Report no. OxPDE-15/07

Oxford Centre for Nonlinear P∂E

THE TRACE THEOREM, THE LUZIN N- AND MORSE-SARD PROPERTIES FOR THE SHARP CASE OF SOBOLEV-LORENTZ MAPPINGS

by

Jan Kristensen
University of Oxford

and

Mikhail V. Korobkov
Novosibirsk State University

Oxford Centre for Nonlinear PDE
Mathematical Institute, University of Oxford
Andrew Wiles Building, Radcliffe Observatory Quarter,
Woodstock Road,
Oxford, UK
OX2 6GG

July 2015

THE TRACE THEOREM, THE LUZIN N- AND MORSE–SARD PROPERTIES FOR THE SHARP CASE OF SOBOLEV–LORENTZ MAPPINGS

Mikhail V. Korobkov* and Jan Kristensen

August 5, 2015

Abstract

We prove Luzin N- and Morse–Sard properties for mappings $v\colon \mathbb{R}^n \to \mathbb{R}^d$ of the Sobolev–Lorentz class $\mathrm{W}^k_{p,1}$, $p = \frac{n}{k}$ (this is the sharp case that guarantees the continuity of mappings). Our main tool is a new trace theorem for Riesz potentials of Lorentz functions in a limiting case. Using these results, we find also some very natural approximation and differentiability properties for functions in $\mathrm{W}^k_{p,1}$ with exceptional set of small Hausdorff content.

Key words and phrases: *Sobolev–Lorentz space, Luzin N–property, Morse–Sard theorem, trace theorem, Riesz potentials, approximation.*
2010 Mathematics Subject Classification: *58K05, 46E35*

Introduction

In this paper we continue the study of the Luzin N- and Morse–Sard properties for the Sobolev mappings under minimal integrability assumptions initiated in our previous papers [9]–[10], [24], see also [22]. Of course, it is in this context very natural to restrict attention to continuous mappings, and so require from the considered function spaces that the inclusion $v \in \mathrm{W}^k_p(\mathbb{R}^n, \mathbb{R}^d)$ should guarantee at least the continuity of v. For values $k \in \{1, \dots, n-1\}$ it is well–known that $v \in \mathrm{W}^k_p(\mathbb{R}^n, \mathbb{R}^d)$ is continuous for $p > \frac{n}{k}$ and could be discontinuous for $p \le \frac{n}{k}$. So **the borderline case** is $p = p_\circ = \frac{n}{k}$. It is well–known (see for instance [22]) that $v \in \mathrm{W}^k_{p_\circ}(\mathbb{R}^n, \mathbb{R}^d)$ is continuous if the derivatives of k-th order belong to the Lorentz space $\mathrm{L}_{p_\circ,1}$, we will denote the space of such mappings by $\mathrm{W}^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^d)$. We refer to section 1 for relevant definitions and notation.

*The author was partially supported by the Russian Foundations for Basic Research and (Grant No. 14-01-00768-a), by the Grant of the Russian Federation for the State Support of Researches (Agreement No. 14.B25.31.0029), by the Grant of the Russian Federation for the State Support of Researches (Agreement No. 14.B25.31.0029), and by the Dynasty Foundation.

In this paper we prove the following Luzin N property with respect to Hausdorff content:

Theorem 0.1. *Let $k \in \{1, \dots, n\}$, $q \in [p_\circ, n]$, and $v \in \mathrm{W}^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^d)$. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any set $E \subset \mathbb{R}^n$ if $\mathcal{H}^q_\infty(E) < \delta$, then $\mathcal{H}^q_\infty(v(E)) < \varepsilon$. In particular, $\mathcal{H}^q(v(E)) = 0$ whenever $\mathcal{H}^q(E) = 0$.*

Here $\mathcal{H}^q_\infty(E)$ is as usual the q–dimensional Hausdorff content:

$$\mathcal{H}^q_\infty(E) = \inf\left\{\sum_{i=1}^{\infty}(\mathrm{diam}\, E_i)^q \ :\ E \subset \bigcup_{i=1}^{\infty} E_i\right\}.$$

Note that the case $k = 1$ was considered in the paper [22], and the case $k > 1$, $q > p_\circ$ in [24], so we omit them and consider here only the remaining limiting case $q = p_\circ$, $k > 1$.

To study this limiting case, we need a new version of the Sobolev Embedding Theorem that gives inclusions in Lebesgue spaces with respect to suitably general positive measures. This result might also be interesting in its own right, and it is the main contribution of this paper. For $\beta \in (0, n)$ denote by $\mathcal{M}^\beta$ the space of all nonnegative Borel measures μ on $\mathbb{R}^n$ such that

$$\|\mu\|_\beta = \sup_{I \subset \mathbb{R}^n} \ell(I)^{-\beta}\mu(I) < \infty, \tag{0.1}$$

where the supremum is taken over all n–dimensional cubic intervals $I \subset \mathbb{R}^n$ and $\ell(I)$ denotes side–length. Recall the following classical theorem proved by D.R. Adams [2] (see also, e.g., [30, §1.4.1]).

Theorem A. *Let μ be a positive Borel measure on $\mathbb{R}^n$ and $\alpha > 0$, $1 < p < q < \infty$, $\alpha p < n$. Then for any $f \in \mathrm{L}_p(\mathbb{R}^n)$ the estimate*

$$\int |I_\alpha f|^q \,\mathrm{d}\mu \le C\|\mu\|_\beta \cdot \|f\|^q_{\mathrm{L}_p} \tag{0.2}$$

holds with $\beta = (n - \alpha p)\frac{q}{p}$, where C depends on n, p, q, α only.

Here

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|y - x|^{n-\alpha}}\,\mathrm{d}y$$

is the Riesz potential of order α. The above estimate (0.2) fails for the limiting case $q = p$. Namely, there exist functions $f \in \mathrm{L}_p(\mathbb{R}^n)$ such that $I_\alpha f(x) = +\infty$ on some set of positive $(n - \alpha p)$–Hausdorff measure[1], see, e.g., [23]. Nevertheless, we prove the following result for this limiting case $q = p$:

[1]The above estimate (0.2) remains valid for $q = p$ if the measure μ instead of (0.1) satisfies the stronger condition $\mu(K) \le C R_{\alpha,p}(K)$ for all compact sets $K \subset \mathbb{R}^n$, where $R_{\alpha,p}$ is the Riesz capacity: $R_{\alpha,p}(K) = \inf\{\|f\|_{L_p} : f \in L_p(\mathbb{R}^n),\ I_\alpha f(x) \ge 1 \text{ on } K\}$, see [3]. Another geometric criterion for such an estimate (without using of Riesz capacity) was found in [23]. A simpler sufficient condition was found in [19], see also [31, p. 28].

Theorem 0.2. *Let μ be a positive Borel measure on $\mathbb{R}^n$ and $\alpha > 0$, $1 < p < \infty$, $\alpha p < n$. Then for any $f \in \mathrm{L}_{p,1}(\mathbb{R}^n)$ the estimate*

$$\|I_\alpha f\|_{L_p(\mu)} \le C \|\mu\|_\beta^{\frac{1}{p}} \cdot \|f\|_{\mathrm{L}_{p,1}}, \tag{0.3}$$

holds with $\beta = n - \alpha p$, where C depends on $n,\ p,\ \alpha$ only.

In view of the definition of the Lorentz spaces, it is sufficient to prove the above assertion for the simpler case when f coincides with the indicator function of some compact set:

Theorem 0.3. *Let μ be a positive Borel measure on $\mathbb{R}^n$ and $\alpha > 0$, $1 < p < \infty$, $\alpha p < n$. Then for any compact set $E \subset \mathbb{R}^n$ the estimate*

$$\|I_\alpha(1_E)\|_{L_p(\mu)}^p \le C \|\mu\|_\beta \operatorname{meas}(E), \tag{0.4}$$

holds with $\beta = n - \alpha p$, where 1_E is the indicator function of the set E and C depends on $n,\ p,\ \alpha$ only.

We emphasize that our proof of Theorem 0.3, and hence of Theorem 0.2, is self–contained, is independent of the previous proofs of this type of results, and uses only very natural and elementary arguments.

From the definition of the space $\mathrm{W}^k_{p_\circ,1}(\mathbb{R}^n,\mathbb{R}^d)$ of Sobolev–Lorentz mappings and the classical estimate $|\nabla v| \le C|I_{k-1}\nabla^k v|$, Theorem 0.2 implies

Theorem 0.4. *Let μ be a positive Borel measure on $\mathbb{R}^n$, $k \in \{1,\dots,n\}$. Then for any function $v \in \mathrm{W}^k_{p_\circ,1}(\mathbb{R}^n)$ the estimate*

$$\int |\nabla v|^{p_\circ}\, \mathrm{d}\mu \le C \|\mu\|_{p_\circ} \cdot \|\nabla^k v\|^{p_\circ}_{\mathrm{L}_{p_\circ,1}} \tag{0.5}$$

holds, where C depends on $n,\ k$ only.

From these results we deduce also some new differentiability and approximation properties of Sobolev–Lorentz mappings $v \in \mathrm{W}^k_{p_\circ,1}(\mathbb{R}^n)$. Namely, for $m \le n$ the m–order derivatives $\nabla^m v$ are well–defined $\mathcal{H}^{mp_\circ}$-almost everywhere, a function v is m-times differentiable (in the classical Fréchet–Peano sense) $\mathcal{H}^{mp_\circ}$-almost everywhere, and, finally, it coincides with C^m-smooth function on $\mathbb{R}^n \setminus U$, where the open exceptional set U has small $\mathcal{H}^{mp_\circ}_\infty$-Hausdorff content (see Theorems 2.1, 2.2–2.3). Note that for mappings of the classical Sobolev space $\mathrm{W}^k_{p_\circ}(\mathbb{R}^n)$ the corresponding exceptional set U has small Bessel capacity $\mathcal{B}_{k-m,p}(U) < \varepsilon$, and, respectively, the gradients $\nabla^m v$ are well-defined in $\mathbb{R}^n$ except for some exceptional set of zero Bessel capacity $\mathcal{B}_{k-m,p}$ (see, e.g., Chapter 3 in [45] or [7]).

