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TRANSONIC FLOWS WITH SHOCKS PAST CURVED WEDGES FOR THE FULL EULER EQUATIONS

Dedicated to Peter Lax on the occasion of his 90th birthday

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Abstract. We establish the existence, stability, and asymptotic behavior of transonic flows with a transonic shock past a curved wedge for the steady full Euler equations in an important physical regime, which form a nonlinear system of mixed-composite hyperbolic-elliptic type. To achieve this, we first employ the transformation from Eulerian to Lagrangian coordinates and then exploit one of the new equations to identify a potential function in Lagrangian coordinates. By capturing the conservation properties of the system, we derive a single second-order nonlinear elliptic equation for the potential function in the subsonic region so that the transonic shock problem is reformulated as a one-phase free boundary problem for the nonlinear equation with the shock-front as a free boundary. One of the advantages of this approach is that, given the shock location or equivalently the entropy function along the shock-front downstream, all the physical variables can be expressed as functions of the gradient of the potential function, and the downstream asymptotic behavior of the potential function at infinity can be uniquely determined with a uniform decay rate. To solve the free boundary problem, we employ the hodograph transformation to transfer the free boundary to a fixed boundary, while keeping the ellipticity of the nonlinear equation, and then update the entropy function to prove that the updating map has a fixed point. Another advantage in our analysis is in the context of the full Euler equations so that the Bernoulli constant is allowed to change for different fluid trajectories.

1. **Introduction.** We are concerned with the existence, stability, and asymptotic behavior of steady transonic flows with transonic shocks past curved wedges for the

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full Euler equations. The two-dimensional steady, full Euler equations for polytropic gases have the form (*cf.* [13, 14, 24]):

$$\begin{cases} \nabla\cdot(\rho\mathbf{u})=0,\\ \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u})+\nabla p=0,\\ \nabla\cdot\big(\rho\mathbf{u}(E+\frac{p}{\rho})\big)=0, \end{cases} \tag{1.1}$$

where $\nabla=\nabla_{\mathbf{x}}$ is the gradient in $\mathbf{x}=(x_1,x_2)\in\mathbb{R}^2$, $\mathbf{u}=(u_1,u_2)$ the velocity, ρ the density, p the pressure, and

$$E=\frac{1}{2}|\mathbf{u}|^2+\frac{p}{(\gamma-1)\rho}$$

the total energy with adiabatic exponent $\gamma>1$. The sonic speed of the flow is

$$c=\sqrt{\frac{\gamma p}{\rho}}.$$

The flow is subsonic if $|\mathbf{u}|<c$ and supersonic if $|\mathbf{u}|>c$. For a transonic flow, both cases occur in the flow, and then system (1.1) is of mixed-composite hyperbolic-elliptic type, which consists of two equations of mixed elliptic-hyperbolic type and two equations of hyperbolic type.

System (1.1) is a prototype of general nonlinear systems of conservation laws:

$$\nabla_{\mathbf{x}}\cdot\mathbf{F}(U)=0,\qquad \mathbf{x}\in\mathbb{R}^n, \tag{1.2}$$

where $U:\mathbb{R}^n\to\mathbb{R}^m$ is unknown, while $\mathbf{F}:\mathbb{R}^m\to\mathbb{M}^{m\times n}$ is a given nonlinear mapping for the $m\times n$ matrix space $\mathbb{M}^{m\times n}$. For (1.1), we may choose $U=(\mathbf{u},p,\rho)$. The systems with form (1.2) often govern time-independent solutions for multidimensional quasilinear hyperbolic systems of conservation laws; *cf.* Lax [19, 20].

It is well known that, for a steady, upstream uniform supersonic flow past a straight-sided wedge whose vertex angle is less than the critical angle, there exists a shock-front emanating from the wedge vertex so that the downstream state is either subsonic or supersonic, depending on the downstream asymptotic condition at infinity (see Appendix B and Fig. 1 for the shock polar). The study of two-dimensional steady uniform supersonic flows past a straight-side wedge can date back to the 1940s (*cf.* Courant-Friedrichs [14]; also see Prandtl [25] and von Neumann [27]).

For the case of supersonic-supersonic shock (*i.e.* both states of the shock are supersonic), local solutions around the curved wedge vertex were first constructed by Gu [18], Li [22], Schaeffer [26], and the references cited therein. Global potential solutions are constructed in [5, 6, 14, 28] when the wedge has certain convexity or the wedge is a small perturbation of the straight-sided wedge with fast decay in the flow direction. In Chen-Zhang-Zhu [4], two-dimensional steady supersonic flows governed by the full Euler equations past Lipschitz wedges were systematically analyzed, and the existence and stability of supersonic Euler flows were established via a modified Glimm difference scheme (*cf.* [17]), when the total variation of the tangent angle functions along the wedge boundaries is suitably small.

For the case of supersonic-subsonic shock (*i.e.* transonic shock-front), the stability of these fronts under a perturbation of the upstream flow, or a perturbation of wedge boundary, has been studied in Chen-Fang [10] for the potential flow and in Fang [15] for the Euler flow with a uniform Bernoulli constant. In particular, the stability of transonic shocks in the steady Euler flows with a uniform Bernoulli constant was first established in the weighted Sobolev norms in Fang [15], even though the downstream asymptotic decay rate of the shock slope at infinity was not derived.

In this paper, one of our main objectives is to deal with the asymptotic behavior of steady transonic flows with a transonic shock past a curved wedge for the full Euler equations, especially the uniform decay rate of the transonic shock slope and the subsonic flows downstream at infinity. For a fixed uniform supersonic state U_0^-, there is an arc on the shock polar corresponding to the subsonic states; see Fig. 2.1. When the wedge angle is less than the critical angle $\theta_{\rm w}^{\rm c}$, the tangential point T corresponding to the critical angle divides arc $\widehat{HS}$ into the two open arcs $\widehat{TS}$ and $\widehat{TH}$. The nature of these two cases is very different. In this paper, we focus mainly on the stability of transonic shocks in the important physical regime $\widehat{TS}$ when the wedge angle is between the sonic angle $\theta_{\rm w}^{\rm s}$ and the critical angle $\theta_{\rm w}^{\rm c} > \theta_{\rm w}^{\rm s}$.

To achieve this, we first rewrite the problem in Lagrangian coordinates so that the original streamlines in Eulerian coordinates become straight lines and the curved wedge boundary in Eulerian coordinates becomes a horizontal ray in Lagrangian coordinates. Then we exploit one of the new equations to identify a potential function ϕ in Lagrangian coordinates. By capturing the conservation properties of the Euler system, we derive a single second-order nonlinear elliptic equation for the potential function ϕ in the subsonic region as in [1], so that the original transonic shock problem is reformulated as a one-phase free boundary problem for a second-order nonlinear elliptic equation with the shock-front as a free boundary. One of the advantages of this approach is that, given the location of the shock-front, or equivalently the entropy function A (which is constant along the fluid trajectories) along the shock-front downstream, all the physical variables $U = (\mathbf{u}, p, \rho)$ can be expressed as functions of the gradient of ϕ, and the asymptotic behavior ϕ^∞ of the potential ϕ at the infinite exit can be uniquely determined.

To solve the free boundary problem, we have to determine the free boundary, and both the subsonic phase and entropy function defined in the downstream domain with the free boundary as a part of its boundary. We approach this problem by employing the hodograph transformation to transfer the free boundary to a fixed boundary, while keeping the ellipticity of the second order partial differential equations, and then by updating the entropy function A to prove that the updating map for A has a fixed point.

For given entropy function A, we first determine *a priori* the limit function of the potential function downstream at infinity. Then we solve the second-order elliptic equations for the potential function in the unbounded domain with the fixed boundary conditions and the downstream asymptotic condition at infinity. This is achieved through the fixed point argument by designing an appropriate map. In order to define this map, we first linearize the second-order elliptic equation for the identified potential function ϕ based on the limit function ϕ^∞ of ϕ, solve the linearized problem in the fixed region, and then make delicate estimates of the solutions, especially the corner singularity near the intersection between the fixed shock-front and the wedge boundary. Finally, these estimates allow us to prove that the map has a fixed point that is the subsonic solution in the downstream domain. Finally, we prove that the entropy function A is a fixed point via the implicit function theorem.

Since the transformation between the Eulerian and Lagrangian coordinates is invertible, we obtain the existence and uniqueness of solutions of the wedge problem in Eulerian coordinates by transforming back the solutions in Lagrangian coordinates, which is the real subsonic phase for the free boundary problem. The asymptotic behavior of solutions at the infinite exit is also clarified. The stability of transonic shocks and corresponding transonic flows is also established by both employing the transformation from Eulerian to Lagrangian coordinates and developing careful, detailed estimates of the solutions.

Another advantage in our analysis here is in the context of the real full Euler equations so that the solutions do not necessarily obey Bernoulli's law with a *uniform* Bernoulli constant, *i.e.*, the Bernoulli constant is allowed to change for different fluid trajectories (in comparison with the setup in [8, 9, 12, 15]).

By the closeness assumption of solution U to the uniform flow in the subsonic region, we obtain the asymptotic behavior of U as $y_1 \to \infty$. The asymptotic state $U^\infty = (\mathbf{u}^\infty, p^\infty, \rho^\infty)$ is uniquely determined by state U^- of the incoming flow and the wedge angle at infinity.

We remark that, when U_0^+ is on arc $\widehat{TH}$ (see Fig. 2.1 below), the nature of the oblique boundary condition near the origin is significantly different from the case when U_0^+ is on arc $\widehat{TS}$. Such a difference affects the regularity of solutions at the origin in general. It requires a further understanding of global features of the problem, especially the global relation between the regularity near the origin and the decay of solutions at infinity, to ensure the existence of a $C^{1,\alpha}$ solution. A different approach may be required to handle this case, which is currently under investigation.

The organization of this paper is as follows: In §2, we first formulate the wedge problem into a free boundary problem and state the main theorem.

In §3, we reduce the Euler system into a second-order nonlinear elliptic equation in the subsonic region and then reformulate the wedge problem into a one-phase free boundary problem for the second-order nonlinear elliptic equation with the shock-front as a free boundary.

In §4, we use the hodograph transformation to make the free boundary into a fixed boundary, in order to reduce the difficulty of the free boundary. After that, we only need to solve for the unknown entropy function A as a fixed point.

In §5, for a given entropy function A, we solve the reformed fixed boundary value problem in the unbounded domain and determine *a priori* the downstream asymptotic function of the potential function at infinity. Then, in §6, we prove that the entropy function A is a fixed point via the implicit function theorem, which is one of the novel ingredients in this paper.

In §7, we determine the decay of the solution to the asymptotic state in the physical coordinates.

In §8, we establish the stability of the transonic solutions and transonic shocks under small perturbations of the incoming flows and wedge boundaries. We finally give some remarks for the problem when the downstream state of the background solution is on arc $\widehat{TH}$ in §9. In Appendices, we show two comparison principles and derive a criterion for different arcs $\widehat{TS}$ and $\widehat{TH}$ on the shock polar, which are employed in the earlier sections.

Finally, we remark that the stability of conical shock-fronts in three-dimensional flow has also been studied in the recent years. The stability of conical supersonic-supersonic shock-fronts has been studied in Liu-Lien [23] in the class of BV solutions when the cone vertex angle is small, and Chen [7] and Chen-Xin-Yin [11] in the class of smooth solutions away from the conical shock-front when the perturbed cone is sufficiently close to the straight-sided cone. The stability of three-dimensional conical transonic shock-fronts in potential flow has been established in Chen-Fang [2] with respect to the conical perturbation of the cone boundary and the upstream flow in appropriate function spaces. Also see Chen-Feldman [3].

2. **Mathematical Setup and the Main Theorem.** In this section, we first formulate the wedge problem into a free boundary problem for the composite-mixed Euler equations, and state the main theorem.

As is well-known, for a uniform horizontal incoming flow $U_0^- = (u_{10}^-, 0, p_0^-, \rho_0^-)$ past a straight wedge with half-wedge angle θ_0, the downstream constant flow can be determined by the Rankine-Hugoniot conditions, that is, the shock polar (see Appendix B and Fig. 1). According to the shock polar, the two flow angles are important: One is the critical angle $\theta_{\rm w}^{\rm c}$ that ensures the existence of the attached shocks at the wedge vertex, and the other is the sonic angle $\theta_{\rm w}^{\rm s}$ for which the downstream fluid velocity at the sonic speed in the direction. When the straight wedge angle $\theta_{\rm w}$ is between $\theta_{\rm w}^{\rm s}$ and $\theta_{\rm w}^{\rm c}$, there are two subsonic solutions; while the wedge angle $\theta_{\rm w}$ is smaller than $\theta_{\rm w}^{\rm s}$, there are one subsonic solution and one supersonic solution. We focus on the subsonic constant state $U_0^+ = (\mathbf{u}_0^+, p_0^+, \rho_0^+)$ with $\mathbf{u}_0^+ \cdot (\sin\theta_0, -\cos\theta_0) = 0$. Then the transonic shock-front $\mathcal{S}_0$ is also straight, described by $x_1 = s_0(x_2) \equiv s_0 x_2$. The question is whether the transonic shock solution is stable under a perturbation of the incoming supersonic flow and the wedge boundary.

Assume that the perturbed incoming flow U^- is close to U_0^-, which is supersonic and almost horizontal, and the wedge is closed to a straight wedge. Then, for any suitable wedge angle (smaller than a critical angle), it is expected that there should be a shock-front which is attached to the wedge vertex. If we impose the subsonicity condition in the far field downstream after the shock-front, then the flow U between the shock-front and the wedge should be subsonic. Since the upper and lower subsonic regions do not interact with each other, it suffices to study the upper part.

