  for all  $x \in \Omega$  by (12).

We can apply Proposition 10 and obtain that

$$\left\{ (\omega_1, \dots, \omega_{d+1}) \in \mathcal{A}^{d+1} : \min_{\overline{\Omega'}} (|\theta_{\omega_1}| + \dots + |\theta_{\omega_{d+1}}|) > 0 \right\}$$

is open and dense in  $\mathcal{A}^{d+1}$ . Note that the condition  $\min_{\overline{\Omega'}} \sum_{k=1}^{d+1} |\theta_{\omega_k}| > 0$  is equivalent to

$$\text{for all } x \in \overline{\Omega'} \text{ there exists } k \text{ such that } |F(\omega_k, f^1, \dots, f^{d+1})| > 0,$$

which means that  $\{\omega_k\}_k \times \{f^i\}_i$  is a  $F$ -complete set of measurements in  $\Omega'$ . Indeed, defining

$$\Omega'_k = \{x \in \Omega' : |F(\omega_k, f^1, \dots, f^{d+1})| > 0\},$$

we have  $\Omega' = \cup_k \Omega'_k$ . This concludes the proof.  $\square$

The rest of this section is devoted to the proof of Proposition 10, which is based on the structure of analytic varieties.

An *analytic variety* in  $\Omega$  is the set of common zeros of a finite collection of real analytic functions in  $\Omega$ , namely  $\{x \in \Omega : g_1(x) = \dots = g_N(x) = 0\}$ , for some  $g_1, \dots, g_N \in C^A(\Omega; \mathbb{C})$ . For  $\omega_1, \dots, \omega_N \in \mathcal{A}$  we shall consider the analytic variety

$$Z(\omega_1, \dots, \omega_N) = \{x \in \Omega : \theta_{\omega_1}(x) = \dots = \theta_{\omega_N}(x) = 0\} = \bigcap_{i=1}^N Z(\omega_i).$$

Analytic varieties can be stratified into submanifolds of different dimensions.

**Lemma 11** ([22]). *Let  $X$  be an analytic variety in  $\Omega$ . There exists a locally finite collection  $\{A_l\}_l$  of pairwise disjoint connected analytic submanifolds of  $\Omega$  (satisfying Whitney's conditions) such that*

$$X = \bigcup_l A_l.$$

The decomposition  $X = \cup_l A_l$  is called a *Whitney stratification* of  $X$ . With this in mind, we can define the dimension of an analytic variety  $X = \cup_l A_l$  by

$$(13) \quad \dim X := \max_l \dim A_l.$$

The main result leading to the proof of Proposition 10 is the following

**Lemma 12.** *Under the hypotheses of Proposition 10, let  $\Omega''$  be a smooth domain such that  $\Omega'' \Subset \Omega$  and  $X$  be an analytic variety in  $\Omega$  such that  $X \cap \Omega'' \neq \emptyset$ . Then the set*

$$\{\omega \in \mathcal{A} : \dim(Z(\omega) \cap X \cap \Omega'') = \dim(X \cap \Omega'')\}$$

*is finite.*

*Proof.* By contradiction, suppose that the set is infinite. Since  $\mathcal{A}$  is compact, there exist  $\omega_n, \omega \in \mathcal{A}$ ,  $\omega_n \rightarrow \omega$  such that  $\dim(Z(\omega_n) \cap X \cap \Omega'') = \dim(X \cap \Omega'')$  and  $\omega_n \neq \omega$  for all  $n \in \mathbb{N}$ .

Therefore, in view of (13), for each  $n$  there exists a non-empty connected analytic submanifolds  $S_n$  such that

$$(14) \quad S_n \subseteq Z(\omega_n) \cap X \cap \Omega''$$

and

$$(15) \quad \dim S_n = \dim(X \cap \Omega'').$$

Choose an arbitrary  $x_n \in S_n$  for all  $n \in \mathbb{N}$ . Up to a subsequence, we have  $x_n \rightarrow x$ , for some  $x \in \overline{\Omega''}$ . By Lemma 11 applied to  $X$ , there exists an open neighborhood  $U$  of  $x$  in  $\Omega$  and a finite collection  $\{A_l\}_l$  of analytic submanifolds of  $\Omega$  such that

$$X \cap U = \cup_l A_l.$$

Moreover, since  $x_n \in S_n$  and  $x_n \rightarrow x$ , up to a subsequence we have  $S_n \cap U \neq \emptyset$  for all  $n \in \mathbb{N}$ . As  $S_n \subseteq X$ , up to a subsequence (and relabeling the collection  $A_l$ ) we have for all  $n$

$$(16) \quad S_n \cap U \subset A_1.$$

Since by (15),  $\dim(S_n \cap U) = \dim(X \cap U)$ , and  $A_1 \subset X \cap U$ , this implies

$$(17) \quad \dim(S_n \cap U) = \dim A_1.$$

In view of (14) we have  $\theta_{\omega_n}(y) = 0$  for all  $y \in S_n$ . Therefore, by (16), (17), [17, Theorem 1.2] and  $\theta_\omega \in C^A(\Omega; \mathbb{C})$  we obtain  $S_n \cap U = A_1$ , whence

$$A_1 \subseteq Z(\omega_n), \quad n \in \mathbb{N}.$$

As  $x_n \in A_1$  for all  $n$ , we have  $x \in \overline{A_1}$ . Thus, since  $Z(\omega_n)$  is closed, we infer that  $x \in Z(\omega_n)$  for all  $n \in \mathbb{N}$ , namely  $\theta_{\omega_n}(x) = 0$  for all  $n \in \mathbb{N}$ . Since  $\omega \mapsto \theta_\omega(x)$  is holomorphic, this implies  $\theta_0(x) = 0$ , which contradicts the assumptions.  $\square$

We are now in a position to prove Proposition 10.