In the last subsection 2.5 we discuss Morse–Sard type theorems for Sobolev–Lorentz mappings. Namely, for an open set $\Omega \subset \mathbb{R}^n$ and a mapping $v \in \mathrm{W}^k_{p_\circ,1,\mathrm{loc}}(\Omega,\mathbb{R}^d)$ denote $Z_{v,m} = \{x \in \Omega : v \text{ is differentiable at } x \text{ and } \operatorname{rank}\nabla v(x) < m\}$ (recall, that by previous results v is differentiable $\mathcal{H}^{p_\circ}$ a.e.). We state:

Theorem 0.5. *If $k, m \in \{1, \dots, n\}$, Ω is an open subset of $\mathbb{R}^n$, and $v \in \mathrm{W}^k_{p_\circ,1,\mathrm{loc}}(\Omega, \mathbb{R}^d)$, then $\mathcal{H}^{q_\circ}(v(Z_{v,m})) = 0$.*

Here

$$p_\circ = \frac{n}{k} \quad \text{and} \quad q_\circ = m - 1 + \frac{n-m+1}{k} = p_\circ + (m-1)\bigl(1 - k^{-1}\bigr). \tag{0.6}$$

The theorem was proved for C^k-smooth functions by Morse [33] in 1939 for the case $k = n$, $m = d = q_\circ = 1$, and subsequently by Sard [37] in 1942 for $k = n - m + 1$, $m = d = q_\circ$. For arbitrary values $k, n, m \in \mathbb{N}$ and C^k-smooth functions the result was proved almost simultaneously by Dubovitskiĭ [15] in 1967 and Federer [18, Theorem 3.4.3] in 1969[2].

The Morse–Sard Theorem for Sobolev spaces $\mathrm{W}^k_p(\mathbb{R}^n, \mathbb{R}^m)$ with $p > n$ (i.e., when $\mathrm{W}^k_p(\mathbb{R}^n) \hookrightarrow \mathrm{C}^{k-1}(\mathbb{R}^n)$) was obtained in [12] (see also [20] for a simple proof), and for Lipschitz and Hölder continuous mappings $\mathrm{C}^{k,\lambda}$ see, e.g., in [5] and [6] respectively. For further background on these issues the reader is referred to [9], [10], [24], where the above Theorem 0.5 was proved in the Sobolev context $\mathrm{W}^k_{p_\circ}(\mathbb{R}^n)$ for $k, m \in \{2, \dots, n\}$. Since the case $k = 1$ (i.e., $q_\circ = n$) can be considered folklore (see, e.g., [38]) we shall in the present paper only consider the cases $m = 1$, $k > 1$, $q_\circ = p_\circ = \frac{n}{k}$.

Let us end this introduction by noting an interesting phenomenon that occurs for functions of the Sobolev–Lorentz space $\mathrm{W}^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^d)$. On the one hand, the order of integrability of the k–th derivative, Lebesgues index $p_\circ$ and Lorentz index 1, is the minimal one on the Lorentz scale that guarantees **continuity** of mappings. On the other hand, these mappings a posteriori have many additional analytical regularity properties: the Luzin N–property, differentiability and approximation properties, and the Morse–Sard property (see above).

For instance, if $k = n-m+1$, then almost all level sets of mappings $v \in \mathrm{W}^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^m)$ are C^1–smooth manifolds [24]. The result should be contrasted with the fact that mappings of class $\mathrm{W}^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^m)$ are continuous only and need not to be C^1–smooth in general. This property recently found some applications in mathematical fluid mechanics (see [25]).

Acknowledgment. We are grateful to Professor Jean Bourgain for very useful interaction on the subject of this paper. The main part of the paper was written during a visit of MVK to the Oxford Centre for Nonlinear PDE in May 2015.

1 Preliminaries

By an *n–dimensional cubic interval* we mean a closed cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes. If Q is an n–dimensional cubic interval then we write $\ell(Q)$ for its sidelength.

[2] Federer announced [17] his result in 1966, this announcement (without any proofs) was sent on 08.02.1966. For the historical details, Dubovitskiĭ sent his paper [15] (with complete proofs) a month earlier, on 10.01.1966.

For a subset S of $\mathbb{R}^n$ we write $\mathcal{L}^n(S)$ for its outer Lebesgue measure. The m–dimensional Hausdorff measure is denoted by $\mathcal{H}^m$ and the m–dimensional Hausdorff content by $\mathcal{H}^m_\infty$. Recall that for any subset S of $\mathbb{R}^n$ we have by definition

$$\mathcal{H}^m(S) = \lim_{\alpha \searrow 0} \mathcal{H}^m_\alpha(S) = \sup_{\alpha > 0} \mathcal{H}^m_\alpha(S),$$

where for each $0 < \alpha \leq \infty$,

$$\mathcal{H}^m_\alpha(S) = \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} S_i)^m \ : \ \operatorname{diam} S_i \leq \alpha, \quad S \subset \bigcup_{i=1}^{\infty} S_i \right\}.$$

It is well known that $\mathcal{H}^n(S) = \mathcal{H}^n_\infty(S) \sim \mathcal{L}^n(S)$ for sets $S \subset \mathbb{R}^n$.

To simplify the notation, we write $\|f\|_{\mathrm{L}_p}$ instead of $\|f\|_{\mathrm{L}_p(\mathbb{R}^n)}$, etc.

The Sobolev space $\mathrm{W}^k_p(\mathbb{R}^n, \mathbb{R}^d)$ is as usual defined as consisting of those $\mathbb{R}^d$–valued functions $f \in \mathrm{L}_p(\mathbb{R}^n)$ whose distributional partial derivatives of orders $l \leq k$ belong to $\mathrm{L}_p(\mathbb{R}^n)$ (for detailed definitions and differentiability properties of such functions see, e.g., [16], [30], [45], [13]). Denote by $\nabla^k f$ the vector-valued function consisting of all k–th order partial derivatives of f arranged in some fixed order. However, for the case of first order derivatives $k = 1$ we shall often think of $\nabla f(x)$ as the Jacobi matrix of f at x, thus the $d \times n$ matrix whose r–th row is the vector of partial derivatives of the r–th coordinate function.

We use the norm

$$\|f\|_{\mathrm{W}^k_p} = \|f\|_{\mathrm{L}_p} + \|\nabla f\|_{\mathrm{L}_p} + \cdots + \|\nabla^k f\|_{\mathrm{L}_p},$$

and unless otherwise specified all norms on the spaces $\mathbb{R}^s$ ($s \in \mathbb{N}$) will be the usual euclidean norms.

Working with locally integrable functions, we always assume that the precise representatives are chosen. If $w \in L_{1,\mathrm{loc}}(\Omega)$, then the precise representative w^* is defined for *all* $x \in \Omega$ by

$$w^*(x) = \begin{cases} \lim\limits_{r \searrow 0} \fint_{B(x,r)} w(z)\,\mathrm{d}z, & \text{if the limit exists and is finite,} \\ 0 & \text{otherwise,} \end{cases} \tag{1.1}$$

where the dashed integral as usual denotes the integral mean,

$$\fint_{B(x,r)} w(z)\,\mathrm{d}z = \frac{1}{\mathcal{L}^n(B(x,r))} \int_{B(x,r)} w(z)\,\mathrm{d}z,$$

and $B(x,r) = \{y : |y - x| < r\}$ is the open ball of radius r centered at x. Henceforth we omit special notation for the precise representative writing simply $w^* = w$.

We will say that x is an L_p–Lebesgue point of w (and simply a Lebesgue point when $p = 1$), if

$$\fint_{B(x,r)} |w(z) - w(x)|^p\,\mathrm{d}z \to 0 \quad \text{as} \quad r \searrow 0.$$

If $k < n$, then it is well-known that functions from Sobolev spaces $\mathrm{W}^k_p(\mathbb{R}^n)$ are continuous for $p > \frac{n}{k}$ and could be discontinuous for $p \le p_\circ = \frac{n}{k}$ (see, e.g., [30, 45]). The Sobolev–Lorentz space $\mathrm{W}^k_{p_\circ,1}(\mathbb{R}^n) \subset \mathrm{W}^k_{p_\circ}(\mathbb{R}^n)$ is a refinement of the corresponding Sobolev space that for our purposes turns out to be convenient. Among other things functions that are locally in $\mathrm{W}^k_{p_\circ,1}$ on $\mathbb{R}^n$ are in particular continuous.

Given a measurable function $f\colon \mathbb{R}^n \to \mathbb{R}$, denote by $f_*\colon (0,\infty) \to \mathbb{R}$ its distribution function

$$f_*(s) := \mathcal{L}^n\{x \in \mathbb{R}^n : |f(x)| > s\},$$

and by f^* the nonincreasing rearrangement of f, defined for $t > 0$ by

$$f^*(t) = \inf\{s \ge 0 : f_*(s) \le t\}.$$

Since $|f|$ and f^* are equimeasurable, we have for every $1 \le p < \infty$,

$$\Bigg(\int\limits_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x\Bigg)^{1/p} = \Bigg(\int\limits_0^{+\infty} f^*(t)^p \,\mathrm{d}t\Bigg)^{1/p}.$$

The Lorentz space $\mathrm{L}_{p,q}(\mathbb{R}^n)$ for $1 \le p < \infty$, $1 \le q < \infty$ can be defined as the set of all measurable functions $f\colon \mathbb{R}^n \to \mathbb{R}$ for which the expresssion

$$\|f\|_{\mathrm{L}_{p,q}} = \begin{cases} \Bigg(\dfrac{q}{p}\displaystyle\int\limits_0^{+\infty} (t^{1/p} f^*(t))^q \frac{\mathrm{d}t}{t}\Bigg)^{1/q} & \text{if } 1 \le q < \infty \\ \\ \sup\limits_{t>0} t^{1/p} f^*(t) & \text{if } q = \infty \end{cases}$$

is finite. We refer the reader to [28], [40] or [45] for information about Lorentz spaces. However, let us remark that in view of the definition of $\|\cdot\|_{\mathrm{L}_{p,q}}$ and the equimeasurability of f and f^* we have $\|f\|_{\mathrm{L}_p} = \|f\|_{\mathrm{L}_{p,p}}$ so that in particular $\mathrm{L}_{p,p}(\mathbb{R}^n) = \mathrm{L}_p(\mathbb{R}^n)$. Further, for a fixed exponent p and $q_1 < q_2$ we have the estimate $\|f\|_{\mathrm{L}_{p,q_2}} \le \|f\|_{\mathrm{L}_{p,q_1}}$, and, consequently, the embedding $\mathrm{L}_{p,q_1}(\mathbb{R}^n) \subset \mathrm{L}_{p,q_2}(\mathbb{R}^n)$ (see [28, Theorem 3.8(a)]). Finally we recall that $\|\cdot\|_{\mathrm{L}_{p,q}}$ is a norm on $\mathrm{L}_{p,q}(\mathbb{R}^n)$ for all $q \in [1,p]$ and a quasi–norm in the remaining cases $q \in (p,\infty]$ (see [28, Proposition 3.3]).