We now use a function $b(x_1)$ to describe the wedge boundary:

$$\partial\mathcal{W} = \{\mathbf{x} \in \mathbb{R}^2 \,:\, x_2 = b(x_1),\ b(0) = 0\}. \tag{2.1}$$

Along the wedge boundary $\partial\mathcal{W}$, the slip condition is naturally prescribed:

$$\frac{u_2}{u_1}\bigg|_{\partial\mathcal{W}} = b'(x_1). \tag{2.2}$$

Let the shock-front $\mathcal{S}$ be $x_1 = \sigma(x_2)$ with $\sigma(0) = 0$. Then the domain for the subsonic flow is denoted by

$$\Omega_{\mathcal{S}} = \{\mathbf{x} \in \mathbb{R}^2 \,:\, x_1 > \sigma(x_2),\, x_2 > b(x_1)\}, \tag{2.3}$$

and the shock-front $\mathcal{S}$ becomes a free boundary connecting the subsonic flow (elliptic) with the supersonic flow (hyperbolic).

To be a weak solution of the Euler equations (1.1), the Rankine-Hugoniot conditions should be satisfied along the shock-front:

$$\begin{cases} [\,\rho u_1\,] = \sigma'(x_2)[\,\rho u_2\,], \\ [\,\rho u_1^2 + p\,] = \sigma'(x_2)[\,\rho u_1 u_2\,], \\ [\,\rho u_1 u_2\,] = \sigma'(x_2)[\,\rho {u_2}^2 + p\,], \\ [\,\rho u_1(E + \frac{p}{\rho})\,] = \sigma'(x_2)[\,\rho u_2(E + \frac{p}{\rho})\,], \end{cases} \tag{2.4}$$

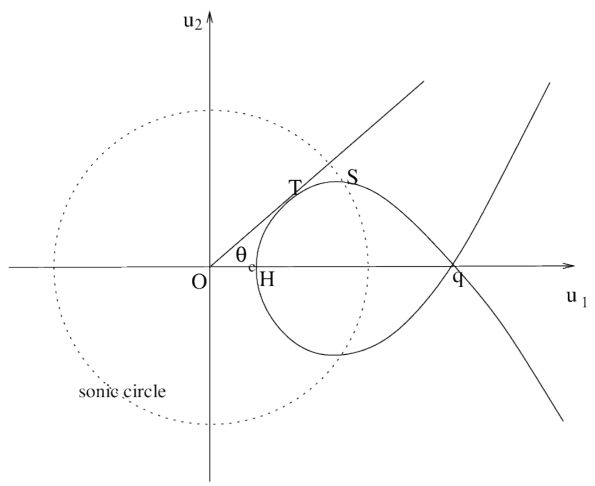


FIGURE 2.1. Two arcs $\widehat{TS}$ and $\widehat{TH}$ on the shock polar

as the free boundary conditions on $\mathcal{S}$, where $[\cdot]$ denotes the jump between the quantity of two states across the shock-front.

For a fixed uniform supersonic state U_0^-, there is an arc on the shock polar corresponding to the subsonic states. When the wedge angle is less than the critical angle $\theta_{\rm w}^{\rm c} > \theta_{\rm w}^{\rm s}$, the tangential point T corresponding to the critical angle divides arc $\widehat{HS}$ into the two open arcs $\widehat{TS}$ and $\widehat{TH}$. The nature of these two cases is very different.

In this paper, we analyze the existence, stability, and asymptotic behavior of steady transonic flows with a transonic shock in the important regime $\widehat{TS}$ for the wedge angle $\theta_{\rm w}$. To state our results, we need the following weighed Hölder norms: For any $\mathbf{x}, \mathbf{x}'$ in a two-dimensional domain E and for a subset P of ∂E, define

$$\delta_{\mathbf{x}} := \min\left\{\frac{\mathrm{dist}(\mathbf{x},P)}{|\mathbf{x}|+1}, 1\right\}, \qquad \delta_{\mathbf{x},\mathbf{x}'} := \min\{\delta_{\mathbf{x}}, \delta_{\mathbf{x}'}\},$$
$$\Delta_{\mathbf{x}} := |\mathbf{x}|+1, \qquad \Delta_{\mathbf{x},\mathbf{x}'} := \min\{|\mathbf{x}|+1, |\mathbf{x}'|+1\}.$$

Let $\alpha \in (0,1)$, $\sigma, \tau \in \mathbb{R}$, and let k be a nonnegative integer. Let $\mathbf{k} = (k_1, k_2)$ be an integer-valued vector, where $k_1, k_2 \geq 0$, $|\mathbf{k}| = k_1 + k_2$, and $D^{\mathbf{k}} = \partial_{x_1}^{k_1}\partial_{x_2}^{k_2}$. We define

$$[f]_{k,0;(\tau);E}^{(\sigma);P} := \sup_{\substack{\mathbf{x}\in E\\ |\mathbf{k}|=k}} \left(\delta_{\mathbf{x}}^{\max\{k+\sigma,0\}}\Delta_{\mathbf{x}}^{\tau+k}|D^{\mathbf{k}}f(\mathbf{x})|\right),$$

$$[f]_{k,\alpha;(\tau);E}^{(\sigma);P} := \sup_{\substack{\mathbf{x},\mathbf{x}'\in E\\ \mathbf{x}\neq\mathbf{x}',\, |\mathbf{k}|=k}} \left(\delta_{\mathbf{x},\mathbf{x}'}^{\max\{k+\alpha+\sigma,0\}}\Delta_{\mathbf{x},\mathbf{x}'}^{\tau+k+\alpha}\frac{|D^{\mathbf{k}}f(\mathbf{x})-D^{\mathbf{k}}f(\mathbf{x}')|}{|\mathbf{x}-\mathbf{x}'|^{\alpha}}\right),$$

$$\|f\|_{k,\alpha;(\tau);E}^{(\sigma);P} := \sum_{i=0}^{k}[f]_{i,0;(\tau);E}^{(\sigma);P} + [f]_{k,\alpha;(\tau);E}^{(\sigma);P}. \tag{2.5}$$

For a vector-valued function $\mathbf{f} = (f_1, f_2, \cdots, f_n)$, we define

$$\|\mathbf{f}\|_{k,\alpha;(\tau);E}^{(\sigma);P} := \sum_{i=1}^{n}\|f_i\|_{k,\alpha;(\tau);E}^{(\sigma);P}.$$

For a function of one variable defined on $(0,\infty)$, we define the Hölder norms with a weight at infinity. The definition above can be reduced to one-dimensional if we keep only the weights at infinity. Then the notation becomes $\|f\|_{k,\alpha;(\tau);(0,\infty)}$.

We also need the norms with weights at infinity which apply only for the derivatives:

$$\|f\|^{*,(\sigma);P}_{k,\alpha;(\tau);E} := \|f\|_{C^0(E)} + \|Df\|^{(\sigma+1);P}_{k-1,\alpha;(\tau+1);E}. \tag{2.6}$$

Similarly, the Hölder norms for a function of one variable on $(0,\infty)$ with only the weights at infinity are denoted by $\|f\|^{*}_{k,\alpha;(\tau);(0,\infty)}$.

In terms of supersonic flows, we prescribe the initial data:

$$U|_{\mathcal{I}} = U_0(x_2) \qquad \text{on } \ \mathcal{I} := \{x_1 = 0\}.$$

Let Ω^- be the domain for the incoming flows defined by

$$\Omega^- = \{\mathbf{x} : 0 < x_1 < 2s_0x_2\}\,. \tag{2.7}$$

For a given shock $\mathcal{S} = \{x_1 = \sigma(x_2)\}$, let

$$\Omega^-_{\mathcal{S}} = \{\mathbf{x} : 0 < x_1 < \sigma(x_2)\}\,. \tag{2.8}$$

We now fix parameters $\alpha, \beta \in (0,1)$ with suitably small β, depending on the background states.

Then we can conclude that there is $\varepsilon > 0$, depending on the background states, such that, when

$$\|U_0 - U_0^-\|_{2,\alpha;(1+\beta);\mathcal{I}} < \varepsilon \qquad \text{for some } \beta > 0, \tag{2.9}$$

there exists a constant $C_0 > 0$, independent of ε, and a unique supersonic solution $U^- = (\mathbf{u}^-, p^-, \rho^-)(x,y)$ of system (1.1) with the initial condition $U^-|_{\mathcal{I}} = U_0$, well defined on Ω^-, such that

$$\|U^- - U_0^-\|_{2,\alpha;(1+\beta);\Omega^-} \le C_0\|U_0 - U_0^-\|_{2,\alpha;(1+\beta);\mathcal{I}}. \tag{2.10}$$

This can be achieved by rewriting the problem as an initial-boundary value problem in the polar coordinates (r,θ) so that system (1.1) is still a hyperbolic system, domain Ω^- becomes a half strip with θ time-like and r space-like, the initial data is on $\{r > 0, \theta = 0\}$, and the boundary data $v = 0$ is on the characteristic line $\{r = 0, 0 \le \theta \le \arctan(2s_0)\}$. This is a standard initial-boundary value problem whose almost-global existence of solutions can be obtained as long as ε is sufficiently small.

Assume that the wedge boundary satisfies

$$\|b - b_0\|^*_{1,\alpha;(\beta);\mathbb{R}_+} < \varepsilon. \tag{2.11}$$

Theorem 2.1 (Main Theorem)**.** *Let the background solution $\{U_0^-, U_0^+\}$ satisfy that U_0^+ is on arc $\widehat{TS}$ in Fig. 2.1 for the straight wedge boundary $x_2 = b_0(x_1) = \tan\theta_0\, x_1, x_1 > 0$. Then there is $\varepsilon > 0$ such that, when the initial data U_0 and the wedge boundary $\partial\mathcal{W} = \{x_2 = b(x_1), b(0) = 0\}$ satisfy (2.9) and (2.11) respectively, there exist a strong transonic shock $\mathcal{S} := \{x_1 = \sigma(x_2)\}$, a transonic solution $\{U^-, U\}$ of the Euler equations (1.1) in $\Omega_{\mathcal{S}}$, and an asymptotic downstream state $U^\infty = (\mathbf{u}^\infty, p_0^+, \rho^\infty) = V^\infty(x_2 - \tan\theta_0\, x_1)$ for an appropriate function $V^\infty : \mathbb{R}^+ \to \mathbb{R}^4$ with $\mathbf{u}^\infty \cdot (\sin\theta_0, -\cos\theta_0) = 0$ for the wedge angle θ_0 such that*

(i) *U^- is a supersonic flow in $\Omega^-_{\mathcal{S}}$, and U is a subsonic solution in $\Omega_{\mathcal{S}}$;*

(ii) *The Rankine-Hugoniot conditions (2.4) hold along the shock-front $\mathcal{S}$;*

(iii) *The slip condition (2.2) holds along the wedge boundary $\partial\mathcal{W}$;*

(iv) *The following estimates hold:*

$$\begin{aligned}&\|U^- - U_0^-\|_{2,\alpha;(1+\beta);\Omega_{\mathcal{S}}^-} + \|U - U^\infty\|_{1,\alpha;(1+\beta);\Omega_{\mathcal{S}}}^{(-\alpha);\partial\mathcal{W}} \\ &\quad + \|s - s_0\|_{2,\alpha;(\beta);\mathbb{R}_+}^{*,(-1-\alpha);\{0\}} + \|V^\infty - U_0^+\|_{1,\alpha;(1+\beta);\mathbb{R}_+}^{(-\alpha);\{0\}} \\ &\le C\left(\|U_0 - U_0^-\|_{2,\alpha;(1+\beta);\mathcal{I}} + \|b - b_0\|_{1,\alpha;(\beta);\mathbb{R}_+}^{*}\right),\end{aligned} \tag{2.12}$$

where C is a constant depending only on $U_\pm^0$, but independent of ε.

Moreover, the solution U is unique within the class of transonic solutions such that the left-hand side of estimate (2.12) is less than $C\varepsilon$.

Remark 1. Estimate (2.12) implies that the downstream flow and the transonic shock-front are close to the background transonic solution with downstream decaying to U^∞ at rate $|\mathbf{x}|^{-1-\beta}$ near infinity. Thus, the transonic shock-front is conditionally stable with respect to the perturbation of the wedge boundary and the upstream flow. In particular, it is clear that the slope of the shock-front tends asymptotically to the slope of the unperturbed shock-front and the subsonic flow downstream tends asymptotically to U^∞ at a uniform decay rate $|\mathbf{x}|^{-1-\beta}$.

Remark 2. Theorem 2.1 indicates that the asymptotic downstream state U^∞ generally is not a uniform constant state. If the $\mathbf{x}$-coordinates are rotated with angle θ_0 into the new coordinates $(\hat{x}_1, \hat{x}_2)$ so that the unperturbed wedge boundary $\partial\mathcal{W}_0 = \{x_2 - \tan\theta_0\, x_1\}$ becomes the $\hat{x}_1$-axis:

$$(\hat{x}_1, \hat{x}_2) = (\cos\theta_0\, x_1 + \sin\theta_0\, x_2, -\sin\theta_0\, x_1 + \cos\theta_0\, x_2),$$

then $V^\infty = V^\infty(\hat{x}_2)$. In Lagrangian coordinates, $\mathbf{y} = (y_1, y_2)$, determined by (3.1) in §3, the asymptotic downstream state is a function of y_2 in general: $U^\infty = U^\infty(y_2) = (\mathbf{u}^\infty(y_2), p_0^+, \rho^\infty(y_2))$. However, our argument also shows that, in the isentropic case with a constant Bernoulli quantity B (see (3.14)), the asymptotic state must be uniform and equal to the background state. Also see Chen-Chen-Feldman [1].

3. **Reduction of the Euler System and Reformulation of the Wedge Problem.**

In this section, we first reduce the Euler system into a second-order nonlinear elliptic equation and then reformulate the wedge problem into a free boundary problem for the nonlinear elliptic equation with the shock-front as a free boundary.

From the first equation in (1.1), there exists a unique stream function ψ in domain $\Omega^- \cup \Omega_{\mathcal{S}}$ such that

$$\nabla\psi = (-\rho u_2, \rho u_1)$$

with $\psi(\mathbf{0}) = 0$.