*Proof of Proposition 10.* Since the map  $(\omega_1, \dots, \omega_{d+1}) \mapsto \min_{\overline{\Omega'}}(|\theta_{\omega_1}| + \dots + |\theta_{\omega_{d+1}}|)$  is continuous, the set  $G = \{(\omega_1, \dots, \omega_{d+1}) \in \mathcal{A}^{d+1} : \min_{\overline{\Omega'}}(|\theta_{\omega_1}| + \dots + |\theta_{\omega_{d+1}}|) > 0\}$  is open. It remains to show that  $G$  is dense in  $\mathcal{A}^{d+1}$ .

Take  $(\tilde{\omega}_1, \dots, \tilde{\omega}_{d+1}) \in \mathcal{A}^{d+1}$  and  $\varepsilon > 0$ . Let  $\Omega''$  be such that  $\Omega' \Subset \Omega'' \Subset \Omega$ . We equip  $\mathcal{A}^{d+1}$  with the norm

$$\|(\omega_1, \dots, \omega_{d+1})\| = \max_k |\omega_k|.$$

We now want to construct an element  $(\omega_1, \dots, \omega_{d+1}) \in G$  such that

$$(18) \quad \|(\omega_1, \dots, \omega_{d+1}) - (\tilde{\omega}_1, \dots, \tilde{\omega}_{d+1})\| < \varepsilon.$$

If  $\dim Z(\omega_1) \leq d-1$ , set  $\omega_1 = \tilde{\omega}_1$ ; obviously we have  $|\omega_1 - \tilde{\omega}_1| < \varepsilon$ . If  $\dim Z(\omega_1) = d$ , by Lemma 12 we obtain that the set

$$\{\omega \in \mathcal{A} : \dim(Z(\omega) \cap \Omega'') = d\}$$

is finite. Therefore, we can choose  $\omega_1 \in \mathcal{A}$  such that

$$\dim(Z(\omega_1) \cap \Omega'') \leq d-1$$

and  $|\omega_1 - \tilde{\omega}_1| < \varepsilon$ . Suppose now that we have constructed  $\omega_1, \dots, \omega_k$  such that  $|\omega_j - \tilde{\omega}_j| < \varepsilon$  for all  $j = 1, \dots, k$ . Let us describe how to construct  $\omega_{k+1}$ . If



$Z(\omega_1, \dots, \omega_k) \cap \Omega'' = \emptyset$ , then it is enough to choose  $\omega_{k+1} = \tilde{\omega}_{k+1}$ . Otherwise, applying Lemma 12 with  $X = Z(\omega_1, \dots, \omega_k)$ , we obtain that the set

$$\{\omega \in \mathcal{A} : \dim(Z(\omega) \cap Z(\omega_1, \dots, \omega_k) \cap \Omega'') = \dim(Z(\omega_1, \dots, \omega_k) \cap \Omega'')\}$$

is finite. Therefore, we can choose  $\omega_{k+1} \in \mathcal{A}$  such that

$$\dim(Z(\omega_1, \dots, \omega_{k+1}) \cap \Omega'') < \dim(Z(\omega_1, \dots, \omega_k) \cap \Omega'')$$

and  $|\omega_{k+1} - \tilde{\omega}_{k+1}| < \varepsilon$ . Therefore, as we have  $\dim(Z(\omega_1) \cap \Omega'') \leq d - 1$ , we obtain  $\dim(Z(\omega_1, \dots, \omega_{d+1}) \cap \Omega'') < 0$ , namely  $Z(\omega_1, \dots, \omega_{d+1}) \cap \Omega'' = \emptyset$ . In other words,  $(\omega_1, \dots, \omega_{d+1}) \in G$ . By construction, (18) is satisfied. This concludes the proof.  $\square$

### 3. CONCLUSIONS

In this work we have showed that, under the assumption of real analytic coefficients, almost any  $d + 1$  frequencies in a fixed range give the required constraints, where  $d$  is the dimension of the ambient space. The proof is based on the structure of analytic varieties, and so the hypothesis of real analytic coefficients is crucial. To prove (or disprove) this result under weaker hypothesis on the coefficients a different approach is required.

While this result seems optimal for an a priori fixed number of boundary conditions and for a somewhat arbitrary constraint function  $F$ , it could be that less than  $d + 1$  frequencies are required if more boundary conditions are allowed (e.g., a set of  $d$  frequencies and  $d \times (d + 1)$  boundary conditions to choose from).

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