Here we shall mainly be concerned with the Lorentz space $\mathrm{L}_{p,1}$, and in this case one may rewrite the norm as (see for instance [28, Proposition 3.6])

$$\|f\|_{p,1} = \int\limits_0^{+\infty} \big[\mathcal{L}^n(\{x \in \mathbb{R}^n : |f(x)| > t\})\big]^{\frac{1}{p}} \,\mathrm{d}t. \tag{1.2}$$

We record the following subadditivity property of the Lorentz norm for later use.

Lemma 1.1 (see, e.g., [36] or [28])**.** *Suppose that $1 \le p < \infty$ and $E = \bigcup_{j\in\mathbb{N}} E_j$, where E_j are measurable and mutually disjoint subsets of $\mathbb{R}^n$. Then for all $f \in \mathrm{L}_{p,1}$ we have*

$$\sum_j \|f \cdot 1_{E_j}\|^p_{\mathrm{L}_{p,1}} \le \|f \cdot 1_E\|^p_{\mathrm{L}_{p,1}},$$

where 1_E denotes the indicator function of the set E.

Denote by $\mathrm{W}^k_{p,1}(\mathbb{R}^n)$ the space of all functions $v \in \mathrm{W}^k_p(\mathbb{R}^n)$ such that in addition the Lorentz norm $\|\nabla^k v\|_{\mathrm{L}_{p,1}}$ is finite.

For a mapping $u \in \mathrm{L}_1(Q, \mathbb{R}^d)$, $Q \subset \mathbb{R}^n$, $m \in \mathbb{N}$, define the polynomial $P_{Q,m}[u]$ of degree at most m by the following rule:

$$\int_Q y^\alpha \left(u(y) - P_{Q,m}[u](y)\right) \mathrm{d}y = 0 \tag{1.3}$$

for any multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ of length $|\alpha| = \alpha_1 + \dots + \alpha_n \le m$. Denote $P_Q[u] = P_{Q,k-1}[u]$.

The following well–known bound will be used on several occasions.

Lemma 1.2. *Suppose $v \in \mathrm{W}^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^d)$ with $k \in \{1, \dots, n\}$. Then v is a continuous mapping and for any n-dimensional cubic interval $Q \subset \mathbb{R}^n$ the estimate*

$$\sup_{y\in Q}\left|v(y) - P_Q[v](y)\right| \le C\|1_Q \cdot \nabla^k v\|_{\mathrm{L}_{p_\circ,1}} \tag{1.4}$$

holds, where C is a constant depending on n, d only. Moreover, the mapping $v_Q(y) = v(y) - P_Q[v](y)$, $y \in Q$, can be extended from Q to the whole of $\mathbb{R}^n$ such that the extension (denoted again) $v_Q \in \mathrm{W}^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^d)$ and

$$\|\nabla^k v_Q\|_{\mathrm{L}_{p_\circ,1}(\mathbb{R}^n)} \le C_0\|\nabla^k v\|_{\mathrm{L}_{p_\circ,1}(Q)}, \tag{1.5}$$

where C_0 also depends on n, d only.

Proof. For continuity and the estimate (1.4) see [24, Lemma 1.3]. Because of coordinate invariance of estimate (1.5), it is sufficient to prove the assertions about extension for the case when Q is a unit cube: $Q = [0, 1]^n$. Put $u(y) = v_Q(y) = v(y) - P_Q[v](y)$ for $y \in Q$.

By Peetre theorem (see Theorem 6.5 in [28, page 10]) it is easy to deduce that

$$\|\nabla^m u\|_{L_{p_\circ,1}(Q)} \le C\|\nabla^k u\|_{L_{p_\circ,1}(Q)} \qquad \forall m = 0, 1, \dots, k-1. \tag{1.6}$$

Using the standard Extension operator for Sobolev spaces (the well-known "finite-order reflection" procedure, see, e.g., [30, §1.1.17]), function u on the unit cube $Q = [0, 1]^n$ can be extended to a function $U \in \mathrm{W}^k_{p_\circ,1}(\mathbb{R}^n)$ such that the estimate

$$\|\nabla^k U\|_{\mathrm{L}_{p_\circ,1}(\mathbb{R}^n)} \le C' \sum_{m=0}^k \|\nabla^m u\|_{L_{p_\circ,1}(Q)}$$

holds. Taking into account the identity $\nabla^k u \equiv \nabla^k v$ on Q and (1.6), we obtain the required estimate (1.5). $\square$

Corollary 1.1 (see, e.g., [24]). *Suppose $v \in \mathrm{W}^k_{p_\circ,1}(\mathbb{R}^n,\mathbb{R}^d)$ with $k \in \{1,\dots,n\}$. Then v is a continuous mapping and for any n-dimensional cubic interval $Q \subset \mathbb{R}^n$ the estimate*

$$\operatorname{diam} v(Q) \le C\left(\frac{\|\nabla v\|_{\mathrm{L}_1(Q)}}{\ell(Q)^{n-1}} + \|1_Q\cdot\nabla^k v\|_{\mathrm{L}_{p_\circ,1}}\right) \le C\left(\frac{\|\nabla v\|_{\mathrm{L}_{p_\circ}(Q)}}{\ell(Q)^{k-1}} + \|1_Q\cdot\nabla^k v\|_{\mathrm{L}_{p_\circ,1}}\right) \quad (1.7)$$

holds.

The above results can easily be adapted to give that $v \in \mathrm{C}_0(\mathbb{R}^n)$, the space of continuous functions on $\mathbb{R}^n$ that vanish at infinity (see for instance [28, Theorem 5.5]).

Analogously, from previous estimates one could deduce

Corollary 1.2. *Suppose $v \in \mathrm{W}^k_{p_\circ,1}(\mathbb{R}^n,\mathbb{R}^d)$ with $k \in \{1,\dots,n\}$. Then for all $m \in \{1,\dots,k\}$ and for any n-dimensional cubic interval $Q \subset \mathbb{R}^n$ the estimate*

$$\sup_{y\in Q}\bigl|v(y) - P_{Q,m-1}[v](y)\bigr| \le C\left(\frac{\|\nabla^m v\|_{\mathrm{L}_{p_\circ}(Q)}}{\ell(Q)^{k-m}} + \|1_Q\cdot\nabla^k v\|_{\mathrm{L}_{p_\circ,1}}\right) \quad (1.8)$$

holds.

Theorem 1.1 (Boundedness of the maximal operator, see [28]). *Let $f \in \mathrm{L}_{p,q}(\mathbb{R}^n)$, $1 < p < \infty$, $1 \le q < \infty$. Then*

$$\|\mathcal{M}f\|_{\mathrm{L}_{p,q}} \le C\|f\|_{\mathrm{L}_{p,q}}.$$

Here

$$\mathcal{M}f(x) = \sup_{r>0} r^{-n}\int\limits_{B(x,r)} |f(y)|\,\mathrm{d}y$$

is the usual Hardy–Littlewood maximal function of f.

Corollary 1.3 (Regularization in Lorentz spaces [28]). *Let $f \in \mathrm{L}_{p,q}(\mathbb{R}^n)$, $1 < p < \infty$, $1 \le q < \infty$. Suppose that $f \in \mathrm{L}_{p,q}(\mathbb{R}^n)$ and $\psi \in \mathrm{C}_0^\infty(\mathbb{R}^n)$ is a standard mollifier. Then $\psi_\delta * f \to f$ in $\mathrm{L}_{p,q}(\mathbb{R}^n)$ as $\delta \to 0$.*

Here and henceforth $\mathrm{C}_0^\infty(\mathbb{R}^n)$ denotes the space of C^∞ smooth and compactly supported functions on $\mathbb{R}^n$.

Corollary 1.4 (Regularization in Sobolev–Lorentz spaces). *If $f \in \mathrm{W}^k_{p,q}(\mathbb{R}^n)$, $1 < p < \infty$, $1 \le q < \infty$, then there exists a sequence of smooth functions $f_i \in \mathrm{C}_0^\infty(\mathbb{R}^n)$ such that $\|\nabla^m(f-f_i)\|_{\mathrm{L}_p(\mathbb{R}^n)} \to 0$ for $m = 0,1,\dots,k$, $\|\nabla^k(f-f_i)\|_{\mathrm{L}_{p,q}(\mathbb{R}^n)} \to 0$ as $i \to \infty$.*

Remark 1.1. By Sobolev inequality, under conditions of Corollary 1.4, if, in addition $1 \le q \le p$ and $(k-m)p < n$ for some $m \in \{0,1,\dots,k-1\}$, then we have also the convergence $\|\nabla^m(f-f_i)\|_{L_{p_m,q}(\mathbb{R}^n)} \to 0$, where p_m is a Sobolev exponent $p_m = \frac{np}{n-(k-m)p}$ (see, e.g., [28, §8]).

We need also the following important Adams strong-type estimates for maximal functions.

Theorem 1.2 (see Theorem A, Proposition 1 and its Corollary in [1]). *Let $\beta \in (0, n)$. Then for nonnegative functions $f \in \mathrm{C}_0(\mathbb{R}^n)$ the estimates*

$$\begin{aligned}\int_0^\infty \mathcal{H}^\beta_\infty(\{x \in \mathbb{R}^n : \mathcal{M}f(x) \geq t\})\,\mathrm{d}t &\leq C_1 \int_0^\infty \mathcal{H}^\beta_\infty(\{x \in \mathbb{R}^n : f(x) \geq t\})\,\mathrm{d}t \\ &\leq C_2 \sup\left\{\int f\,\mathrm{d}\mu : \mu \in \mathcal{M}^\beta,\ \|\mu\|_\beta \leq 1\right\},\end{aligned}$$

hold, where the constants C_1, C_2 depend on β, n only.

We need also the following classical fact (cf. with [8]).

Lemma 1.3 (see Lemma 2 in [13]). *Let $u \in \mathrm{W}^m_1(\mathbb{R}^n)$, $m \leq n$. Then for any n–dimensional cubic interval $Q \subset \mathbb{R}^n$, $x \in Q$, and for any $j = 0, 1, \dots, m-1$ the estimate*

$$\left|\nabla^j u(x) - \nabla^j P_{Q,m-1}[u](x)\right| \leq C\ell(Q)^{m-j}(\mathcal{M}\nabla^m u)(x) \tag{1.9}$$

holds, where the constant C depends on n, m only.