To simplify the analysis, we employ the following coordinate transformation to the Lagrangian coordinates:

$$\begin{cases} y_1 = x_1, \\ y_2 = \psi(x_1, x_2), \end{cases} \tag{3.1}$$

under which the original curved streamlines become straight. In the new coordinates $\mathbf{y} = (y_1, y_2)$, we still denote the unknown variables $U(\mathbf{x}(\mathbf{y}))$ by $U(\mathbf{y})$ for simplicity of notation.

The original Euler equations in (1.1) become the following equations in divergence form:

$$\Big(\frac{1}{\rho u_1}\Big)_{y_1} - \Big(\frac{u_2}{u_1}\Big)_{y_2} = 0, \tag{3.2}$$

$$\Big(u_1 + \frac{p}{\rho u_1}\Big)_{y_1} - \Big(\frac{p u_2}{u_1}\Big)_{y_2} = 0, \tag{3.3}$$

$$(u_2)_{y_1} + p_{y_2} = 0, \tag{3.4}$$

$$\Big(\frac{1}{2}|\mathbf{u}|^2 + \frac{\gamma p}{(\gamma-1)\rho}\Big)_{y_1} = 0. \tag{3.5}$$

Let $\mathcal{T} : y_1 = \hat{\sigma}(y_2)$ be a shock-front. Then, from the above equations, we can derive the Rankine-Hugoniot conditions along $\mathcal{T}$:

$$\Big[\frac{1}{\rho u_1}\Big] = -\Big[\frac{u_2}{u_1}\Big]\hat{\sigma}'(y_2), \tag{3.6}$$

$$\Big[u_1 + \frac{p}{\rho u_1}\Big] = -\Big[\frac{p u_2}{u_1}\Big]\hat{\sigma}'(y_2), \tag{3.7}$$

$$[\, u_2\,] = [\, p\,]\hat{\sigma}'(y_2), \tag{3.8}$$

$$\Big[\frac{1}{2}|\mathbf{u}|^2 + \frac{\gamma p}{(\gamma-1)\rho}\Big] = 0. \tag{3.9}$$

The background shock-front now is $\mathcal{T}_0 : y_1 = s_1 y_2$, with $\frac{1}{s_1} = \rho_0^+ u_{10}^+(\frac{1}{s_0} - \tan\theta_0) > 0$. Without loss of generality, we assume that the supersonic solution U^- exists in domain $\mathbb{D}^-$ defined by

$$\mathbb{D}^- = \{\mathbf{y} \,:\, 0 < y_1 < 2 s_1 y_2\}\,. \tag{3.10}$$

For a given shock function $\hat{\sigma}(y_2)$, let

$$\mathbb{D}^-_{\hat{\sigma}} \;=\; \{\mathbf{y} : 0 < y_1 < \hat{\sigma}(y_2)\}\,, \tag{3.11}$$

$$\mathbb{D}_{\hat{\sigma}} \;=\; \{\mathbf{y} : 0 < y_2, \hat{\sigma}(y_2) < y_1\}\,. \tag{3.12}$$

In either the supersonic or subsonic region, x_2 can be solved as a function of $\mathbf{y}$ since $\psi_{x_2} = \rho u_1 \neq 0$. Let $x_2 := \phi(\mathbf{y})$ in the subsonic region $\mathbb{D}_{\hat{\sigma}}$ and $x_2 := \phi^-(\mathbf{y})$ in the supersonic region $\mathbb{D}^-_{\hat{\sigma}}$. Given U^-, we can find the corresponding function ϕ^-. We now use function $\phi(\mathbf{y})$ to reduce the original Euler system to an elliptic equation in the subsonic region.

By the definition of coordinate transformation (3.1), we have

$$\phi_{y_1} = \frac{u_2}{u_1}, \quad \phi_{y_2} = \frac{1}{\rho u_1}, \tag{3.13}$$

that is, $\phi(\mathbf{y})$ is the potential function of the vector field $(\frac{u_2}{u_1}, \frac{1}{\rho u_1})$.

Equation (3.5) implies Bernoulli's law:

$$\frac{1}{2}|\mathbf{u}|^2 + \frac{\gamma p}{(\gamma-1)\rho} = B(y_2), \tag{3.14}$$

where $B = B(y_2)$ is completely determined by the incoming flow U^- at the initial position $\mathcal{I}$, because of the Rankine-Hugoniot condition (3.9).

From equations (3.2)–(3.5), we find

$$(\gamma \ln \rho - \ln p)_{y_1} = 0,$$

which implies

$$p = A(y_2)\rho^{\gamma} \qquad \text{in the subsonic region } \mathbb{D}_{\hat{\sigma}}. \tag{3.15}$$

With equations (3.13) and (3.15), we can rewrite Bernoulli's law into the following form:

$$\frac{\phi_{y_1}^2 + 1}{2\phi_{y_2}^2} + \frac{\gamma}{\gamma - 1} A\rho^{\gamma+1} = B\rho^2. \tag{3.16}$$

In the subsonic region, $|\mathbf{u}| < c := \sqrt{\frac{\gamma p}{\rho}}$. Therefore, Bernoulli's law (3.14) implies

$$\rho^{\gamma-1} > \frac{2(\gamma - 1)B}{\gamma(\gamma + 1)A}. \tag{3.17}$$

Condition (3.17) guarantees that ρ can be solved from (3.16) as a smooth function of $(A, B, \nabla\phi)$.

Assume that $A = A(y_2)$ has been known. Then $(\mathbf{u}, p, \rho)$ can be expressed as functions of $\nabla\phi$:

$$\rho = \rho(A, B, \nabla\phi), \quad \mathbf{u} = (\frac{1}{\rho\phi_{y_2}}, \frac{\phi_{y_1}}{\rho\phi_{y_2}}), \quad p = A\rho^{\gamma}, \tag{3.18}$$

since $B = B(y_2)$ is given by the incoming flow.

Similarly, in the supersonic region $\mathbb{D}^-$, we employ the corresponding variables (A^-, B, ϕ^-) to replace U^-, where B is the same as in the subsonic region because of the Rankine-Hugoniot condition (3.9).

We now choose (3.4) to derive a second-order nonlinear elliptic equation for ϕ so that the full Euler system is reduced to this equation in the subsonic region. Set

$$N^1 = u_2, \quad N^2 = p. \tag{3.19}$$

Then we obtain the *second-order nonlinear equation for* ϕ:

$$(N^1)_{y_1} + (N^2)_{y_2} = 0, \tag{3.20}$$

where $N^i = N^i(A(y_2), B(y_2), \nabla\phi), i = 1, 2$, are given by

$$\begin{aligned} N^1(A, B, \nabla\phi) &= \frac{\phi_{y_1}}{\phi_{y_2}\rho(A(y_2), B(y_2), \nabla\phi)}, \\ N^2(A, B, \nabla\phi) &= A(y_2)\rho(A(y_2), B(y_2), \nabla\phi)^{\gamma}. \end{aligned} \tag{3.21}$$

Let $q = \sqrt{u_1^2 + u_2^2}$. Then a careful calculation shows that

$$N^1_{\phi_{y_1}} = \frac{u_1(c^2 - u_1^2)}{c^2 - q^2}, \tag{3.22}$$

$$N^1_{\phi_{y_2}} = N^2_{\phi_{y_1}} = -\frac{c^2\rho u_1 u_2}{c^2 - q^2}, \tag{3.23}$$

$$N^2_{\phi_{y_2}} = \frac{c^2\rho^2 q^2 u_1}{c^2 - q^2}. \tag{3.24}$$

Thus, the discriminant

$$N^1_{\phi_{y_1}} N^2_{\phi_{y_2}} - N^1_{\phi_{y_2}} N^2_{\phi_{y_1}} = \frac{c^2\rho^2 u_1^2}{c^2 - q^2} > 0 \tag{3.25}$$

in the subsonic region with $\rho u_1 \neq 0$. Therefore, when solution ϕ is sufficiently close to ϕ_0^+ (determined by the subsonic background state U_0^+) in the C^1 norm, equation (3.20) is uniformly elliptic, and the Euler system (3.2)–(3.5) is reduced to the elliptic equation (3.20) in domain $\mathbb{D}_{\hat{\sigma}}$, where $\hat{\sigma}$ is the function for the transonic shock.

The boundary condition for ϕ on the wedge boundary $\{y_2 = 0\}$ is

$$\phi(y_1, 0) = b(y_1). \tag{3.26}$$

The condition on $\mathcal{T}$ is derived from the Rankine-Hugoniot conditions (3.6)–(3.8). Condition (3.6) is equivalent to the continuity of ϕ across $\mathcal{T}$:

$$[\phi]|_{\mathcal{T}} = 0. \tag{3.27}$$

It also gives

$$\hat{\sigma}'(y_2) = -\frac{[\phi_{y_2}]}{[\phi_{y_1}]}(\hat{\sigma}(y_2), y_2). \tag{3.28}$$

Replacing $\hat{\sigma}'(y_2)$ in (3.7) and (3.8) with (3.28) gives rise to the conditions on $\mathcal{T}$:

$$G(U^-, A, \nabla\phi) \equiv [\phi_{y_1}]\Big[\frac{1}{\rho\phi_{y_2}} + A\rho^\gamma\phi_{y_2}\Big] - [\phi_{y_2}][A\rho^\gamma\phi_{y_1}] = 0, \tag{3.29}$$

$$H(U^-, A, \nabla\phi) \equiv [\phi_{y_1}][N^1] + [\phi_{y_2}][N^2] = 0. \tag{3.30}$$

We will combine the above two conditions into the boundary condition for (3.20) by eliminating A.

By calculation, we have

$$N_A^1 = \frac{\gamma}{\gamma-1}\frac{\rho^{\gamma-1}u_2}{c^2-q^2}, \tag{3.31}$$

$$N_A^2 = -\frac{\rho^\gamma(q^2+\frac{c^2}{\gamma-1})}{c^2-q^2}. \tag{3.32}$$

Thus, we obtain

$$\begin{aligned} H_A &= N_A^1[\phi_{y_1}] + N_A^2[\phi_{y_2}] \\ &= \frac{\gamma}{\gamma-1}\frac{\rho^{\gamma-1}u_2}{c^2-q^2}\Big[\frac{u_2}{u_1}\Big] - \frac{\rho^\gamma(q^2+\frac{c^2}{\gamma-1})}{c^2-q^2}\Big[\frac{1}{\rho u_1}\Big] \\ &> 0, \end{aligned}$$

and

$$\begin{aligned} G_A &= [\phi_{y_1}]\Big(\frac{N_A^1}{\phi_{y_1}} + \phi_{y_2}N_A^2\Big) - [\phi_{y_2}]\phi_{y_1}N_A^2 \\ &= \frac{u_2\rho^\gamma(q^2+\frac{c^2}{\gamma-1})}{u_1(c^2-q^2)}\Big[\frac{1}{\rho u_1}\Big] - \frac{\rho^{\gamma-1}}{u_1(c^2-q^2)}\Big({u_2}^2 + \frac{c^2-u_1^2}{\gamma-1}\Big)\Big[\frac{u_2}{u_1}\Big] \\ &< 0, \end{aligned}$$

since $[\frac{1}{\rho u_1}] < 0$ and u_{2-} is close to 0.

Therefore, both equations (3.29) and (3.30) can be solved for A to obtain $A = g_1(U^-, \nabla\phi)$ and $A = g_2(U^-, \nabla\phi)$, respectively. Then we obtain our desired condition on the free boundary (*i.e.* the shock-front):

$$\bar{g}(U^-, \nabla\phi) := (g_2 - g_1)(U^-, \nabla\phi) = 0. \tag{3.33}$$

Then the original transonic problem is reduced to the elliptic equation (3.20) with the fixed boundary condition (3.26) and the free boundary conditions (3.27) and (3.33), and A is determined through either of (3.29)–(3.30).

4. **Hodograph Transformation and Fixed Boundary Value Problem.** In order to reduce the difficulty of the free boundary, we employ the hodograph transformation to make the shock-front into a fixed boundary. After that, we only need to solve for the unknown function A.

We now extend the domain of ϕ^- from $\mathbb{D}^-$ to the first quadrant $\mathbb{D}^- \cup \mathbb{D}_{\hat{\sigma}}$. Let $\phi_0^- = \frac{1}{\rho_0^- u_{20}^-} y_2$, which is the background potential function. We can extend ϕ^- into $\mathbb{D}^- \cup \mathbb{D}_{\hat{\sigma}}$ such that

$$\phi^- = \phi_0^- \qquad \text{when } \ 0 < 2\sigma_1 y_2 < y_1 - 1.$$

We then use the following partial hodograph transformation:

$$\begin{cases} z_1 = \phi - \phi^-, \\ z_2 = y_2, \end{cases} \tag{4.1}$$

so that y_1 is a function of (z_1, z_2): $y_1 = \varphi(z_1, z_2)$.

Let

$$\begin{aligned} M^1(U^-, A, \nabla\phi) &= N^1(A, B, \nabla\phi) + N^2(A, B, \nabla\phi)\frac{[\phi_{y_2}]}{[\phi_{y_1}]}, \\ M^2(U^-, A, \nabla\phi) &= \frac{N^2(A, B, \nabla\phi)}{[\phi_{y_1}]}, \end{aligned}$$

and

$$\begin{aligned} &\overline{M}^i(\mathbf{z}, A, \varphi, \nabla\varphi) \\ &= -M^i\Big(U^-(\varphi, z_2), A, \partial_{y_1}\phi^-(\varphi, z_2) + \frac{1}{\varphi_{z_1}}, \partial_{y_2}\phi^-(\varphi, z_2) - \frac{\varphi_{z_2}}{\varphi_{z_1}}\Big), \quad i = 1, 2. \end{aligned}$$

Therefore, equation (3.20) becomes

$$\Big(\overline{M}^1(\mathbf{z}, A, \varphi, \nabla\varphi)\Big)_{z_1} + \Big(\overline{M}^2(\mathbf{z}, A, \varphi, \nabla\varphi)\Big)_{z_2} = 0. \tag{4.2}$$

Notice that

$$\overline{M}^1_{\varphi_{z_1}} = [\phi_{y_1}]^2 N^1_{\phi_{y_1}} + 2N^1_{\phi_{y_2}}[\phi_{y_1}][\phi_{y_2}] + N^2_{\phi_{y_2}}[\phi_{y_2}]^2, \tag{4.3}$$

$$\overline{M}^1_{\varphi_{z_2}} = N^1_{\phi_{y_2}}[\phi_{y_1}] + N^2_{\phi_{y_2}}[\phi_{y_2}] + N^2, \tag{4.4}$$

$$\overline{M}^2_{\varphi_{z_1}} = N^1_{\phi_{y_2}}[\phi_{y_1}] + N^2_{\phi_{y_2}}[\phi_{y_2}] - N^2, \tag{4.5}$$

$$\overline{M}^2_{\varphi_{z_2}} = N^2_{\phi_{y_2}}. \tag{4.6}$$

Also

$$\overline{M}^1_{\varphi_{z_1}}\overline{M}^2_{\varphi_{z_2}} - \overline{M}^1_{\varphi_{z_2}}\overline{M}^2_{\varphi_{z_1}} = (N^2)^2 + [\phi_{y_1}]^2\Big(N^1_{\phi_{y_1}}N^2_{\phi_{y_2}} - (N^1_{\phi_{y_2}})^2\Big) > 0,$$

which implies that equation (4.2) is uniformly elliptic, for any solution φ that is close to φ_0^+ (determined by the background solution U_0^+) in the C^1 norm.

Then the unknown shock-front $\mathcal{T}$ becomes a fixed boundary, which is the z_2-axis. Along the z_2-axis, condition (3.33) is now

$$\tilde{g}(\mathbf{z}, \varphi, \nabla\varphi) \equiv \bar{g}\Big(U^-(\varphi, z_2), \partial_{y_1}\phi^-(\varphi, z_2) + \frac{1}{\varphi_{z_1}}, \partial_{y_2}\phi^-(\varphi, z_2) - \frac{\varphi_{z_2}}{\varphi_{z_1}}\Big) = 0. \tag{4.7}$$

We also convert condition (3.30) into the $\mathbf{z}$-coordinates:

$$\widetilde{H}(\mathbf{z}, A, \varphi, \nabla\varphi) \\ := H(U^-(\varphi, z_2), A, \partial_{y_1}\phi^-(\varphi, z_2) + \frac{1}{\varphi_{z_1}}, \partial_{y_2}\phi^-(\varphi, z_2) - \frac{\varphi_{z_2}}{\varphi_{z_1}}) = 0 \quad (4.8)$$

along the z_2-axis.

The condition on the z_1-axis can be derived from (3.26) as follows: Restricted on $z_2 = 0$, the coordinate transformation (4.1) becomes

$$z_1 = b(y_1) - \phi_-(y_1, 0).$$

Then y_1 can be solved in terms of z_1 so that

$$y_1 = \varphi(z_1, 0) = \widetilde{b}(z_1). \quad (4.9)$$

Let Q be the first quadrant. Then the original wedge problem is now reduced to both solving equation (4.2) for φ in the unbounded domain Q with the boundary conditions (4.7) and (4.9) and determining A via (4.8).

This will be achieved by the following fixed point arguments. Consider a Banach space:

$$X = \{\lambda \,:\, \lambda(0) = 0, \|\lambda\|^{(-\alpha);\{0\}}_{1,\alpha;(1+\beta);(0,\infty)} < \infty\}$$

as defined in (6.3) below. Then we define our iteration map $\mathcal{J} : X \longrightarrow X$ through the following two steps:

1. Consider any $A = A(z_2)$ so that $A - w_t \in X$ satisfying

$$\|A - A_0^+\|^{(-\alpha);\{0\}}_{1,\alpha;(1+\beta);(0,\infty)} \le C_0\varepsilon \quad (4.10)$$

for some fixed constant $C_0 > 0$, where $w_t = w_t(z_2)$ is determined by (6.2) below. With this A, we solve equation (4.2) for $\varphi = \varphi_A$ in the unbounded domain Q with the boundary conditions (4.7) and (4.9) in a compact and convex set:

$$\Sigma_\delta = \{\varphi \,:\, \|\varphi - \varphi^\infty\|^{(-1-\alpha);\partial\mathcal{W}}_{2,\alpha;(\beta);Q} \le \delta\} \qquad \text{for sufficiently small } \delta > 0 \quad (4.11)$$

in the Banach space:

$$\mathcal{B} = \{\varphi \,:\, \|\varphi - \varphi^\infty\|^{(-1-\alpha')}_{2,\alpha';(\beta');Q} < \infty\} \qquad \text{with } \ 0 < \alpha' < \alpha,\, 0 < \beta' < \beta, \quad (4.12)$$

where φ is determined by (5.2). Equation (4.2) is uniformly elliptic for $\varphi \in \Sigma$ for small $\delta > 0$. The existence of solution $\varphi_A \in \Sigma_\delta$ will be established by the Schauder fixed point theorem in §5.

2. With this $\varphi = \varphi_A$, we solve (4.8) to obtain a unique $\tilde{A}$ that defines $\mathcal{J}(A - w_t) = \tilde{A} - w_t$.

Finally, by the implicit function theorem, we prove that $\mathcal{J}$ has a fixed point $A - w_t$ in §6, for which A satisfies (4.10).

5. **An Elliptic Problem to Determine φ in Domain Q.** In this section, for given A satisfying (4.10), we solve equation (4.2) for φ in the unbounded domain Q with boundary conditions (4.7) and (4.9). Before this, we determine *a priori* the limit function φ_∞ at infinity.

5.1. **Determine *a priori* the limit function φ^∞ at infinity.** First, we assume that the asymptotic downstream state U^∞ depends only on y_2, which will be verified later. Then we determine the limit function ϕ^∞ for ϕ. From (3.2), we expect the flow direction at infinity is the same as that of the wedge. That is,

$$\phi_{y_1} = \frac{u_2}{u_1} \to \tan\theta_0 = \frac{u_{20}^+}{u_{10}^+} \qquad \text{as } y_1 \to \infty.$$

Then

$$\phi^\infty = \tan\theta_0\, y_1 + l(y_2).$$

Replacing ϕ with ϕ^∞ in Bernoulli's law (3.16), we obtain

$$\frac{(\tan\theta_0)^2+1}{2\big(l'(y_2)\big)^2} + \frac{\gamma}{\gamma-1}A\rho^{\gamma+1} = B\rho^2.$$

From (3.4), we expect that pressure $p \to p_0^+$ and then relation (3.15) becomes $p_0^+ = A(\rho^\infty)^\gamma$ so that $A = A(y_2)$ and $\rho^\infty(y_2) = (\frac{p_0^+}{A(y_2)})^{1/\gamma}$. Therefore, the above equation becomes

$$\frac{(\tan\theta_0)^2+1}{2\big(l'(y_2)\big)^2} + \frac{\gamma}{\gamma-1}A\Big(\frac{p_0^+}{A}\Big)^{(\gamma+1)/\gamma} = B\Big(\frac{p_0^+}{A}\Big)^{2/\gamma}. \tag{5.1}$$

This equation gives the expression for $l'(y_2)$. We can find $l(y_2)$ by integration with $l(0) = w_0$, where w_0 is the limit of $b - b_0$ as $y_1 \to \infty$.

Then we employ

$$z_1 = (\phi^\infty - \phi^-)(\varphi^\infty, z_2) \tag{5.2}$$

to solve for φ^∞. Also, equation (5.2) restricted on $z_2 = 0$ gives rise to

$$z_1 = \tan\theta_0\, \tilde b_0 + w_0 - \phi^-(\tilde b_0, 0),$$

from which we can solve for $\tilde b_0$.

By the definition of φ^∞, we know that φ^∞ satisfies (4.2). That is,

$$\Big(\overline{M}^1(\mathbf{z}, A, \varphi^\infty, \nabla\varphi^\infty)\Big)_{z_1} + \Big(\overline{M}^2(\mathbf{z}, A, \varphi^\infty, \nabla\varphi^\infty)\Big)_{z_2} = 0. \tag{5.3}$$

5.2. **Linearization.** Let

$$\Sigma_\delta = \Big\{ w : \|w\|^{(-1-\alpha);\partial\mathcal{W}}_{2,\alpha;(\beta);Q} \le \delta \Big\}, \tag{5.4}$$

where the wedge boundary $\partial\mathcal{W}$ is the z_1-axis. We will omit $\partial\mathcal{W}$ in the norm when no confusion arises.

To solve equation (4.2) in the first quadrant Q, we first linearize (4.2) and solve the linearized equation in bounded domains, and then take the limit to obtain a solution in the unbounded domain Q.

For given φ such that $\varphi - \varphi^\infty \in \Sigma_\delta$, we define a map in Σ_δ and show that there exists a fixed point that is a solution for equation (4.2).

We use a straight line $L^R := \{z_2 = -k(z_1 - R)\}$ to cut off Q into a triangular domain $Q^R := \{0 < z_2 < -k(z_1 - R), z_1 > 0\}$, where k is a positive number depending on the background state $U_0^\pm$.

Let

$$v = \tilde\varphi - \varphi^\infty, \qquad \zeta = \tilde b - \tilde b_0.$$

Taking the difference of equations (4.2) and (5.3) and linearizing the resulting equation lead to

$$\sum_{i,j=1,2} (a_{ij}^\varphi v_{z_i} + b_j^\varphi v)_{z_j} = 0, \tag{5.5}$$

where

$$a_{ij}^{\varphi} = \int_0^1 \overline{M}^i_{\varphi_{z_j}}(\mathbf{z}, A, \varphi^\infty + s(\varphi-\varphi^\infty), \nabla(\varphi^\infty + s(\varphi-\varphi^\infty)))\,\mathrm{d}s, \quad (5.6)$$

$$b_j^{\varphi} = \int_0^1 \overline{M}^j_{\varphi}(\mathbf{z}, A, \varphi^\infty + s(\varphi-\varphi^\infty), \nabla(\varphi^\infty + s(\varphi-\varphi^\infty)))\,\mathrm{d}s \quad (5.7)$$

for $i, j = 1, 2$, which are all bounded in the Hölder norm $\|\cdot\|^{(-1-\alpha);\partial\mathcal{W}}_{2,\alpha;(\beta);Q}$. Also, the uniform ellipticity of equation (5.5) follows from (5.6) and the uniform ellipticity of (4.2) for the solutions close to φ_0^+, provided that δ in (5.4) is chosen sufficiently small.

The boundary condition on the z_1-axis is

$$v|_{z_2=0} = \zeta. \quad (5.8)$$

On the cutoff line L^R, we prescribe the condition:

$$v|_{L^R} = \zeta(R), \quad (5.9)$$

which is compatible with the condition on the z_1-axis at point $(R, 0)$.

Condition (4.7) on the z_2-axis can be linearized as follows: Condition (4.7) can be rewritten as

$$\tilde{g}(\mathbf{z}, \varphi, \nabla\varphi) - \tilde{g}(\mathbf{z}, \varphi^\infty, \nabla\varphi^\infty) = -\tilde{g}(\mathbf{z}, \varphi^\infty, \nabla\varphi^\infty).$$

Therefore, we derive the oblique condition:

$$\sum_{i=1,2} \nu_i^{\varphi} v_{z_i} + c^{\varphi} v = -\tilde{g}(\mathbf{z}, \varphi^\infty, \nabla\varphi^\infty) =: g_0, \quad (5.10)$$

where

$$\nu_i^{\varphi} = \int_0^1 \tilde{g}_{\varphi_{z_i}}(\mathbf{z}, \varphi^\infty + s(\varphi-\varphi^\infty), \nabla(\varphi^\infty + s(\varphi-\varphi^\infty)))\,\mathrm{d}s,$$

$$c^{\varphi} = \int_0^1 \tilde{g}_{\varphi}(\mathbf{z}, \varphi^\infty + s(\varphi-\varphi^\infty), \nabla(\varphi^\infty + s(\varphi-\varphi^\infty)))\,\mathrm{d}s,$$

which have all the corresponding bounded Hölder norms.

When U_0^+ is on arc $\widehat{TS}$, the direction of $\boldsymbol{\nu} = (\nu_1, \nu_2)$ is

$$\begin{aligned}\nu_1 &= \tilde{g}_{\varphi_{z_1}} \\ &= \frac{-\rho^{\gamma-1}}{(\gamma-1)u_1^2 G_A H_A (c^2-q^2)}\Big(\Big[\frac{1}{\rho u_1}\Big]^2 u_1^2\rho^2c^2 - 2c^2\rho u_1 u_2\Big[\frac{1}{\rho u_1}\Big] + u_2^2(c^2-u_1^2)\Big) \\ &> 0\end{aligned}$$

since $[\frac{1}{\rho u_1}] < 0$, and

$$\nu_2 = \tilde{g}_{\varphi_{z_2}} = \frac{-\rho^{\gamma-1}u_2}{(\gamma-1)u_1 G_A H_A (c^2-q^2)} C_p,$$

where

$$C_p = [p]\Big(c^2 + (\gamma-1)q^2 - \gamma u_1^2\Big) + (\gamma-1)\rho q^2 u_2^2 + \Big[\frac{1}{\rho u_1}\Big]\rho^2c^2u_1q^2. \quad (5.11)$$

Since, on arc $\widehat{TS}$, $C_p < 0$ from (B.11), we have

$$\nu_2 < 0.$$

In particular, if δ is small, then

$$\nu_2 \le \frac{1}{2}\nu_{20}^{+} < 0,$$

where ν_{20}^{+} is the quantity ν_2 for the background subsonic state. This implies that condition (5.10) is uniformly oblique.