2 Proofs of the main results

2.1 The trace theorem

Theorem 0.3 plays the key role among other results. Its proof splits into a number of lemmas. Fix parameters $m > 0$, $1 < p < \infty$, $0 < \alpha p < n$, and a positive Borel measure μ on $\mathbb{R}^n$ satisfying

$$\mu(B(x, r)) \leq r^{n-\alpha p} \tag{2.1}$$

for every ball $B(x, r) \subset \mathbb{R}^n$. Fix also a compact set $E \subset \mathbb{R}^n$. Denote by I_E the corresponding Riesz potential $I_\alpha(1_E)$.

It is very easy to check by standard calculation that

$$0 \leq I_E(x) \leq C_0|E|^{\frac{\alpha}{n}}, \tag{2.2}$$

where the constant C_0 depends on n, α only.

Denote also $t_m = 2^m$ (here $m \in \mathbb{Z}$),

$$E_m = \{x \in E : I_E(x) \in [t_m, 2t_m]\},$$

$$E'_m = \{x \in E : I_E(x) \leq t_m\}, \qquad E''_m = \{x \in E : I_E(x) > t_m\}.$$

In this section we will write $f \lesssim g$, if $f \leq Cg$, where C depends on n, α, p only (really, most of the corresponding constants below up to Lemma 2.6 depends on n, α only).

Lemma 2.1. *There exists a positive constant $m_0 \in \mathbb{N}$ depending on n, α only such that for any $m \in \mathbb{Z}$ and $x \in \mathbb{R}^n$ if $I_E(x) \geq t_m$, then $I_{E''_{m-m_0}}(x) \gtrsim t_m$.*

Proof. The claim follows from the well-known maximum principle: $I_{E'_m}(x) \le 2^{n-\alpha} t_m$ for every $m \in \mathbb{Z}$ (see [21, Theorem 5.2]). $\square$

Lemma 2.2. *For any $x, y \in \mathbb{R}^n$ if $I_E(y) = t$ and $|x-y| \le (2t)^{\frac{1}{\alpha}}$ then $I_E(x) \gtrsim t$.*

Proof. Let $I_E(y) = t$ and

$$|y-x| \le (2t)^{\frac{1}{\alpha}}. \tag{2.3}$$

Denote $r = |y-x|$, $B = B(y,r) = \{z \in \mathbb{R}^n : |z-y| < r\}$. Then by construction

$$t = I_E(y) = I_{E\cap B}(y) + I_{E\setminus B}(y). \tag{2.4}$$

Consider two possible situations.

(I). $I_{E\cap B}(y) \le \frac{t}{2}$, then $I_{E\setminus B}(y) \ge \frac{t}{2}$. For any $z \in E \setminus B$ we have $|z-y| \ge r = |x-y|$, thus, $|x-z| \le |x-y| + |z-y| \le 2|z-y|$, consequently,

$$I_E(x) \ge I_{E\setminus B}(x) \ge 2^{n-\alpha} I_{E\setminus B}(y) \ge 2^{n-\alpha-1} t. \tag{2.5}$$

(II). $I_{E\cap B}(y) \ge \frac{t}{2}$. Then (2.2) implies $\frac{t}{2} \le C_0 |B \cap E|^{\frac{\alpha}{n}}$. Since $B \cap E \subset B(x, 2r)$, by elementary estimates we have

$$I_E(x) \ge \frac{|B\cap E|}{(2r)^{n-\alpha}} \ge C' \frac{t^{\frac{n}{\alpha}}}{r^{n-\alpha}} \overset{(2.3)}{\ge} C'' \frac{t^{\frac{n}{\alpha}}}{t^{\frac{n}{\alpha}-1}} = C_2 t.$$

$\square$

Denote $F_m = \{x \in \mathbb{R}^n : I_E(x) \in [t_m, 2t_m]\}$, $\mu_m = \mu(F_m)$, $\mu_m(\cdot) = \mu \llcorner F_m$. By construction,

$$\|I_\alpha(1_E)\|^p_{L_p(\mu)} \sim \sum_{m=-\infty}^{\infty} t_m^p \mu_m.$$

So our main purpose below is to estimate $t_m \mu_m$. Of course, $t_m \mu_m \le \int_{\mathbb{R}^n} I_E(x)\,\mathrm{d}\mu_m(x)$. By Fubini Theorem we have

$$\begin{aligned} \int_{\mathbb{R}^n} I_E(x)\,\mathrm{d}\mu_m(x) &= \int_0^\infty \rho^{-n+\alpha-1} \left[\int_{\mathbb{R}^n} |E \cap B(x,\rho)|\,\mathrm{d}\mu_m(x) \right] \mathrm{d}\rho \\ &= \int_0^\infty \rho^{-n+\alpha-1} \left[\int_E \mu_m\big[B(y,\rho)\big]\,\mathrm{d}y \right] \mathrm{d}\rho. \end{aligned} \tag{2.6}$$

Lemma 2.3. *The estimate*

$$t_m \mu_m \lesssim \int_0^\infty \rho^{-n+\alpha-1} \left[\int_{\mathbb{R}^n} |E''_{m-m_0} \cap B(x,\rho)|\,\mathrm{d}\mu_m(x) \right] \mathrm{d}\rho \tag{2.7}$$

holds, where m_0 is a constant from Lemma 2.1.

Proof. By Lemma 2.1, $I_{E''_{m-m_0}} \geq C_1 t_m$ on F_m, therefore $t_m \mu_m \leq C \int_{\mathbb{R}^n} I_{E''_{m-m_0}}(x)\,\mathrm{d}\mu_m(x)$, and the last inequality implies in conjunction with Fubini's Theorem (2.7). □

Lemma 2.4. *There exists a constant $m_1 \in \mathbb{N}$ such that*

$$t_m \mu_m \lesssim \int_{t_{m-m_1}^{\frac{1}{\alpha}}}^{\infty} \rho^{-n+\alpha-1} \left[\int_{\mathbb{R}^n} |E''_{m-m_0} \cap B(x,\rho)|\,\mathrm{d}\mu_m(x) \right] \mathrm{d}\rho. \tag{2.8}$$

Proof. Let $m_1 \in \mathbb{N}$, its exact value will be specified below. We have $|E \cap B(x,\rho)| \leq \omega_n \rho^n$, where ω_n is a volume of a unit ball in $\mathbb{R}^n$. Thus

$$\int_0^{t_{m-m_1}^{\frac{1}{\alpha}}} \rho^{-n+\alpha-1} \left[\int_{\mathbb{R}^n} |E \cap B(x,\rho)|\, d\mu_m(x) \right] d\rho \leq \omega_n \mu_m \int_0^{t_{m-m_1}^{\frac{1}{\alpha}}} \rho^{\alpha-1}\, d\rho = \frac{\omega_n}{\alpha} \mu_m t_{m-m_1} = \frac{\omega_n}{\alpha} 2^{-m_1} \mu_m t_m.$$

So the target estimate (2.8) follows from (2.7) provided that $\frac{1}{\alpha}\omega_n 2^{-m_1}$ is sufficiently small. □

Lemma 2.5. *There exists a constant $i_0 \in \mathbb{N}$ such that for all $i \geq m - m_1$ the equality*

$$\int_{t_i^{\frac{1}{\alpha}}}^{t_{i+1}^{\frac{1}{\alpha}}} \rho^{-n+\alpha-1} \left[\int_{\mathbb{R}^n} |E''_{m-m_0} \cap B(x,\rho)|\,\mathrm{d}\mu_m(x) \right] \mathrm{d}\rho$$

$$= \sum_{j=m-m_0}^{i+i_0} \int_{t_i^{\frac{1}{\alpha}}}^{t_{i+1}^{\frac{1}{\alpha}}} \rho^{-n+\alpha-1} \left[\int_{\mathbb{R}^n} |E_j \cap B(x,\rho)|\,\mathrm{d}\mu_m(x) \right] \mathrm{d}\rho \tag{2.9}$$

holds, where m_0, m_1 are the constants from Lemma 2.1, 2.4 respectively.

Proof. Let $i \geq m - m_1$,

$$\rho^\alpha \leq t_{i+1}, \tag{2.10}$$

and $y \in E_j \cap B(x,\rho)$, $x \in F_m = \operatorname{supp} \mu_m$. Then by definitions of these sets

$$I_E(x) \leq 2t_m \tag{2.11}$$

and $I_E(y) \geq t_j$. Suppose $j \geq i+1$. Then (2.10) implies $|x-y|^\alpha \leq t_{i+1} \leq t_j$, therefore, by Lemma 2.2 (applying for $t = t_j$) we have $I_E(x) \geq C_2 t_j$. Thus by (2.11) we obtain $j \leq m + m_2$ for some constant m_2 depending on α, n only.

Finally we have $j \leq \max(i+1, m+m_2) \leq \max(i+1, i+m_1+m_2)$ finishing the proof of the Lemma. □

Lemma 2.6. *The estimate*

$$t_m\mu_m \lesssim \sum_{j=m-m_0}^{\infty} |E_j| t_{j-i_0}^{1-p} \tag{2.12}$$

holds for all $m \in \mathbb{Z}$, where m_0, i_0 are the constants from Lemmas 2.1, 2.5, respectively.

Proof. We have

$$\begin{aligned}
&t_m\mu_m \overset{(2.8)}{\lesssim} \sum_{i=m-m_1}^{\infty} \int_{t_i^{\frac{1}{\alpha}}}^{t_{i+1}^{\frac{1}{\alpha}}} \rho^{-n+\alpha-1} \left[\int_{\mathbb{R}^n} |E''_{m-m_0} \cap B(x,\rho)| \, \mathrm{d}\mu_m(x) \right] \mathrm{d}\rho \lesssim \\
&\overset{(2.9)}{\lesssim} \sum_{i=m-m_1}^{\infty} \sum_{j=m-m_0}^{i+i_0} \int_{t_i^{\frac{1}{\alpha}}}^{t_{i+1}^{\frac{1}{\alpha}}} \rho^{-n+\alpha-1} \left[\int_{\mathbb{R}^n} |E_j \cap B(x,\rho)| \, \mathrm{d}\mu_m(x) \right] \mathrm{d}\rho = \\
&\overset{\text{Fubini}}{=} \sum_{i=m-m_1}^{\infty} \sum_{j=m-m_0}^{i+i_0} \int_{t_i^{\frac{1}{\alpha}}}^{t_{i+1}^{\frac{1}{\alpha}}} \rho^{-n+\alpha-1} \left[\int_{E_j} \mu_m\big[B(y,\rho)\big] \, \mathrm{d}y \right] \mathrm{d}\rho \lesssim \\
&\overset{(2.1)}{\lesssim} \sum_{i=m-m_1}^{\infty} \sum_{j=m-m_0}^{i+i_0} \int_{t_i^{\frac{1}{\alpha}}}^{t_{i+1}^{\frac{1}{\alpha}}} \rho^{-n+\alpha-1+(n-\alpha p)} |E_j| \, \mathrm{d}\rho \lesssim \\
&\lesssim \sum_{i=m-m_1}^{\infty} \sum_{j=m-m_0}^{i+i_0} |E_j| (t_i)^{1-p} \overset{\text{changing order of summation } i\leftrightarrow j}{\leq} \sum_{j=m-m_0}^{\infty} |E_j| \sum_{i=j-i_0}^{\infty} (t_i)^{1-p} \lesssim \\
&\overset{\text{geometric progression}}{\lesssim} \sum_{j=m-m_0}^{\infty} |E_j| (t_{j-i_0})^{1-p}.
\end{aligned} \tag{2.13}$$

□

Lemma 2.7. *The estimate*

$$\sum_{m=-\infty}^{\infty} t_m^p \mu_m \lesssim |E| \tag{2.14}$$

holds.