Set

$$\varepsilon = \|U_0 - U_0^-\|_{2,\alpha;(1+\beta);\mathbb{D}^-} + \|b - b_0\|^*_{2,\alpha;(\beta);(0,\infty)}.$$

Now, for any function f of (U^-, U), we use $\hat{f}$ to denote the value at the background states: $\hat{f} = f(U_0^-, U_0^+)$. We also omit domain Q^R and boundary $\partial\mathcal{W}$ in the norms when no confusion arises.

5.3. C^0 **estimate for** v**.** We employ the comparison principles, Theorem A.1 and Theorem A.2, to estimate v.

We decompose matrix $\mathbf{A} = (\hat{a}_{ij})$ into $\mathbf{A} = KK^\top$, where

$$K = (k_{ij}) = \begin{pmatrix} \sqrt{\frac{\hat{a}_{11}\hat{a}_{22} - \hat{a}_{12}^2}{\hat{a}_{22}}} & \frac{\hat{a}_{12}}{\sqrt{\hat{a}_{22}}} \\ 0 & \sqrt{\hat{a}_{22}} \end{pmatrix}.$$

We define the transformation $\mathbf{z} = K\bar{\mathbf{z}}$, where $\bar{\mathbf{z}} = (\bar{z}_1, \bar{z}_2)$ is a new coordinate system. Then $\sum_{j=1,2} k_{ij}\bar{z}_j = z_i$ implies

$$\sum_{i,j=1,2} \hat{a}_{ij}\partial^2_{z_i z_j} = \Delta_{\bar{\mathbf{z}}}.$$

We use the polar coordinates (r,θ) for $\bar{\mathbf{z}}$ to construct a comparison function for v. That is,

$$r = |\bar{\mathbf{z}}|, \qquad \theta = \arctan\Big(\frac{\bar{z}_2}{\bar{z}_1}\Big).$$

Let $\bar{\theta} = t\theta + \tau$. Define

$$\bar{v} = r^s \sin\bar{\theta}, \tag{5.12}$$

where $\tau > 0$, and s and t will be chosen later.

We compute

$$\sum_{i,j=1,2} \hat{a}_{ij}\partial^2_{z_i z_j}\bar{v} = \Delta_{\bar{\mathbf{z}}}\bar{v} = (s^2 - t^2)\, r^{s-2}\sin\bar{\theta}. \tag{5.13}$$

Let $s = -\beta$, $t = \alpha$, and $0 < \beta < \alpha$ in (5.12). We set $v_1 = r^{-\beta}\sin(\alpha\theta + \tau)$. Since

$$\|a^{\varphi}_{ij} - \hat{a}_{ij}\|^{(-\alpha)}_{1,\alpha;(1+\beta)} \le C\delta, \qquad \|b^{\varphi}_i\|_{1,\alpha;(2+\beta)} < C\varepsilon,$$

we have

$$\begin{aligned} L^{\varphi}v_1 &= \sum_{i,j=1,2}\Big(\hat{a}_{ij}\partial^2_{z_i z_j} + (a^{\varphi}_{ij} - \hat{a}_{ij})\partial^2_{z_i z_j}\Big)v_1 \\ &\qquad + \sum_{i=1,2}\Big(\sum_{j=1,2}(a^{\varphi}_{ij} - \hat{a}_{ij})_{z_j} + b^{\varphi}_i\Big)\partial_{z_i}v_1 + \sum_{i=1,2}(b^{\varphi}_i)_{z_i}v_1 \\ &= (\beta^2 - \alpha^2)\, r^{-\beta-2}\sin\bar{\theta} + \mathrm{O}(\varepsilon + \delta)\, r^{-\beta-2} + \mathrm{O}(\delta)\, r^{-2-\beta} z_2^{\alpha-1}. \end{aligned}$$

Let $v_2 = r^{-\beta}\sin^{\alpha}\theta$. Then

$$\Delta_{\bar{\mathbf{z}}}v_2 = (\beta^2 - \alpha^2)r^{-\beta-2}\sin^{\alpha}\theta - \alpha(1-\alpha)r^{-\beta-2}\sin^{\alpha-2}\theta.$$

Set $v_3 = v_1 + v_2$. Thus, we have

$$L^\varphi v_3 < -c_0 r^{-\beta-2} < 0.$$

By Theorem A.1, $\frac{v}{v_3}$ achieves its positive maximum on the boundary.

On $z_2 = 0$ and L^R, $\frac{v}{v_3} \le C\varepsilon$.

Let $\theta_0 = \arctan(-\frac{k_{11}}{k_{12}})$. We compute $\nabla_{\mathbf{z}} \bar{v}$ on the z_2-axis:

$$\bar{v}_{\bar{z}_1} = r^{s-1}(s\cos\theta_0 \sin\bar{\theta} - t\sin\theta_0\cos\bar{\theta}),$$
$$\bar{v}_{\bar{z}_2} = r^{s-1}(s\sin\theta_0 \sin\bar{\theta} + t\cos\theta_0\cos\bar{\theta}).$$

Then

$$\nabla_{\mathbf{z}}\bar{v} = (K^{-1})^\top \nabla_{\bar{\mathbf{z}}}\bar{v} = r^{s-1}\begin{pmatrix} \frac{s\cos\theta_0\sin\bar{\theta} - t\sin\theta_0\cos\bar{\theta}}{k_{11}} \\ \frac{s\sin\bar{\theta}}{k_{22}\sin\theta_0} \end{pmatrix},$$

$$(v_3)_{z_1} = -\frac{\beta\cos\theta_0\sin\bar{\theta} + \alpha\sin\theta_0\cos\bar{\theta} + (\alpha+\beta)\sin^\alpha\theta_0\cos\theta_0}{k_{11}} r^{-\beta-1},$$

$$(v_3)_{z_2} = -\frac{\beta(\sin\bar{\theta} + \sin^\alpha\theta_0)}{k_{22}\sin\theta_0} r^{-\beta-1}.$$

Then, when β is suitably small, we have

$$D_{\boldsymbol{\nu}}(v_1 + v_2) < -c_1 r^{-\beta-1}.$$

Assume that $\frac{v}{v_3}$ achieves its maximum εM at some point P on the z_2-axis. We know that $D_{\boldsymbol{\nu}}(\frac{v}{v_3})(P) \le 0$.

Since $|g_0| = |\tilde{g}(\mathbf{z}, \varphi_0, \nabla\varphi_0) - \tilde{g}(\mathbf{z}, \varphi^\infty, \nabla\varphi^\infty)| \le C\varepsilon r^{-\beta-1}$, we obtain that, at point P,

$$\begin{aligned} 0 &\ge D_{\boldsymbol{\nu}} v - \frac{v}{v_3} D_{\boldsymbol{\nu}} v_3 \\ &= g_0 - c^\varphi v - \varepsilon M D_{\boldsymbol{\nu}} v_3 \\ &\ge -C\varepsilon(1+\varepsilon M) r^{-\beta-1} + M\varepsilon c_1 r^{-\beta-1}. \end{aligned} \tag{5.14}$$

This implies that $M \le \frac{2C}{c_1}$ for sufficiently small ε.

Thus, we obtain the estimate for v:

$$|v| \le C\varepsilon r^{-\beta}. \tag{5.15}$$

5.4. $C^{1,\alpha}$ **estimate for** v **at corner** O**.** In (5.12), let $s = 1+\alpha$ and $t = 1+\alpha+\tau$. We define

$$v_4 = r^{1+\alpha}\sin\Big((1+\alpha+\tau)\theta + \tau\Big). \tag{5.16}$$

By (5.13), it is easy to check

$$L^\varphi v_4 < -c_2\, r^{\alpha-1}.$$

On the z_2-axis, we have

$$\begin{aligned} D_{\boldsymbol{\nu}} v_4 &= r^\alpha\left(\frac{\nu_1}{k_{11}}\Big((\alpha+1)\sin((\alpha+\tau)\theta_0+\tau) - \tau\sin\theta_0\cos\bar{\theta}\Big) + \nu_2\frac{(\alpha+1)\sin\bar{\theta}}{k_{22}\sin\theta_0}\right) \\ &< -cr^\alpha, \end{aligned}$$

provided that α and τ are suitably small.

Then we can use $\varepsilon C v_4$ as a comparison function to control $w \equiv v - v(\mathbf{0}) - D_{\mathbf{z}} v(\mathbf{0}) \cdot \mathbf{z}$ for $r < 2$.

Denote any quarter ball $B_r(\mathbf{0}) \cap Q$ with radius r by B_r^+. In B_2^+,

$$\begin{aligned} L^\varphi w &= \sum_{j=1,2} (f_j)_{z_j} := -\sum_{j=1,2} \Big(\sum_{i=1,2} a_{ij}^\varphi v_{z_i}(\mathbf{0}) + b_j^\varphi (v(\mathbf{0}) + D_{\mathbf{z}} v(\mathbf{0}) \cdot \mathbf{z}) \Big)_{z_j} \\ &\geq -C\varepsilon r^{\alpha-1} \geq L^\varphi (C \varepsilon v_4). \end{aligned} \tag{5.17}$$

By Theorem A.2, we have

$$\sup_{B_2^+} \Big(\frac{w}{\varepsilon C v_4} \Big) \leq \sup_{\partial B_2^+} \Big(\frac{w^+}{\varepsilon C v_4} \Big) + 1.$$

On $\partial B_2^+ \cap (\{z_2 = 0\} \cup \{|\mathbf{z}| = 2\})$, we see that $\frac{w}{\varepsilon C v_4} \leq C$.

Assume that $\frac{w}{C \varepsilon v_4}$ achieves its maximum M at a point P on the z_2-axis. The oblique condition (5.10) implies

$$\sum_{i=1,2} \nu_i^\varphi w_{z_i} + c^\varphi w = \bar{g}_0 = \mathrm{O}(\varepsilon r^\alpha).$$

The same argument as in (5.14) implies that, at the maximum point P,

$$\begin{aligned} 0 &\geq D_{\boldsymbol{\nu}} (\frac{w}{v_4}) \\ &= \frac{1}{v_4} \Big(D_{\boldsymbol{\nu}} w - \frac{w}{v_4} D_{\boldsymbol{\nu}} v_4 \Big) \\ &\geq \frac{1}{v_4} \Big(-c^\varphi w - \varepsilon C r^\alpha + \varepsilon M c_0 r^\alpha \Big), \end{aligned}$$

which implies that $M \leq \frac{C}{c_0}$. Thus, $w \leq \varepsilon C r^{1+\alpha}$ in B_2^+.

Similarly, we obtain the corresponding lower bound.

Therefore, we conclude

$$|w(\mathbf{z})| \leq \varepsilon C r^{1+\alpha} \qquad \text{for any } \mathbf{z} \in B_2^+. \tag{5.18}$$

With estimate (5.18), we can use the scaling technique to obtain the $C^{1,\alpha}$ estimate for w up to the corner. More precisely, for any point $P_* \in B_1^+$ with polar coordinates (d_*, θ_*), we consider two cases for different values of θ_*.

Case 1: Interior estimate for $\theta_* \in [\frac{\pi}{6}, \frac{\pi}{3}]$. Set $B_1 = B_{\frac{d_*}{6}}(P_*)$ and $B_2 = B_{\frac{d_*}{3}}(P_*)$. Then $B_1 \subset B_2 \subset B_2^+$. By the Schauder interior estimates (*cf.* (4.45) and Theorem 8.33 in [16]), we have

$$\|w\|_{1,\alpha;B_2}^{(0)} \leq C \Big(\|w\|_{0,0;B_2} + \sum_{i=1,2} \|f_i\|_{0,\alpha;B_2}^{(1)} \Big),$$

where f_i is defined in (5.17), C is a constant independent of d_*, and the weight of the norm is up to ∂B_2. Therefore, by (5.18), we conclude

$$\|w\|_{1,\alpha;B_1} \leq d_*^{-(1+\alpha)} \|w\|_{1,\alpha;B_2}^{(0)} \leq C \varepsilon. \tag{5.19}$$

Case 2: Boundary estimate for $\theta_* > \frac{\pi}{3}$ or $\theta_* < \frac{\pi}{6}$. Denote $B_3 = Q \cap B_{\frac{2d_*}{3}}(P_*)$. By the Schauder boundary estimate (*cf.* (4.46) and Theorem 8.33 in [16]), we have

$$\begin{aligned}
&\|w\|^{(0)}_{1,\alpha;B_3}\\
&\le C\Big(\|w\|_{0,0;B_3} + \sum_{i=1,2}\|f_i\|^{(1)}_{0,\alpha;B_3} + \|\zeta\|_{1,\alpha;\overline{B_3}\cap\{z_2=0\}} + \|\bar{g}_0\|^{(1)}_{0,\alpha;\overline{B_3}\cap\{z_1=0\}}\Big)\\
&\le \varepsilon C d_*^{1+\alpha}.
\end{aligned}$$

Combining Case 1 with Case 2 yields the corner estimate:

$$\|v\|_{1,\alpha;B_1^+} = \|w + v(\mathbf{0}) + D_{\mathbf{z}}v(\mathbf{0})\cdot\mathbf{z}\|_{1,\alpha;B_1^+} \le C\varepsilon. \tag{5.20}$$

The other two corners can be treated in the same way.

For any point $P_* \in Q^R$ with polar coordinates (R_*, θ_*) for $\frac{1}{2} < R_* < R$, we employ the same scaling arguments as above in Cases 1–2 to obtain the estimates with decay rate β. In other words, for $B_* := B_{\frac{R_*}{4}}(P_*) \cap Q$, if $\theta_* \in [\frac{\pi}{6}, \frac{\pi}{3}]$ and $R_* < \frac{R}{2}$, we employ the Schauder interior estimate; otherwise, we employ the Schauder boundary estimate. Therefore, we have

$$\begin{aligned}
\|v\|^{(0)}_{1,\alpha;B_*} &\le C\Big(\|w\|_{0,0;B_*} + \sum_{i=1,2}\|f_i\|^{(1)}_{0,\alpha;B_*} + \|\zeta\|_{1,\alpha;(\frac{R_*}{4},2R_*)} + \|\bar{g}_0\|^{(1)}_{0,\alpha;(\frac{R_*}{4},2R_*)}\Big)\\
&\le C\varepsilon R_*^{-\beta}.
\end{aligned}$$

Then the estimate for v in Q^R is

$$\|v\|_{1,\alpha;(\beta);Q^R} \le C\varepsilon. \tag{5.21}$$

5.5. $C^{2,\alpha}$ **regularity.** For the $C^{2,\alpha}$ estimates with a weight to the z_1-axis, we rewrite equation (5.5) into the non-divergence form:

$$\sum_{i,j=1,2} a^{\varphi}_{ij} v_{z_i z_j} = f_1, \tag{5.22}$$

with the boundary condition on the z_2-axis:

$$\sum_{i,j=1,2} \hat{\nu}_i v_{z_i} = g_1, \tag{5.23}$$

where

$$\begin{aligned}
f_1 &= -\sum_{i=1,2}\Big(\big(\sum_{j=1,2}(a^{\varphi}_{ij})_{z_j} + b^{\varphi}_i\big)v_{z_i} + (b^{\varphi}_i)_{z_i}v\Big),\\
g_1 &= g_0 - c^{\varphi}v + \sum_{i=1,2}(\hat{\nu}_i - \nu^{\varphi}_i)v_{z_i}.
\end{aligned}$$

For any point $\mathbf{z}^* = (z_1^*, z_2^*) \in Q^{R/2}$ with $z_2^* < 1$, set

$$B_1 := B_{z_2^*/2}(\mathbf{z}^*) \cap Q, \qquad B_2 = B_{z_2^*}(\mathbf{z}^*) \cap Q, \qquad T = B_{z_2^*}(\mathbf{z}^*) \cap \{z_1 = 0\}.$$

The Schauder interior and boundary estimates (*cf.* Theorem 6.26 in [16]) imply

$$\|v\|^{(0)}_{2,\alpha;B_2} \le C\left(\|v\|_{0,0;B_2} + \|g_1\|^{(1)}_{1,\alpha;T} + \|f_1\|^{(2)}_{0,\alpha;B_2}\right). \tag{5.24}$$

For $\theta_* \ge \frac{\pi}{6}$, we have

$$\begin{aligned}
\|g_1\|^{(1)}_{1,\alpha;T} &\le C\Big(|g_0|^{(1)}_{1,\alpha;T} + \varepsilon\|v\|^{(0)}_{2,\alpha;B_2}\Big)\\
&\le C\varepsilon(|\mathbf{z}^*|^{-\beta} + \|v\|^{(0)}_{2,\alpha;B_2}),
\end{aligned}$$

which leads to

$$\|v\|^{(0)}_{2,\alpha;B_2} \le C\varepsilon|\mathbf{z}^*|^{-\beta}.$$

For $\theta_* < \frac{\pi}{6}$, B_2 has no intersection with the z_2-axis. Therefore, the term of g_1 vanishes in (5.24). Define $\bar{v}(\mathbf{z}) = v(\mathbf{z}) - v(z_1^*, 0) - \nabla v(z_1^*, 0) \cdot \mathbf{z}$, $B_3^+ = B_{3z_2^*/2}(z_1^*, 0) \cap Q$. Then

$$\begin{aligned} \|\bar{v}\|_{0,0;B_2} &\le C[v]_{1,\alpha;B_3^+}|z_2^*|^{1+\alpha} \\ &\le C\varepsilon|\mathbf{z}^*|^{-1-\alpha-\beta}|z_2^*|^{1+\alpha} \\ &= C\varepsilon|\mathbf{z}^*|^{-\beta}(|z_2^*|/|\mathbf{z}^*|)^{1+\alpha}. \end{aligned}$$

Estimate (5.24) for $\bar{v}$, together with (5.15), leads to

$$\|v\|^{(-1-\alpha)}_{2,\alpha;(\beta);Q^{R/2}} \le C\varepsilon. \tag{5.25}$$

Solution v depends on R, which is denoted by v^R. By compactness of v^R, we can find a subsequence converging to $\tilde{v}$ such that

$$\|\tilde{v}\|^{(-1-\alpha)}_{2,\alpha;(\beta);Q} \le C\varepsilon.$$

When $C\varepsilon < \delta$, then $\tilde{v} \in \Sigma_\delta$, and $\tilde{v}$ is a solution of equation (5.5).

5.6. **Uniqueness.** Because of the decay of $\tilde{v}$ at infinity, we can obtain the uniqueness of $\tilde{v}$ by the comparison principle as follows:

Suppose that v_1 and v_2 are two solutions of (5.5). The difference $w = v_1 - v_2$ satisfies the same equation and boundary conditions on the z_2-axis, and $w = 0$ on the z_1-axis.

For any small positive constant τ, we let R be large enough such that $|w| \le \tau$ on the cutoff boundary L^R. Similar to (5.14), we employ Theorem A.1 to obtain $|w| \le \tau$ in Q^R. Let $R \to \infty$ and $\tau \to \infty$, we conclude that $w \equiv 0$, which implies the uniqueness.

5.7. **Determination of φ as a fixed point.** We define a map $\mathcal{Q} : \Sigma_\delta \to \Sigma_\delta$ by

$$\mathcal{Q}(w) \equiv \tilde{v} \qquad \text{for any} \quad w = \varphi - \varphi^\infty,$$

where the closed set Σ_δ is defined in (4.11). We employ the Schauder fixed point theorem to prove the existence of a fixed point for $\mathcal{Q}$. That is, we need to verify the following facts:

(i) Σ_δ is a compact and convex set in a Banach space $\mathcal{B}$;

(ii) $\mathcal{Q} : \Sigma_\delta \to \Sigma_\delta$ is continuous in $\mathcal{B}$.

Choose the Banach space $\mathcal{B}$ as defined in (4.12). Then Σ_δ is compact and convex in $\mathcal{B}$.

For the continuity of $\mathcal{Q}$, we make the following contradiction argument. Let $w_0, w^n \in \Sigma_\delta$ and $w^n \to w_0$ in $\mathcal{B}$. Then $v^n \equiv \mathcal{Q}(w^n)$ in Σ_δ and $v_0 \equiv \mathcal{Q}(w_0)$ in Σ_δ. We want to prove that $v^n \to v_0$ in $\mathcal{B}$.

Assume that $v^n \nrightarrow v_0$. Then there exist $c_0 > 0$ and a subsequence $\{v^{n_k}\}$ such that $\|v^{n_k} - v_0\|_{\mathcal{B}} \ge c_0$. Since $\{v^{n_k}\} \subset \Sigma_\delta$ is compact in $\mathcal{B}$, we can find another subsequence, again denoted by $\{v^{n_k}\}$, converging to some $v_1 \in \Sigma_\delta$. Then v_0 and v_1 satisfy the same equation (5.5), where $\varphi = \varphi^\infty + w_0$, which contradicts with the uniqueness of solutions for (5.5). Therefore, $\mathcal{Q}$ is continuous in $\mathcal{B}$.

Thus, we have a fixed point v for $\mathcal{Q}$, which gives a solution $\varphi \equiv \varphi^\infty + v$ for the nonlinear equation (4.2). The solution is unique by applying the same comparison principle as for the linear equation.

Therefore, for given (A, U^-, b), we have determined φ.

6. **Determination of the Entropy Function A as a Fixed Point.** In this section, we employ the implicit function theorem to prove the existence of a fixed point A.

6.1. **Setup for the implicit function theorem for A.** Through the shock polar, we can determine the values of U at O, and hence $A(0) = A_t$ is fixed, depending on the values of $U_-(O)$ and $b'(0)$. Then we solve (4.8) to obtain a unique solution $\tilde{A} = h(\mathbf{z}, \varphi, \nabla\varphi)$ that defines the iteration map. To complete the proof, we need to prove that the iteration map exists and has a fixed point by the implicit function theorem.

In order to employ the implicit function theorem, we need to set up a Banach space for A. To realize this, we perform the following normalization for (A, U^-, b).

Let $A_0^+ = \frac{p_0^+}{(\rho_0^+)^\gamma}$. Define a smooth cutoff function χ on $[0, \infty)$ such that

$$\chi(s) = \begin{cases} 1, & 0 \le s < 1, \\ 0, & s > 2. \end{cases}$$

Let $\omega = U^- - U_0^-$ and $\mu = b - b_0$. Set

$$A(0) := t(\omega(0), \mu'(0)), \tag{6.1}$$

where t is a function determined by the Rankine-Hugoniot conditions (3.6)–(3.9).

Set $\lambda = A - w_t$ with

$$w_t(z_2) = A_0^+ + \Big(t(\omega(0), \mu'(0)) - A_0^+\Big)\chi(z_2). \tag{6.2}$$

Then $\lambda(0) = 0$.

Given (λ, ω, μ), we can compute $\tilde{A} - w_t = \mathcal{J}(A - w_t)$ that defines the iteration map, by constructing a map $\tilde{\lambda} = \tilde{A} - w_t \equiv \mathcal{P}(\lambda, \omega, \mu)$.

We will prove that equation $\mathcal{P}(\lambda, \omega, \mu) - \lambda = 0$ is solvable for λ, given parameters (ω, μ) near $(0, 0)$. This is obtained by the implicit function theorem.

6.2. **Properties of the operator $\mathcal{P}$.** We first define some Banach spaces for operator $\mathcal{P}$. Set

$$X = \{\lambda \,:\, \lambda(0) = 0, \|\lambda\|_X < \infty\} \tag{6.3}$$

with

$$\|\lambda\|_X \equiv \|\lambda\|^{(-\alpha);\{0\}}_{1,\alpha;(1+\beta);(0,\infty)}, \tag{6.4}$$

$$Y = \{\omega \,:\, \|\omega\|_Y < \infty\} \tag{6.5}$$

with

$$\|\omega\|_Y \equiv \|\omega\|_{2,\alpha;(\beta+1);\Omega_-} \tag{6.6}$$

for a vector-valued function ω, and

$$Z = \{\mu \,:\, \mu(0) = 0, \|\mu\|_Z < \infty\} \tag{6.7}$$

with

$$\|\mu\|_Z \equiv \|\mu\|^*_{1,\alpha;(\beta);(0,\infty)}. \tag{6.8}$$

Clearly, X, Y, and Z are Banach spaces. Operator $\mathcal{P}$ is a map from $X \times Y \times Z$ to X.

We now define a linear operator $D_\lambda\mathcal{P}(\lambda,\omega,\mu)$ and show that it is the partial differential of $\mathcal{P}$ with respect to λ. When no confusion arises, we may drop the variables (λ,ω,μ) in $D_\lambda\mathcal{P}(\lambda,\omega,\mu)$.

We divide the proof into four steps.

1. *Definition of a linear operator* $D_\lambda\mathcal{P}(\lambda,\omega,\mu)$. Given $\delta\lambda\in X$, we solve the following equation for $\delta\varphi$:

$$\sum_{i=1,2}\Big(\sum_{j=1,2}a_{ij}^\lambda(\delta\varphi)_{z_j}+b_i^\lambda\delta\varphi+d_i^\lambda\delta\lambda\Big)_{z_i}=0, \tag{6.9}$$

with boundary conditions:

$$\delta\varphi|_{z_2=0}=0, \tag{6.10}$$

$$\Big(\sum_{i=1,2}\nu_i^\lambda(\delta\varphi)_{z_i}+c^\lambda\delta\varphi\Big)|_{z_1=0}=0, \tag{6.11}$$

where

$$a_{ij}^\lambda=\overline{M}^i_{\varphi_{z_j}}(\mathbf{z},w_t+\lambda,\varphi,\nabla\varphi),\qquad b_i^\lambda=\overline{M}^i_{\varphi}(\mathbf{z},w_t+\lambda,\varphi,\nabla\varphi),$$
$$d_i^\lambda=\overline{M}^i_{A}(\mathbf{z},w_t+\lambda,\varphi,\nabla\varphi),\quad \nu_i^\lambda=g_{\varphi_{z_i}}(\mathbf{z},\varphi,\nabla\varphi),\quad c^\lambda=g_\varphi(\mathbf{z},\varphi,\nabla\varphi).$$

Once we have known $\delta\varphi$, we define

$$\widetilde{\delta\lambda}=D_\lambda\mathcal{P}(\lambda,\omega,\mu)(\delta\lambda):=\sum_{i=1,2}e_i^\lambda(\delta\varphi)_{z_i}+e_0^\lambda\delta\varphi, \tag{6.12}$$

where

$$e_i^\lambda=h_{\varphi_{z_i}}(\mathbf{z},\varphi,\nabla\varphi),\qquad e_0^\lambda=h_\varphi(\mathbf{z},\varphi,\nabla\varphi).$$

It is easy to see that $\widetilde{\delta\lambda}(0)=0$. Then $D_\lambda\mathcal{P}(\lambda,\omega,\mu)$ is a linear operator from X to X.

2. *Show that* $D_\lambda\mathcal{P}(\lambda,\omega,\mu)$ *is the partial differential of* $\mathcal{P}$ *with respect to* λ *at* (λ,ω,μ).

For fixed (ω,μ), we need to estimate $\mathcal{P}(\lambda+\delta\lambda,\omega,\mu)-\mathcal{P}(\lambda,\omega,\mu)-D_\lambda\mathcal{P}(\lambda,\omega,\mu)(\delta\lambda)$ to be $\mathrm{o}(\delta\lambda)$.

For λ, we define φ by following the definition of $\mathcal{P}$, *i.e.*, we solve the following equation, an alternative form from (4.2):

$$\sum_{i=1,2}\Big(\overline{M}^i(\mathbf{z},w_t+\lambda,\varphi,\nabla\varphi)\Big)_{z_i}=0, \tag{6.13}$$

with boundary conditions (4.7) and (4.9).