Proof. We have

$$\sum_{m=-\infty}^{\infty} t_m^p \mu_m \overset{(2.12)}{\lesssim} \sum_{m=-\infty}^{\infty} \sum_{j=m-m_0}^{\infty} |E_j| \big(\tfrac{t_m}{t_{j-i_0}}\big)^{p-1} \lesssim$$

$$\overset{\text{changing order of summation } m\leftrightarrow j}{\le} \sum_{j=-\infty}^{\infty} |E_j| \sum_{m=-\infty}^{j+m_0} \big(\tfrac{t_m}{t_{j-i_0}}\big)^{p-1} \lesssim \tag{2.15}$$

$$\overset{\text{geometric progression}}{\lesssim} \sum_{j=-\infty}^{\infty} |E_j| \big(\tfrac{t_{j+m_0}}{t_{j-i_0}}\big)^{p-1} \overset{\text{definition of } t_j}{=} \sum_{j=-\infty}^{\infty} |E_j| 2^{(m_0+i_0)(p-1)} \lesssim |E|.$$

□

2.2 On approximation of Sobolev–Lorentz mappings

Using the established Theorem 0.2 and Adam's estimate from Theorem 1.2 with $\beta = n-(k-l)p$, we obtain the following estimates, which are key ingredients in the proof of N–property.

Corollary 2.1. *Let $p \in (1,\infty)$, $k,l \in \{1,\dots,n\}$, $l \le k$, $(k-l)p < n$. Then for any function $f \in \mathrm{W}^k_{p,1}(\mathbb{R}^n)$ the estimates*

$$\|\nabla^l f\|^p_{L_p(\mu)} \le C\|\mu\|_\beta \|\nabla^k f\|^p_{\mathrm{L}_{p,1}} \quad \forall \mu \in \mathcal{M}^\beta, \tag{2.16}$$

$$\int_0^\infty \mathcal{H}^\beta_\infty(\{x\in\mathbb{R}^n : \mathcal{M}\big(|\nabla^l f|^p\big)(x) \ge t\})\,\mathrm{d}t \le C\|\nabla^k f\|^p_{\mathrm{L}_{p,1}} \tag{2.17}$$

hold, where $\beta = n-(k-l)p$ and the constant C depends on n,k,p only.

The main result of this subsection is the following

Theorem 2.1. *Let $p \in (1,\infty)$, $k,l \in \{1,\dots,n\}$, $l \le k$, $(k-l)p < n$. Then for any $f \in \mathrm{W}^k_{p,1}(\mathbb{R}^n)$ and for each $\varepsilon > 0$ there exist an open set $U \subset \mathbb{R}^n$ and a function $g \in \mathrm{C}^l(\mathbb{R}^n)$ such that*

(i) $\mathcal{H}^{n-(k-l)p}_\infty(U) < \varepsilon$;

(ii) each point $x \in \mathbb{R}^n\setminus U$ is an L_p-Lebesgue point for $\nabla^j f$, $j=0,\dots,l$;

(iii) $f \equiv g$, $\nabla^j f \equiv \nabla^j g$ on $\mathbb{R}^n \setminus U$ for $j=1,\dots,l$.

Note that in the analogous theorem for the case of Sobolev mappings $f \in \mathrm{W}^k_p(\mathbb{R}^n)$ the assertion (i) should be reformulated as follows:

(i') $\mathcal{B}_{k-l,p}(U) < \varepsilon$ if $l < k$, where $\mathcal{B}_{\alpha,p}(U)$ denotes the Bessel capacity of the set U (see, e.g., Chapter 3 in [45] or [7]).

Recall that for $1 < p < \infty$ and $0 < n - \alpha p < n$ the smallness of $\mathcal{H}^{n-\alpha p}_\infty(U)$ implies the smallness of $\mathcal{B}_{\alpha,p}(U)$, but that the opposite is false since $\mathcal{B}_{\alpha,p}(U) = 0$ whenever $\mathcal{H}^{n-\alpha p}(U) < \infty$. On the other hand, for $1 < p < \infty$ and $0 < n - \alpha p < \beta \leq n$ the smallness of $\mathcal{B}_{\alpha,p}(U)$ implies the smallness of $\mathcal{H}^{\beta}_\infty(U)$ (see, e.g., [4]). So the usual assertion (i') is essentially weaker than (i).

Proof of Theorem 2.1. Let the assumptions of the Theorem be fulfilled. By Theorem 1.1 and Corollary 1.4, we can choose the sequence of mappings $f_i \in C^\infty_0(\mathbb{R}^n)$ such that $\|\nabla^k f - \nabla^k f_i\|_{L_{p,1}(\mathbb{R}^n)} < 4^{-i}$. Denote $\tilde{f}_i = f - f_i$. Then by Corollary 2.1

$$\mathcal{H}^{n-(k-l)p}_\infty\big(\{x \in \mathbb{R}^n : \mathcal{M}\big(|\nabla^l \tilde{f}_i|^p\big)(x) \geq 2^{-i}\}\big) < C\,2^{-i}.$$

Then one could repeat almost word by word the proof of Theorem 3.1 in [10]. Since there are no essential differences, we omit the detailed calculations here. $\square$

2.3 On differentiability properties of Sobolev–Lorentz mappings

We start with the following simple technical observation.

Lemma 2.8 (see, e.g., Lemma 4.1 in [24]). *If $l, k \in \{1, \dots, n\}$, $l < k$, and $v \in \mathrm{W}^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^d)$, then for any $\varepsilon > 0$ there exists an open set $U \subset \mathbb{R}^n$ such that $\mathcal{H}^{lp_\circ}_\infty(U) < \varepsilon$ and the uniform convergence*

$$r^{-l}\|1_{B(x,r)} \cdot \nabla^k v\|_{\mathrm{L}_{p_\circ,1}} \to 0 \qquad \text{as } r \searrow 0$$

holds for $x \in \mathbb{R}^n \setminus U$.

Proof. The proof of the Lemma follows standard arguments, we reproduce it here for reader's convenience. Fix $\sigma > 0$. Let $\{B_\alpha\}$ be a family of disjoint balls $B_\alpha = B(x_\alpha, r_\alpha)$ such that

$$\|1_{B_\alpha} \cdot \nabla^k v\|_{\mathrm{L}_{p_\circ,1}} \geq \sigma r_\alpha^l$$

and $\sup_\alpha r_\alpha < \delta$ for some $\delta > 0$, where δ is chosen small enough to guarantee that $\sup_\alpha \|1_{B_\alpha} \cdot \nabla^k v\|_{\mathrm{L}_{p_\circ,1}} < 1$. Then by Lemma 1.1 we have

$$\sum_\alpha r_\alpha^{lp_\circ} \leq \sigma^{-p_\circ} \sum_\alpha \|1_{B_\alpha} \cdot \nabla^k v\|^{p_\circ}_{\mathrm{L}_{p_\circ,1}} \leq \sigma^{-p_\circ} \|1_{\bigcup_\alpha B_\alpha} \cdot \nabla^k v\|^{p_\circ}_{\mathrm{L}_{p_\circ,1}}. \tag{2.18}$$

Since the last term tends to 0 as $\mathcal{L}^n(\bigcup_\alpha B_\alpha) \to 0$, and $\mathcal{L}^n(\bigcup_\alpha B_\alpha) \leq c\,\delta^{n-lp_\circ} \sum_\alpha r_\alpha^{lp_\circ}$, we get easily that $\sum_\alpha r_\alpha^{lp_\circ} \to 0$ as $\delta \searrow 0$. Using this fact and some standard covering lemmas we infer in a routine manner that for a set

$$A_{\sigma,\delta} := \{x \in \mathbb{R}^n : \exists r \in (0,\delta] \qquad r^{-l}\|1_{B(x,r)} \cdot \nabla^k v\|_{\mathrm{L}_{p_\circ,1}} > \sigma\}$$

the convergence

$$\mathcal{H}^{lp_\circ}_\infty\big(A_{\sigma,\delta}\big) \to 0 \qquad \text{as } \delta \searrow 0$$

holds for any fixed $\sigma > 0$. The rest part of the proof of the lemma is straight forward, so we omit it here. $\square$

From the last lemma (for $l=1$), Theorem 2.1 (ii) and estimate (1.7) we obtain the following result:

Theorem 2.2. *Let $k\in\{1,\dots,n\}$ and $v\in \mathrm{W}^k_{p_\circ,1}(\mathbb{R}^n,\mathbb{R}^d)$. Then there exists a Borel set $A_v\subset\mathbb{R}^n$ such that $\mathcal{H}^{p_\circ}(A_v)=0$ and for any $x\in\mathbb{R}^n\setminus A_v$ the function v is differentiable (in the classical Fréchet sense) at x, furthermore, the classical derivative coincides with $\nabla v(x)$ (x is a Lebesgue point for ∇v).*

The case $k=1$, $p_\circ=n$ is a classical result due to Stein [39] (see also [22]), and for $k=n$, $p_\circ=1$ the result is also proved in [13].

We have the following extension of Theorem 2.2.

Theorem 2.3. *Let $k,l\in\{1,\dots,n\}$, $l\le k$, and $v\in \mathrm{W}^k_{p_\circ,1}(\mathbb{R}^n,\mathbb{R}^d)$. Then there exists a Borel set $A_v\subset\mathbb{R}^n$ such that $\mathcal{H}^{lp_\circ}(A_v)=0$ and for any $x\in\mathbb{R}^n\setminus A_v$ the function v is l-times differentiable (in the classical Fréchet–Peano sense) at x, i.e.,*

$$\lim_{r\searrow 0}\sup_{y\in B(x,r)\setminus\{x\}}\frac{\big|v(y)-T_{v,l,x}(y)\big|}{|x-y|^l}=0,$$

where $T_{v,l,x}(y)$ is the Taylor polynomial of order l for v centered at x (which is well defined $\mathcal{H}^{lp_\circ}$-a.e. by Theorem 2.1).