For $\lambda+\delta\lambda$, the corresponding potential $\bar\varphi$ satisfies

$$\sum_{i=1,2}\Big(\overline{M}^i(\mathbf{z},w_t+\lambda+\delta\lambda,\bar\varphi,\nabla\bar\varphi)\Big)_{z_i}=0, \tag{6.14}$$

with the same boundary conditions (4.7) and (4.9).

Taking the difference of equations (6.13) and (6.14) leads to the following equation:

$$\sum_{i=1,2}\Big(\sum_{j=1,2}a_{ij}^{\delta\lambda}(\bar\varphi-\varphi)_{z_j}+b_i^{\delta\lambda}(\bar\varphi-\varphi)+d_i^{\delta\lambda}\delta\lambda\Big)_{z_i}=0, \tag{6.15}$$

with boundary conditions:

$$(\bar\varphi-\varphi)|_{z_2=0}=0, \tag{6.16}$$

$$\Big(\sum_{i=1,2}\nu_i^{\delta\lambda}(\bar\varphi-\varphi)_{z_i}+c^{\delta\lambda}(\bar\varphi-\varphi)\Big)|_{z_1=0}=0, \tag{6.17}$$

where

$$\begin{aligned}
a_{ij}^{\delta\lambda}&=\int_0^1 \overline{M}^i_{\varphi_{z_i}}(\mathbf{z},w_t+\lambda+s\delta\lambda,\varphi+s(\bar\varphi-\varphi),\nabla(\varphi+s(\bar\varphi-\varphi)))\,\mathrm{d}s,\\
b_i^{\delta\lambda}&=\int_0^1 \overline{M}^i_{\varphi}(\mathbf{z},w_t+\lambda+s\delta\lambda,\varphi+s(\bar\varphi-\varphi),\nabla(\varphi+s(\bar\varphi-\varphi)))\,\mathrm{d}s,\\
d_i^{\delta\lambda}&=\int_0^1 \overline{M}^i_{A}(\mathbf{z},w_t+\lambda+s\delta\lambda,\varphi+s(\bar\varphi-\varphi),\nabla(\varphi+s(\bar\varphi-\varphi)))\,\mathrm{d}s,\\
\nu_i^{\delta\lambda}&=\int_0^1 g_{\varphi_{z_i}}(\mathbf{z},\varphi+s(\bar\varphi-\varphi),\nabla(\varphi+s(\bar\varphi-\varphi)))\,\mathrm{d}s,\\
c^{\delta\lambda}&=\int_0^1 g_{\varphi}(\mathbf{z},\varphi+s(\bar\varphi-\varphi),\nabla(\varphi+s(\bar\varphi-\varphi)))\,\mathrm{d}s.
\end{aligned}$$

Take the difference of (6.15) and (6.9), and let $v:=\bar\varphi-\varphi-\delta\varphi$. Then we have

$$\begin{aligned}
&\sum_{i=1,2}\Big(\sum_{j=1,2}a_{ij}^{\delta\lambda}v_{z_j}+b_i^{\delta\lambda}v\Big)_{z_i}\\
&=-\sum_{i=1,2}\Big(\sum_{j=1,2}(a_{ij}^{\delta\lambda}-a_{ij}^{\lambda})(\delta\varphi)_{z_j}+(b_i^{\delta\lambda}-b_i^{\lambda})\delta\varphi+(d_i^{\delta\lambda}-d_i^{\lambda})\delta\lambda\Big)_{z_i}\\
&\equiv E^{\delta\lambda},
\end{aligned} \tag{6.18}$$

with boundary conditions:

$$\begin{aligned}
&v|_{z_2=0}=0,\\
&\sum_{i=1,2}\nu_i^{\delta\lambda}v_{z_i}+c^{\delta\lambda}v=\sum_{i=1,2}(\nu_i^{\lambda}-\nu_i^{\delta\lambda})(\delta\varphi)_{z_i}+(c^{\lambda}-c^{\delta\lambda})\delta\varphi \qquad \text{for}\quad z_1=0.
\end{aligned}$$

Since $\bar\varphi-\varphi=\mathrm{O}(\delta\lambda)$ and $\delta\varphi=\mathrm{O}(\delta\lambda)$, we conclude that $E^{\delta\lambda}$ and $\sum_{i=1,2}(\nu_i^{\lambda}-\nu_i^{\delta\lambda})(\delta\varphi)_{z_i}+(c^{\lambda}-c^{\delta\lambda})\delta\varphi$ are $\mathrm{o}(\delta\lambda)$. Thus, we obtain that

$$v=\mathrm{o}(\delta\lambda).$$

Then

$$\begin{aligned}
&\mathcal{P}(\lambda+\delta\lambda,\omega,\mu)-\mathcal{P}(\lambda,\omega,\mu)-D_\lambda\mathcal{P}(\lambda,\omega,\mu)(\delta\lambda)\\
&:=\tilde{\tilde{\lambda}}-\tilde\lambda-\widetilde{\delta\lambda}\\
&=h(\mathbf{z},\bar\phi,\nabla\bar\phi)-h(\mathbf{z},\phi,\nabla\phi)-\sum_{i=1,2}e_i^{\lambda}(\delta\varphi)_{z_i}-e_0^{\lambda}\delta\varphi\\
&=\sum_{i=1,2}e_i^{\delta\lambda}v_{z_i}+e_0^{\delta\lambda}v+\sum_{i=1,2}(e_i^{\delta\lambda}-e_i^{\lambda})(\delta\varphi)_{z_i}+(e_0^{\delta\lambda}-e_0^{\lambda})\delta\varphi,
\end{aligned}$$

where

$$e_i^{\delta\lambda} = \int_0^1 h_{\varphi_{z_i}}(\mathbf{z}, \varphi + s(\bar{\varphi} - \varphi), \nabla(\varphi + s(\bar{\varphi} - \varphi)))\,\mathrm{d}s,$$
$$e_0^{\delta\lambda} = \int_0^1 h_{\varphi}(\mathbf{z}, \varphi + s(\bar{\varphi} - \varphi), \nabla(\varphi + s(\bar{\varphi} - \varphi)))\,\mathrm{d}s.$$

Therefore, we conclude that $\tilde{\bar{\lambda}} - \tilde{\lambda} - \widetilde{\delta\lambda} = \mathrm{o}(\delta\lambda)$. Thus, $D_\lambda\mathcal{P}(\lambda, \omega, \mu)$ is *the partial differential* of $\mathcal{P}$ with respect to λ at (λ, ω, μ).

3. *Continuity of $\mathcal{P}$ and $D_\lambda\mathcal{P}$.* It suffices to show the continuity of $\mathcal{P}$ at any point $(\lambda^*, \omega^*, \mu^*)$ near $(0, 0, 0)$ by a contradiction argument, since the same argument applies to $D_\lambda\mathcal{P}$.

Assume that there exists a sequence $(\lambda^n, \omega^n, \mu^n) \to (\lambda^*, \omega^*, \mu^*)$ in $X \times Y \times Z$, while $\|\widetilde{\lambda^n} - \widetilde{\lambda^*}\|_X \geq c_0 > 0$. Using the compactness of $\widetilde{\lambda^n}$ in $\|\cdot\|_{\alpha'}$ and the compactness of φ^n in $\|\cdot\|_{1,\alpha'}$ with $\alpha' < \alpha$, we find a subsequence $\{n_k\}$ such that $\widetilde{\lambda^{n_k}}$ converges to some $\widetilde{\lambda^{**}}$ in $\|\cdot\|_{\alpha'}$ and φ^n converges to some φ^{**} in $\|\cdot\|_{1,\alpha'}$. Now we see that φ^* and φ^{**} satisfy the same equation (4.2) with the same boundary conditions. By the uniqueness of solutions for (4.2), we conclude that $\varphi^* = \varphi^{**}$. This implies that $\widetilde{\lambda^*} = \widetilde{\lambda^{**}}$. However, by assumption, $\|\widetilde{\lambda^{**}} - \widetilde{\lambda^*}\|_X \geq c_0 > 0$. This leads to a contradiction. Therefore, $\mathcal{P}$ is continuous.

4. *Show that, at the background state $(\lambda, \omega, \mu) = (0, 0, 0)$, $D_\lambda\mathcal{P}(0, 0, 0) - I$ is an isomorphism.*

When $(\lambda, \omega, \mu) = (0, 0, 0)$, we solve for $\delta\varphi$:

$$\sum_{i=1,2} \Big(\hat{M}_A^i \delta\lambda + \hat{M}_j^i (\delta\varphi)_{z_j}\Big)_{z_i} = 0 \qquad \text{in } Q, \tag{6.19}$$

with the boundary conditions:

$$\sum_{i=1,2} \hat{g}_i (\delta\varphi)_{z_i}|_{z_1=0} = 0, \tag{6.20}$$

$$\delta\varphi|_{z_2=0} = 0, \tag{6.21}$$

where $(\hat{M}_A^i, \hat{M}_j^i, \hat{g}_i)$ are the corresponding $(\overline{M}_A, \overline{M}^i_{\varphi_{z_j}}, \tilde{g}_{\varphi_{z_i}})$ evaluated at the background state (U_-^0, U_+^0, ζ_0).

Then we have

$$D_\lambda\mathcal{P}(\delta\lambda) := \widetilde{\delta\lambda} = \sum_{i=1,2} \hat{h}_i (\delta\varphi)_{z_i},$$

where $\hat{h}_i := h_{\varphi_{z_i}}$ evaluated at the background state.

We rewrite the system in the following way: Let

$$m = \frac{\hat{M}_A^2}{\hat{M}_2^2}, \qquad \overline{\delta\varphi} = \delta\varphi + m\int_0^{z_2} \delta\lambda(s)\,ds.$$

Then (6.19)–(6.21) become

$$\sum_{i,j=1,2} \hat{M}^i_j(\overline{\delta\varphi})_{z_iz_j} = 0 \qquad \text{in } Q, \tag{6.22}$$

$$\sum_{i=1,2} \hat{g}_i(\overline{\delta\varphi})_{z_i}|_{z_1=0} = m\hat{g}_2\delta\lambda, \tag{6.23}$$

$$\overline{\delta\varphi}|_{z_2=0} = 0. \tag{6.24}$$

Then

$$\widetilde{\delta\lambda} = \sum_{i=1,2} \hat{h}_i(\overline{\delta\varphi})_{z_i}(0,z_2) - \hat{h}_2 m\delta\lambda. \tag{6.25}$$

Equations (6.23) and (6.25) give rise to

$$(\hat{g}_2\hat{h}_1 - \hat{g}_1\hat{h}_2)(\overline{\delta\varphi})_{z_1} = \hat{g}_2\widetilde{\delta\lambda}. \tag{6.26}$$

Noticing that $\widetilde{\delta\lambda}(0) = 0$ and $\hat{g}_2\hat{h}_1 - \hat{g}_1\hat{h}_2 \neq 0$, the boundary conditions (6.26) and (6.24) are compatible to guarantee the unique solution of (6.22) for arbitrary $\widetilde{\delta\lambda}$. This implies that $I - D_\lambda\mathcal{P}$ is onto.

When $\delta\lambda - D_\lambda\mathcal{P}(\delta\lambda) = 0$, (6.25) becomes

$$\sum_{i=1,2} \hat{h}_i(\overline{\delta\varphi})_{z_i}(0,z_2) = (1 + \hat{h}_2 m)\delta\lambda. \tag{6.27}$$

Cancelling $\delta\lambda$ in (6.23) and (6.27) implies

$$\Big(\hat{g}_1 + m(\hat{g}_1\hat{h}_2 - \hat{g}_2\hat{h}_1)\Big)(\overline{\delta\varphi})_{z_1} + \hat{g}_2(\overline{\delta\varphi})_{z_2} = 0. \tag{6.28}$$

The oblique boundary condition (6.28) above is nondegenerate, since $\hat{g}_2 \neq 0$. Therefore, solving equation (6.22) with boundary conditions (6.24) and (6.28) leads to $\overline{\delta\varphi} \equiv 0$. Using (6.23), we conclude that $\delta\lambda \equiv 0$, which implies that $I - D_\lambda\mathcal{P}$ is one-to-one. Thus, $I - D_\lambda\mathcal{P}$ is an isomorphism.

Therefore, given U^- and b, operator $\mathcal{P}$ has a fixed point λ, which determines $A = w_t + \lambda$.

With A from λ, we obtain a unique potential φ so that the subsonic flow U can be expressed by (φ, A, B).

This completes the existence part of Theorem 2.1.

7. Decay of the Solution to the Asymptotic State in the Physical Coordinates.

Now we determine the decay of the solution to the asymptotic state U^∞ in the **x**-coordinates. We divide the proof into four steps.

1. For the fixed point established in §5.7,

$$\varphi - \varphi^\infty \in \Sigma_\delta.$$

Then the change of variables from the **z**-coordinates to **y**-coordinates yields

$$\|\hat{\sigma} - \hat{\sigma}_0\|^{*,(-1-\alpha);\{0\}}_{2,\alpha;(\beta);\mathbb{R}^+} + \|\phi - \phi^\infty\|^{(-1-\alpha);\partial\mathcal{W}}_{2,\alpha;(1+\beta);\mathbb{D}_{\hat{\sigma}}} \leq C\delta, \tag{7.1}$$

where $\mathbb{D}_{\hat{\sigma}}$ is the subsonic region defined in (3.12).