Proof. We consider only the case $l<n$; for $l=n$ the arguments are similar and becomes even simpler. Below we follow methods of [9, proof of Lemma 5.5] and [10, proof of Theorem 3.1]. By Theorem 2.1 of the present paper, there exists a set A_l such that $\mathcal{H}^{lp_\circ}(A_l)=0$ and the derivatives $\nabla^j v(x)$ are well-defined for all $x\in\mathbb{R}^n\setminus A_l$ and $j=0,1,\dots,l$. Further, by Lemma 2.8 there exists a sequence of open sets $U_i\subset\mathbb{R}^n$ such that $U_i\supset U_{i+1}$, $\mathcal{H}^{lp_\circ}_\infty(U_i)<2^{-i}$ and the *uniform* convergence

$$r^{-l}\|1_{B(x,r)}\cdot\nabla^k v\|_{\mathrm{L}_{p_\circ,1}}\to 0\qquad \text{as } r\searrow 0$$

holds for $x\in\mathbb{R}^n\setminus U_i$. It means that there exists a function $\omega_i\colon(0,+\infty)\to(0,+\infty)$ such that $\omega_i(r)\to 0$ as $r\searrow 0$ and

$$r^{-l}\|1_{B(x,r)}\cdot\nabla^k v\|_{\mathrm{L}_{p_\circ,1}}\le\omega_i(r)\qquad \forall x\in\mathbb{R}^n\setminus U_i. \tag{2.19}$$

Take a sequence of mappings $v_i\colon\mathbb{R}^n\to\mathbb{R}^d$ from Corollary 1.4, i.e., $v_i\in\mathrm{C}^\infty_0(\mathbb{R}^n)$ and $\|\nabla^k(v-v_i)\|_{L_{p_\circ,1}(\mathbb{R}^n)}<4^{-i}$. Denote $\tilde v_i=v-v_i$ and

$$B_i=\big\{x\in\mathbb{R}^n:\mathcal{M}\big(|\nabla^l\tilde v_i|^{p_\circ}\big)(x)\ge 2^{-ip_\circ}\big\},\qquad G_i=A_l\cup U_i\cup\Bigg(\bigcup_{j=i}^\infty B_j\Bigg).$$

Then by estimate (2.17) we have

$$\mathcal{H}^{lp_\circ}_\infty(B_i)\le c2^{-i}, \tag{2.20}$$

therefore,

$$\mathcal{H}^{lp_\circ}_\infty(G_i) \le C2^{-i}. \tag{2.21}$$

By construction,

$$|\nabla^l \tilde{v}_j(x)|^{p_\circ} \le \mathcal{M}\big(|\nabla^l \tilde{v}_j|^{p_\circ}\big)(x) \le 2^{-jp_\circ} \tag{2.22}$$

for all $x \in \mathbb{R}^n \setminus G_i$ and all $j \ge i$. Moreover, since $v_j \in \mathrm{C}_0^\infty(\mathbb{R}^n)$, there exists constants M_j such that $|\nabla^k v_j(x)| \le M_j \ \ \forall x \in \mathbb{R}^n$, this fact and (2.19) implies

$$r^{-l}\|1_{B(x,r)} \cdot \nabla^k \tilde{v}_j\|_{\mathrm{L}_{p_\circ,1}} \le \omega_i(r) + M_j r^{n-l} \qquad \forall x \in \mathbb{R}^n \setminus G_i. \tag{2.23}$$

We start by estimating the remainder term $\tilde{v}_j(y) - T_{\tilde{v}_j,l,x}(y)$. Fix $y \in \mathbb{R}^n$, $x \in \mathbb{R}^n \setminus G_i$, $j \ge i$, and an n–dimensional cubic interval Q such that $x, y \in Q$, $\ |x-y| \sim \ell(Q)$. By construction and Lemma 1.3, for any multi–index α with $|\alpha| \le l$ we have

$$\big|\partial^\alpha \tilde{v}_j(x) - \partial^\alpha P_{Q,l-1}[\tilde{v}_j](x)\big| \le C\ell(Q)^{l-|\alpha|}(\mathcal{M}\nabla^l \tilde{v}_j)(x) \overset{(2.22)}{\le} Cr^{l-|\alpha|}2^{-j}, \tag{2.24}$$

where $r = |x-y|$. Consequently,

$$\begin{aligned}
|\tilde{v}_j(y) - T_{l,\tilde{v}_j,x}(y)| &\le |\tilde{v}_j(y) - P_{Q,l-1}[\tilde{v}_j](y)| + |P_{Q,l-1}[\tilde{v}_j](y) - T_{l,\tilde{v}_j,x}(y)| \\
&\overset{(1.8),\ (2.22),\ (2.23)}{\le} \big[C2^{-j}r^l + \omega_i(r)r^l + M_j r^n\big] \\
&\quad + \sum_{|\alpha|\le l} \frac{1}{\alpha!}\big|\big(\partial^\alpha \tilde{v}_j(x) - \partial^\alpha P_{Q,l-1}[\tilde{v}_j](x)\big)\cdot (y-x)^\alpha\big| \\
&\overset{(2.24)}{\le} \big(C_1 2^{-j} + \omega_i(r) + M_j r^{n-l}\big) r^l.
\end{aligned} \tag{2.25}$$

Finally from the last estimate and equality $v = \tilde{v}_j + v_j$ we have

$$\begin{aligned}
|v(y) - T_{l,v,x}(y)| &\le |\tilde{v}_j(y) - T_{l,\tilde{v}_j,x}(y)| + |v_j(y) - T_{l,v_j,x}(y)| \\
&\le \big(C_1 2^{-j} + \omega_i(r) + M_j r^{n-l}\big) r^l + \omega_{v_j}(r) r^l \\
&= \big(C_1 2^{-j} + \omega_i(r) + M_j r^{n-l} + \omega_{v_j}(r)\big) r^l,
\end{aligned}$$

where $\omega_i(r) \to 0$ and $\omega_{v_j}(r) \to 0$ as $r \to 0$ (the latter holds since $v_j \in \mathrm{C}_0^\infty(\mathbb{R}^n)$). We emphasize that the last inequality is valid for all $y \in \mathbb{R}^n$, $j \ge i$, and $x \in \mathbb{R}^n \setminus G_i$. Therefore

$$\frac{|v(y) - T_{l,v,x}(y)|}{|x-y|^l} \to 0 \qquad \text{as } y \to x$$

uniformly for all $x \in \mathbb{R}^n \setminus G_i$. This means, that v is uniformly l-times differentiable (in the classical Fréchet–Peano sense) at every $x \in \mathbb{R}^n \setminus G_i$. Then the estimate (2.21) finishes the proof. $\square$

2.4 Proof of the N–property

In this subsection we aim to prove the assertion of Theorem 0.1, namely the Luzin N–property for $\mathrm{W}^k_{p_\circ,1}$–mappings with respect to Hausdorff content $\mathcal{H}^{p_\circ}_\infty$ (i.e., when $q = p_\circ = \frac{n}{k}$). Let us for emphasis restate the result:

Theorem 2.4. *Let $k \in \{1,\dots,n\}$, and $v \in \mathrm{W}^k_{p_\circ,1}(\mathbb{R}^n,\mathbb{R}^d)$. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any set $E \subset \mathbb{R}^n$ if $\mathcal{H}^{p_\circ}_\infty(E) < \delta$, then $\mathcal{H}^{p_\circ}_\infty(v(E)) < \varepsilon$. In particular, $\mathcal{H}^{p_\circ}(v(E)) = 0$ whenever $\mathcal{H}^{p_\circ}(E) = 0$.*

Recall that for the case $k = 1$ this assertion was proved in [22], and for $k = n$ it was proved in [10], so we omit these cases. Our proof here for the remaining cases follows and expands on the ideas from [10].
For the remainder of this section we fix $k \in \{2,\dots,n-1\}$, and a mapping v in $\mathrm{W}^k_{p_\circ,1}(\mathbb{R}^n,\mathbb{R}^d)$. To prove Theorem 2.4, we need some preliminary lemmas that we turn to next.

Applying Corollary 2.1 for the case $p = p_\circ = \frac{n}{k}$, $l = 1$, we obtain

$$\|\nabla v\|^{p_\circ}_{L_{p_\circ}(\mu)} \leq C \|\mu\|_{p_\circ} \|\nabla^k v\|^{p_\circ}_{\mathrm{L}_{p_\circ,1}} \quad \forall \mu \in \mathcal{M}^{p_\circ}, \tag{2.26}$$

where C depends on $n, p_\circ, d$ only.
By *a dyadic interval* we understand a cubic interval of the form $[\frac{k_1}{2^m},\frac{k_1+1}{2^m}]\times\dots\times[\frac{k_n}{2^m},\frac{k_n+1}{2^m}]$, where k_i, m are integers. The following assertion is straightforward, and hence we omit its proof here.

Lemma 2.9. *For any n–dimensional cubic interval $J \subset \mathbb{R}^n$ there exist dyadic intervals $Q_1,\dots,Q_{2^n}$ such that $J \subset Q_1 \cup \dots \cup Q_{2^n}$ and $\ell(Q_1) = \dots = \ell(Q_{2^n}) \leq 2\ell(J)$.*

Let $\{Q_\alpha\}_{\alpha\in A}$ be a family of n-dimensional dyadic intervals. We say that the family $\{Q_\alpha\}$ is *regular*, if for any n-dimensional dyadic interval Q the estimate

$$\ell(Q)^{p_\circ} \geq \sum_{\alpha : Q_\alpha \subset Q} \ell(Q_\alpha)^{p_\circ} \tag{2.27}$$

holds. Since dyadic intervals are either nonoverlapping or contained in one another, (2.27) implies that any regular family $\{Q_\alpha\}$ must in particular consist of nonoverlapping intervals.

Lemma 2.10 (see Lemma 2.3 in [10])**.** *Let $\{Q_\alpha\}$ be a family of n–dimensional dyadic intervals. Then there exists a regular family $\{J_\beta\}$ of n–dimensional dyadic intervals such that $\bigcup_\alpha Q_\alpha \subset \bigcup_\beta J_\beta$ and*

$$\sum_\beta \ell(J_\beta)^{p_\circ} \leq \sum_\alpha \ell(Q_\alpha)^{p_\circ}.$$

Lemma 2.11. *For each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, v) > 0$ such that for any regular family $\{Q_\alpha\}$ of n–dimensional dyadic intervals we have if*

$$\sum_\alpha \ell(Q_\alpha)^{p_\circ} < \delta, \tag{2.28}$$

then

$$\sum_\alpha \|1_{Q_\alpha}\cdot\nabla^k v\|^{p_\circ}_{\mathrm{L}_{p_\circ,1}} < \varepsilon \tag{2.29}$$

and

$$\sum_\alpha \frac{1}{\ell(Q_\alpha)^{n-p_\circ}}\int_{Q_\alpha}|\nabla v|^{p_\circ} < \varepsilon. \tag{2.30}$$

Proof. Fix $\varepsilon\in(0,1)$ and let $\{Q_\alpha\}$ be a regular family of n–dimensional dyadic intervals satisfying (2.28), where $\delta>0$ will be specified below.

Let us start by checking (2.29). We have

$$\sum_\alpha \|1_{Q_\alpha}\cdot\nabla^k v\|^{p_\circ}_{\mathrm{L}_{p_\circ,1}} \overset{\text{Lemma 1.1}}{\leq} \|1_{\bigcup_\alpha Q_\alpha}\cdot\nabla^k v\|^{p_\circ}_{\mathrm{L}_{p_\circ,1}}.$$

Using (1.2), we can rewrite the last estimate as

$$\sum_\alpha \|1_{Q_\alpha}\cdot\nabla^k v\|^{p_\circ}_{\mathrm{L}_{p_\circ,1}} \leq \biggl(\int\limits_0^{+\infty}\bigl[\mathcal{L}^n(\{x\in\bigcup_\alpha Q_\alpha:\,|\nabla^k v(x)|>t\})\bigr]^{\frac{1}{p_\circ}}\,\mathrm{d}t\biggr)^{p_\circ}. \tag{2.31}$$

Since

$$\int\limits_0^{+\infty}\bigl[\mathcal{L}^n(\{x\in\mathbb{R}^n:|\nabla^k v(x)|>t\})\bigr]^{\frac{1}{p_\circ}}\,\mathrm{d}t<\infty,$$

it follows that the integral on the right–hand side of (2.31) tends to zero as $\mathcal{L}^n(\bigcup_\alpha Q_\alpha)$ tends to zero. In particular, it will be less than ε if the condition (2.28) is fulfilled with a sufficiently small δ. Thus (2.29) is established for all $\delta\in(0,\delta_1]$, where $\delta_1=\delta_1(\varepsilon,v)>0$.

Next we check (2.30). By virtue of Corollary 1.4, applied coordinate–wise, we can find a decomposition $v=v_0+v_1$, where $\|\nabla v_0\|_{\mathrm{L}^\infty}\leq K=K(\varepsilon,v)$ and

$$\|\nabla^k v_1\|_{\mathrm{L}_{p_\circ,1}}<\varepsilon. \tag{2.32}$$

Assume that $\delta\in(0,\delta_1]$ and

$$\sum_\alpha \ell(Q_\alpha)^{p_\circ}<\delta<\tfrac{1}{K^{p_\circ}+1}\varepsilon. \tag{2.33}$$

Define the measure μ by

$$\mu=\biggl(\sum_\alpha\frac{1}{\ell(Q_\alpha)^{n-p_\circ}}1_{Q_\alpha}\biggr)\mathcal{L}^n, \tag{2.34}$$

where 1_{Q_α} denotes the indicator function of the set Q_α.

Claim. The estimate

$$\sup_J \bigl\{ \ell(J)^{-p_\circ} \mu(J) \bigr\} \le 2^{n+p_\circ} \tag{2.35}$$

holds, where the supremum is taken over all n–dimensional cubic intervals.
Indeed, write for a dyadic interval Q

$$\mu(Q) = \sum_{\alpha : Q_\alpha \subset Q} \ell(Q_\alpha)^{p_\circ} + \sum_{\alpha : Q_\alpha \not\subseteq Q} \frac{\ell(Q \cap Q_\alpha)^n}{\ell(Q_\alpha)^{n-p_\circ}}.$$

By regularity of $\{Q_\alpha\}$ the first sum is bounded above by $\ell(Q)^{p_\circ}$. If the second sum is nonzero then there must exist an index α such that $Q_\alpha \not\subseteq Q$ and Q_α, Q overlap. But as the intervals $\{Q_\alpha\}$ are nonoverlapping and dyadic we must then precisely have one such interval Q_α and $Q_\alpha \supset Q$. But then the first sum is empty and the second sum has only the one term $\ell(Q)^n / \ell(Q_\alpha)^{n-p_\circ}$, hence is at most $\ell(Q)^{p_\circ}$. Thus the estimate $\mu(Q) \le \ell(Q)^{p_\circ}$ holds for every dyadic Q. The inequality (2.35) in the case of a general cubic interval J follows from the above dyadic case and Lemma 2.9. The proof of the claim is complete.

Now return to (2.30). By properties (2.26), (2.32) and (2.33) (applied to the mapping v_1), we have

$$\begin{aligned} \sum_\alpha \frac{1}{\ell(Q_\alpha)^{n-p_\circ}} \int_{Q_\alpha} |\nabla v|^{p_\circ} &\le \frac{2^{p_\circ -1} K^{p_\circ}}{K^{p_\circ}+1} \varepsilon + \sum_\alpha \frac{2^{p_\circ -1}}{\ell(Q_\alpha)^{n-p_\circ}} \int_{Q_\alpha} |\nabla v_1|^{p_\circ} \\ &\le C' \left(\varepsilon + \int |\nabla v_1|^{p_\circ} \, d\mu \right) \le C'' \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the proof of Lemma 2.11 is complete. □

Proof of Theorem 2.4. Fix $\varepsilon > 0$ and take $\delta = \delta(\varepsilon, v)$ from Lemma 2.11. Then by Corollary 1.1 for any regular family $\{Q_\alpha\}$ of n–dimensional dyadic intervals we have if $\sum_\alpha \ell(Q_\alpha)^{p_\circ} < \delta$, then $\sum_\alpha \bigl(\operatorname{diam} v(Q_\alpha)\bigr)^{p_\circ} < C\varepsilon$. Now we may conclude the proof of Theorem 2.4 by use of Lemmas 2.9 and 2.10. Indeed they allow us to find a $\delta_0 > 0$ such that if for a subset E of $\mathbb{R}^n$ we have $\mathcal{H}^{p_\circ}_\infty(E) < \delta_0$, then E can be covered by a regular family $\{Q_\alpha\}$ of n– dimensional dyadic intervals with $\sum_\alpha \ell(Q_\alpha)^{p_\circ} < \delta$. □

Remark 2.1. Note that the order of integrability $p_\circ$ is sharp: for example, the Luzin N–property fails in general for continuous mappings $v \in \mathrm{W}^1_n(\mathbb{R}^n, \mathbb{R}^n)$ (here $k = 1, q = p_\circ = n$), see, e.g., [27].

2.5 Morse–Sard–Dubovitskiĭ–Federer theorem for Sobolev mappings

Let $k, m \in \{1, \ldots, n\}$ and $v \in \mathrm{W}^k_{p_\circ,1,\mathrm{loc}}(\Omega, \mathbb{R}^d)$, where Ω is an open subset of $\mathbb{R}^n$. Then, by Theorem 2.1 (ii), there exists a Borel set A_v such that $\mathcal{H}^{p_\circ}(A_v) = 0$ and all points of the complement $\Omega \setminus A_v$ are $\mathrm{L}_{p_\circ}$-Lebesgue points for the gradient $\nabla v(x)$. Moreover, v is differentiable (in the classical Fréchet sense) at every point of $\Omega \setminus A_v$.

Denote $Z_{v,m} = \{x \in \Omega \setminus A_v : \mathrm{rank}\nabla v(x) < m\}$. The purpose of this section is to prove the assertion of Theorem 0.5:

$$\mathcal{H}^{q_\circ}(v(Z_{v,m})) = 0. \tag{2.36}$$

The exponents occurring in the theorem are the critical exponents that were defined in (0.6):

$$p_\circ = \frac{n}{k} \quad \text{and} \quad q_\circ = m - 1 + \frac{n-m+1}{k}.$$

By an easy calculation, assumptions $n \geq m \geq 1$, $k \geq 1$ imply

$$p_\circ \leq q_\circ \leq n. \tag{2.37}$$

Note that in the double inequality (2.37) we have equality in the first inequality iff $m = 1$ or $k = 1$, while in the second inequality equality holds iff $k = 1$. In particular,

$$p_\circ < q_\circ < n \quad \text{ for } k, m \in \{2, \ldots, n\}.$$

By results obtained in the previous papers [9]–[10], [24] (see commentary to the Theorem 0.5 in the Introduction), we need only consider the case

$$m = 1, \quad q_\circ = p_\circ = \frac{n}{k}.$$

Before embarking on the detailed proof let us make some preliminary observations that will enable us to make some convenient additional assumptions. Namely because the result is local we can without loss in generality assume that $\Omega = \mathbb{R}^n$. For the remainder of the section we fix $k \in \{2, \ldots n\}$ and a mapping $v \in \mathrm{W}^k_{p_\circ,1}(\mathbb{R}^n, \mathbb{R}^d)$. In view of the definition of critical set we have for $m = 1$

$$Z_v = Z_{v,1} = \{x \in \mathbb{R}^n \setminus A_v : \nabla v(x) = 0\}.$$

The following lemma provides the main step in the proof of Theorem 0.5.

Lemma 2.12. *For any n-dimensional dyadic interval $Q \subset \mathbb{R}^n$ the estimate*

$$\mathcal{H}^{p_\circ}_\infty(v(Z_v \cap Q)) \leq C\, \|\nabla^k v\|^{p_\circ}_{\mathrm{L}_{p_\circ,1}(Q)} \tag{2.38}$$

holds, where the constant C depends on n, m, k, d only.

Proof. By virtue of (1.5) it suffices to prove that

$$\mathcal{H}^{p_\circ}_\infty(v(Z_v \cap Q)) \leq C\|\nabla^k v_Q\|^{p_\circ}_{\mathrm{L}_{p_\circ,1}(\mathbb{R}^n)} \tag{2.39}$$

for the mapping v_Q defined in Lemma 1.2, where $C = C(n, m, k, d)$ is a constant. To establish (2.39) it is possible to repeat almost verbatim the proof of Lemma 3.2 in [24]. One must observe the following minor changes: first $q_\circ = p_\circ$, and next, instead of Corollary 1.8 from [24] one must use Corollary 2.1 established above. Note that in the present situation the calculations simplify since for $m = 1$ many of terms from [24, proof of Lemma 3.2] disappear. □

Corollary 2.2. *For any $\varepsilon > 0$ there exists $\delta > 0$ such that for every subset E of $\mathbb{R}^n$ we have $\mathcal{H}^{p_\circ}_\infty(v(Z_v \cap E)) \le \varepsilon$ provided $\mathcal{L}^n(E) \le \delta$. In particular, $\mathcal{H}^{p_\circ}(v(Z_v \cap E)) = 0$ whenever $\mathcal{L}^n(E) = 0$.*

Proof. Let $\mathcal{L}^n(E) \le \delta$, then we can find a family of nonoverlapping n-dimensional dyadic intervals Q_α such that $E \subset \bigcup_\alpha Q_\alpha$ and $\sum\limits_\alpha \ell^n(Q_\alpha) < C\delta$. Of course, for sufficiently small δ the estimate $\|\nabla^k v\|_{\mathrm{L}_{p_\circ,1}(Q_\alpha)} < 1$ is fulfilled for every α. Then in view of Lemma 1.1 we have

$$\sum_\alpha \|\nabla^k v\|^{p_\circ}_{\mathrm{L}_{p_\circ,1}(Q_\alpha)} \le \|\nabla^k v\|^{p_\circ}_{\mathrm{L}_{p_\circ,1}(\bigcup Q_\alpha)} \tag{2.40}$$

This estimate together with Lemma 2.12 allow us to conclude the required smallness of

$$\sum_\alpha \mathcal{H}^{p_\circ}_\infty(Z_v \cap Q_\alpha) \ge \mathcal{H}^{p_\circ}_\infty(Z_v \cap E).$$

□

Invoking Dubovitskiĭ–Federer's Theorem (see commentary to the Theorem 0.5 in the Introduction) for the smooth case $g \in \mathrm{C}^k(\mathbb{R}^n, \mathbb{R}^d)$, Theorem 2.1 (iii) (applied to the case $l = k$) implies

Corollary 2.3 (see, e.g., [12])**.** *There exists a set $\widetilde{Z}_v$ of n-dimensional Lebesgue measure zero such that $\mathcal{H}^{p_\circ}(v(Z_v \setminus \widetilde{Z}_v)) = 0$. In particular, $\mathcal{H}^{p_\circ}(v(Z_v)) = \mathcal{H}^{p_\circ}(v(\widetilde{Z}_v))$.*

From Corollaries 2.2 and 2.3 we conclude that $\mathcal{H}^{p_\circ}(v(Z_v)) = 0$, and this ends the proof of Theorem 0.5.

References

[1] Adams D. R., A note on Choquet integrals with respect to Hausdorff capacity, in "Function Spaces and Applications," Lund 1986, Lecture Notes in Math. **1302**, Springer-Verlag, 1988, pp. 115–124.

[2] Adams D. R., A trace inequality for generalized potentials, Studia Math., 48 (1973), 99-105.

[3] Adams D. R., On the existence of capacitary strong type estimates in $\mathbb{R}^n$, Ark. Mat., 14 (1976), 125-140.

[4] Aikawa H., Bessel capacity, Hausdorff content and the tangential boundary behavior of harmonic functions, Hiroshima Math. J., **26** (1996), no. 2, 363–384.

[5] Bates S.M., Toward a precise smoothness hypothesis in Sard's theorem, Proc. Amer. Math. Soc. **117** (1993), no. 1, 279–283.

[6] Bojarski B., Hajlasz P., and Strzelecki P., Sard's theorem for mappings in Hölder and Sobolev spaces, Manuscripta Math., **118**, (2005), 383–397.

[7] Bojarski B., Hajlasz P., Strzelecki P., Improved $C^{k,\lambda}$ approximation of higher order Sobolev functions in norm and capacity, Indiana Univ. Math. J. **51** (2002), no. 3, 507–540.

[8] Bojarski B., Hajlasz P., Pointwise inequalities for Sobolev functions and some applications, Studia Math. **106** (1993), 77–92.

[9] Bourgain J., Korobkov M.V., Kristensen J., On the Morse– Sard property and level sets of Sobolev and BV functions, Rev. Mat. Iberoam. **29** (2013), no. 1, 1–23.

[10] Bourgain J., Korobkov M.V., Kristensen J., On the Morse–Sard property and level sets of $\mathrm{W}^{n,1}$ Sobolev functions on $\mathbb{R}^n$, Journal fur die reine und angewandte Mathematik (Crelles Journal), **2015** (2015), no. 700, 93–112. (Online first 2013), http://dx.doi.org/10.1515/crelle-2013-0002

[11] Bucur D., Giacomini A., and Trebeschi P., Whitney property in two dimensional Sobolev spaces, Proc. Amer. Math. Soc. **136** (2008), no. 7, 2535–2545.

[12] De Pascale L., The Morse–Sard theorem in Sobolev spaces, Indiana Univ. Math. J. **50** (2001), 1371–1386.

[13] Dorronsoro J.R., Differentiability properties of functions with bounded variation, Indiana Univ. Math. J. **38** (1989), no. 4, 1027–1045.

[14] Dubovitskiĭ A.Ya., Structure of level sets for differentiable mappings of an n-dimensional cube into a k-dimensional cube (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **21** (1957), no. 3, 371–408.

[15] Dubovitskiĭ A.Ya., On the set of dengenerate points, (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **31** (1967), no. 1, 27–36. English Transl.: Math. USSR Izv. **1**, no. 1 (1967), 25–33. http://dx.doi.org/10.1070/IM1967v001n01ABEH000545

[16] Evans L.C., Gariepy R.F., *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[17] Federer H., Two theorems in geometric measure theory, Bull. Amer. Math. Soc. **72** (1966), P. 719.

[18] Federer H., *Geometric Measure Theory*, Springer- Verlag, New York; Heidelberg; Berlin (1969).

[19] Feffermann C.L., The Uncertainty Principle, Bull. Amer. Math. Soc. (N.S.) **9** (1983), no. 2, 129–206.

[20] Figalli A., A simple proof of the Morse–Sard theorem in Sobolev spaces, Proc. Amer. Math. Soc. **136** (2008), 3675–3681.

[21] Hayman W.K., Kennedy P.B. *Subharmonic functions,* Academic Press Inc., London (1976).

[22] Kauhanen J., Koskela P., and Maly J., On functions with derivatives in a Lorentz space, Manuscripta Math., **100** (1999), no. 1, 87–101.

[23] Kerman R. and Sawyer Eric T., The trace inequality and eigenvalue estimates for Schrödinger operators, Annales de l'institut Fourier, **36** (1986), no. 4, 207–228.

[24] Korobkov M. V. and Kristensen J., On the Morse-Sard Theorem for the sharp case of Sobolev mappings, Indiana Univ. Math. J., **63** (2014), no. 6, 1703–1724. http://dx.doi.org/10.1512/iumj.2014.63.5431

[25] Korobkov M. V., Pileckas K., Russo R., Solution of Leray's problem for stationary Navier-Stokes equations in plane and axially symmetric spatial domains, Ann. of Math., **181** (2015), no. 2, 769–807. http://dx.doi.org/10.4007/annals.2015.181.2.7

[26] Landis E.M., On functions representative as the difference of two convex functions (Russian), Doklady Akad. Nauk SSSR (N.S.) **80** (1951), 9–11.

[27] Maly J. and Martio O., Luzin's condition N and mappings of the class $W^{1,n}$, J. Reine Angew. Math. **458** (1995), 19–36.

[28] Maly J., Advanced theory of differentiation – Lorentz spaces, March 2003 http://www.karlin.mff.cuni.cz/∼maly/lorentz.pdf.

[29] Maly J., Coarea properties of Sobolev functions, Function Spaces, Differential Operators and Nonlinear Analysis, The Hans Triebel Anniversary Volume, Birkhäuser Verlag, Basel, 2003, 371–381.

[30] Maz'ya V.G., *Sobolev Spaces.* Springer-Verlag, 1985.

[31] Maz'ya Vl.G. and Shaposhnikova T.O., *Theory of Sobolev multipliers. With applications to differential and integral operators*, Grundlehren der Mathematischen Wissenschaft 337, Berlin-Heidelberg-New York: Springer-Verlag, 2009, pp. xiii+609.

[32] C.G.T. De A. Moreira, Hausdorff measures and the Morse–Sard theorem, Publ. Mat. **45** (2001), 149–162.

[33] Morse A.P., The behavior of a function on its critical set, Ann. of Math. **40** (1939), 62–70.

[34] Norton A., A critical set with nonnull image has large Hausdorff dimension, Trans. Amer. Math. Soc. **296** (1986), 367–376.

[35] Pavlica D., Zajíček L., Morse–Sard theorem for d.c. functions and mappings on $\mathbb{R}^2$, Indiana Univ. Math. J. **55** (2006), no. 3, 1195–1207.

[36] Romanov A.S., Absolute continuity of the Sobolev type functions on metric spaces, Siberian Math. J., **49** (2008), no. 5, 911–918.

[37] Sard A., The measure of the critical values of differentiable maps, Bull. Amer. Math. Soc., **48** (1942), 883–890.

[38] Saks S., *Theory of the Integral.* Dover Books on Mathematics, 2005 (first published in 1937).

[39] Stein E. M., *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J. 1970.

[40] Stein E. M. and Weiss G., *Introduction to Fourier Analysis on Eucledian Spaces.* Princeton Mathematical Series, No. 32. Princeton University Press, Princeton, N.J. 1971.

[41] Van der Putten R.,, The Morse-Sard theorem for Sobolev spaces in a borderline case, Bull. Sci. Math. **136** (2012), no. 4, 463–475.

[42] Van der Putten R., The Morse-Sard theorem in $W^{n,n}(\Omega)$: a simple proof, Bull. Sci. Math. **136** (2012), 477–483.

[43] Whitney H., A function not constant on a connected set of critical points, Duke Math. J. **1** (1935), 514–517.

[44] Yomdin Y., The geometry of critical and near-critical values of differentiable mappings, Math. Ann. **264** (1983), no. 4, 495–515.

[45] Ziemer W.P., *Weakly Differentiable Functions. Sobolev Spaces and Functions of Bounded Variation*, Graduate Texts in Math. **120**, Springer-Verlag, New York, 1989.

Sobolev Institute of Mathematics, Acad. Koptyuga pr., 4, and Novosibirsk State University, Pirogova Str. 2, 630090 Novosibirsk, Russia
e-mail: *korob@math.nsc.ru*

Mathematical Institute, University of Oxford, Oxford OX2 6GG, England
e-mail: *kristens@maths.ox.ac.uk*