2. From (6.4), (7.1), and Step 4 in §6.2, we have

$$\|A^\infty - A_0^+\|^{(-\alpha);\{0\}}_{1,\alpha;(1+\beta);\mathbb{R}^+} \leq C\delta. \tag{7.2}$$

Then, from §5.1, we obtain that, for $U^\infty = U^\infty(y_2) = (\mathbf{u}^\infty, p_0^+, \rho^\infty)(y_2)$,

$$\|U^\infty - U_0^+\|_{1,\alpha;(1+\beta);\mathbb{R}^+} \leq C\delta, \tag{7.3}$$

and

$$\mathbf{u}^\infty \cdot (\sin\theta_0, -\cos\theta_0) = 0. \tag{7.4}$$

3. Since $x_2 = \phi(\mathbf{y})$, we now estimate $\phi(\mathbf{y}) - \tan\theta_0\, y_1$, which is $x_2 - \tan\theta_0\, x_1$. From §5.1 and (7.1),

$$\phi^\infty - \tan\theta_0\, y_1 = l(y_2),$$

so that

$$\|(\phi - \tan\theta_0\, y_1) - l(y_2)\|^{(-1-\alpha);\partial\mathcal{W}}_{2,\alpha;(1+\beta);\mathbb{D}_{\hat{\sigma}}} \le C\delta. \tag{7.5}$$

In particular, this implies that, for each $y_2 > 0$,

$$l'(y_2) = \lim_{y_1\to\infty} \partial_{y_2}(\phi(\mathbf{y}) - \tan\theta_0\, y_1) = \lim_{y_1\to\infty} \phi_{y_2}(\mathbf{y}). \tag{7.6}$$

By (3.13), $\phi_{y_2} = \frac{1}{\rho u_1}$. By (7.2),

$$\phi_{y_2} \ge \frac{1}{\rho u_{10}^+} - C\delta \ge \frac{1}{2\rho u_{10}^+} \qquad \text{for small } \delta > 0.$$

Therefore, there exists $C > 0$ such that, for any $y_2 > 0$,

$$\frac{1}{C} \le l'(y_2) \le C.$$

Since $\psi(\mathbf{0}) = 0$, we conclude that $l(0) = w_0$, where w_0 is the limit of $b - b_0$ (see §5.1).

Furthermore, by (7.5)–(7.6), and (5.1) with (7.2), we have

$$\|l - \frac{y_2}{\rho_0^+ u_{10}^+}\|^{*,(-1-\alpha);\{0\}}_{2,\alpha;(1+\beta);\mathbb{R}^+} \le C\delta. \tag{7.7}$$

Then there exists $g : [w_0, \infty) \to [0, \infty)$ with $g(w_0) = 0$ such that $g = l^{-1}$:

$$g(l(y_2)) = y_2 \qquad \text{on } (0, \infty)$$

and

$$\frac{1}{C} \le g'(s) \le C,$$

so that $g(\cdot)$ satisfies (7.7). Therefore, by (7.5), we have

$$\|g(\phi(\mathbf{y}) - \tan\theta_0\, y_1) - y_2\|^{(-1-\alpha);\partial\mathcal{W}}_{2,\alpha;(1+\beta);\mathbb{D}_{\hat{\sigma}}} \le C\delta.$$

Define

$$V^\infty(s) = U^\infty(g(s)).$$

Then we employ (7.3) to obtain

$$\|V^\infty(\phi(\mathbf{y}) - \tan\theta_0\, y_1) - U^\infty(y_2)\|^{(-\alpha);\partial\mathcal{W}}_{1,\alpha;(1+\beta);\mathbb{D}_{\hat{\sigma}}} \le C\delta. \tag{7.8}$$

4. Next, we use that the change of variables $\mathbf{y} \to \mathbf{x}$ is globally bi-Lipschitz, which follows from (7.1) and (3.13) that implies the Jacobian:

$$J = \frac{1}{\rho^\infty u_1^\infty} \ge \frac{1}{C} > 0$$

if δ is small, by (7.3).

We also note that, in the $\mathbf{y}$-coordinates, (7.1) implies for $U^\infty = U^\infty(y_2)$ that

$$\|U - U^\infty\|^{(-\alpha);\partial\mathcal{W}}_{1,\alpha;(1+\beta);\mathbb{D}_{\hat{\sigma}}} \le C\delta.$$

Then, changing the variables from $\mathbf{y}$ to $\mathbf{x}$ (which is bi-Lipschitz) and using (7.8), we obtain that, in the $\mathbf{x}$-coordinates with $x_2 = \phi(\mathbf{y})$,

$$\|U - V^\infty(x_2 - \tan\theta_0\, x_1)\|^{(-\alpha);\partial\mathcal{W}}_{1,\alpha;(1+\beta);\Omega_{\mathcal{S}}} \le C\delta.$$

This completes the proof for the decay of the solution $U(\mathbf{x})$ to the asymptotic state U^∞.

8. **Stability of Solutions.** In this section, we prove that the subsonic solutions are stable under small perturbations of the incoming flows and the wedges as stated in Theorem 2.1. We modify operator $\mathcal{P}$ into $\overline{\mathcal{P}}$ as follows:

We first modify the definitions of the spaces in (6.4)–(6.7) in §6.2 by discarding the constraints:

$$\overline{X} = \{\lambda : \|\lambda\|_X < \infty\}, \qquad \overline{Z} = \{\mu : \|\mu\|_Z < \infty\},$$

where norms $\|\cdot\|_X$ and $\|\cdot\|_Z$ are defined in (6.4) and (6.8), respectively. We still use the same space Y and the related norm as in (6.5) and (6.6).

Let $\omega = U^- - U_0^-$, $\lambda = A - A_0^+$, and $\mu = b - b_0$. Given (λ, ω, μ), we define $\overline{\mathcal{P}}(\lambda, \omega, \mu)$ in the same way as for $\mathcal{P}$ by the end of §5, except that we do not restrict the value of $A(0)$ by (6.1). The restriction for $A(0)$ is essential for the isomorphism of $D_\lambda \mathcal{P}$. To prove the stability, we need to eliminate this restriction so that the differentiability in ω can be achieved in a larger space.

Equation (4.2) can be written as

$$\sum_{i=1,2} \overline{M}^i(U_-(\varphi, z_2), A_0^+ + \lambda, \nabla\varphi)_{z_i} = 0. \tag{8.1}$$

Given $(\delta\lambda, \delta\omega, \delta\mu) \in \overline{X} \times Y \times \overline{Z}$, define $D\overline{\mathcal{P}}(\delta\lambda, \delta\omega, \delta\mu)$ in the following way: We solve the following equation for $\delta\varphi$:

$$\sum_{i=1,2} \Big(\sum_{j=1,2} a_{ij}^\lambda (\delta\varphi)_{z_j} + b_i^\lambda \delta\varphi + d_i^\lambda \delta\lambda + f_i^\lambda \cdot \delta\omega \Big)_{z_i} = 0 \tag{8.2}$$

with the boundary conditions:

$$\delta\varphi|_{z_2=0} = -\frac{\delta\mu(\tilde{b}) - \delta\phi^-(\tilde{b}, 0)}{b'(\tilde{b}) - (\phi^-)_{y_1}(\tilde{b}, 0)}, \tag{8.3}$$

$$\Big(\sum_{i=1,2} \nu_i^\lambda (\delta\varphi)_{z_i} + c^\lambda \delta\varphi + w^\lambda \cdot \delta\omega \Big)|_{z_1=0} = 0, \tag{8.4}$$

where

$$\begin{aligned}
a_{ij}^\lambda &= \overline{M}^i_{\varphi_{z_i}}(U^-(\varphi, z_2), A_0^+ + \lambda, \varphi, \nabla\varphi), \\
b_i^\lambda &= \overline{M}^i_{U^-}(U^-(\varphi, z_2), A_0^+ + \lambda, \varphi, \nabla\varphi) \cdot (U^-)_{y_1}(\varphi, z_2), \\
d_i^\lambda &= \overline{M}^i_A(U^-(\varphi, z_2), A_0^+ + \lambda, \varphi, \nabla\varphi), \\
f_i^\lambda &= \overline{M}^i_{U^-}(U^-(\varphi, z_2), A_0^+ + \lambda, \varphi, \nabla\varphi), \\
\nu_i^\lambda &= g_{\varphi_{z_i}}(U^-(\varphi, z_2), \nabla\varphi), \\
c^\lambda &= g_{U^-}(U^-(\varphi, z_2), \nabla\varphi) \cdot (U^-)_{y_1}(\varphi, z_2), \\
w^\lambda &= g_{U^-}(U^-(\varphi, z_2), \nabla\varphi).
\end{aligned}$$

Then we define

$$\widetilde{\delta\lambda} = D\overline{\mathcal{P}}(\lambda, \omega, \mu)(\delta\lambda, \lambda\omega, \lambda\mu) := \sum_{i=1,2} e_i^\lambda (\delta\varphi)_{z_i} + e_0^\lambda \delta\varphi + w_1^\lambda \cdot \delta\omega, \tag{8.5}$$

where

$$\begin{aligned} e_i^\lambda &= h_{\varphi_{z_i}}(U^-(\varphi, z_2), \nabla\varphi), \\ e_0^\lambda &= h_{U^-}(U^-(\varphi, z_2), \nabla\varphi)\cdot (U^-)_{y_1}(\varphi, z_2), \\ w_1^\lambda &= h_{U^-}(U^-(\varphi, z_2), \nabla\varphi). \end{aligned}$$

Following the same estimates as in §6.2, we can verify that $D\overline{\mathcal{P}}$ is the differential of $\overline{\mathcal{P}}$.

Let $\mathcal{R} := \overline{\mathcal{P}} - I$. In §6.2, given (ω, μ), we can find the fixed point for A. Therefore, λ is a function of (ω, μ), denoted by $\lambda(\omega, \mu)$. Therefore, we have

$$\mathcal{R}(\lambda(\omega, \mu), \omega, \mu) = 0. \tag{8.6}$$

Suppose that there is another parameter $(\bar\omega, \bar\mu)$ so that

$$\mathcal{R}(\lambda(\bar\omega, \bar\mu), \bar\omega, \bar\mu) = 0. \tag{8.7}$$

Taking the difference of equations (8.6) and (8.7), we have

$$D_\lambda\mathcal{R}(\lambda, \omega, \mu)(\bar\lambda - \lambda) + D_{(\omega,\mu)}(\bar\omega - \omega, \bar\mu - \mu) + \mathrm{o}(\bar\lambda - \lambda, \bar\omega - \omega, \bar\mu - \mu) = 0,$$

where $\bar\lambda = \lambda(\omega, \mu)$. Since $D_\lambda\mathcal{R}(\lambda, \omega, \mu)$ is an isomorphism near the background state, by inverting $D_\lambda\mathcal{R}$, we obtain

$$\bar\lambda - \lambda = -(D_\lambda\mathcal{R})^{-1}D_{(\omega,\mu)}(\bar\omega - \omega, \bar\mu - \mu) + \mathrm{o}(\bar\lambda - \lambda, \bar\omega - \omega, \bar\mu - \mu).$$

Therefore, we obtain the following inequality:

$$\|\bar\lambda - \lambda\|_X \le C\Big(\|\bar\omega - \omega\|_Y + \|\bar\mu - \mu\|_Z\Big), \tag{8.8}$$

which implies the stability of the solutions depending on the perturbation of both the incoming flows and wedge boundaries.

9. Remarks on the Transonic Shock Problem when U_0^+ Is on Arc $\widehat{TH}$.

In this case, $\nu_1^\varphi > 0$ and $\nu_2^\varphi > 0$ in the boundary condition (5.10), which makes a significant difference from the case when U_0^+ is on arc $\widehat{TS}$. Such a difference may affect the estimates, hence the smoothness of the solutions, in general.

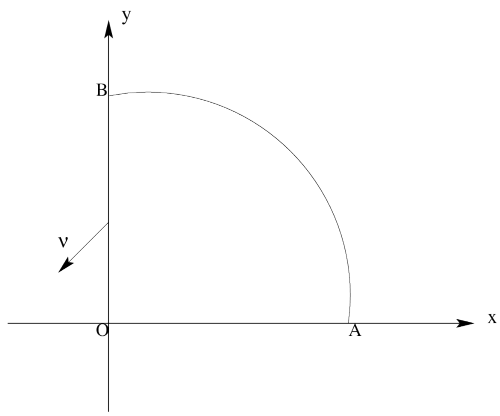
In particular, one may not expect a solution for the case $\widehat{TS}$ is $C^{1,\alpha}$; it is generically only in C^α.

For example, in the first quadrant, let domain OAB be the quarter of the unit disc. The oblique direction $\boldsymbol{\nu} = (-1, -1)$. Let $u = r^{\frac{1}{2}}\sin(\frac{\theta}{2})$, where (r, θ) are the polar coordinates. In OAB, u satisfies the Laplace equation:

$$\Delta u = 0, \tag{9.1}$$

and the boundary conditions: $u = 0$ on OA, and $\nabla u \cdot \boldsymbol{\nu} = 0$ on OB. However, u is Hölder continuous only in $C^{\frac{1}{2}}$.

Therefore, it requires a further understanding of global features of the problem, especially the global relation between the regularity near the origin and the decay of solutions at infinity, to ensure the existence of a smooth solution, more regular than the Hölder continuity. A different approach may be required to handle this case.



Appendix A. **Two Comparison Principles.** In this appendix, we establish two comparison principles.

Suppose that Ω is a bounded, connected, and open set in $\mathbb{R}^n$. Define a uniformly elliptic operator

$$L \equiv \sum_{i=1,2} \partial_{x_i}\Big(\sum_{j=1,2} a_{ij}(\mathbf{x})\partial_{x_j} + b_i(\mathbf{x})\Big) \qquad \text{in } \Omega$$

in the following sense:

$$\sum_{i,j=1,2} a_{ij}(\mathbf{x})\xi_i\xi_j \geq \lambda|\xi|^2 \qquad \text{for any } \mathbf{x} \in \Omega \text{ and } \xi \in \mathbb{R}^n,$$

where λ is a positive constant. Assume that $a_{ij}, b_i \in C^1(\Omega) \cap C(\bar{\Omega})$.

Theorem A.1. *Suppose that $v, w \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy*

(i) *$Lv \geq 0$ and $Lw \leq 0$ in Ω;*
(ii) *$w > 0$ in $\bar{\Omega}$.*

Then $\frac{v}{w}$ achieves its positive maximum on the boundary:

$$\sup_{\Omega}\Big(\frac{v}{w}\Big) \leq \sup_{\partial\Omega}\Big(\frac{v^+}{w}\Big). \tag{A.1}$$

Proof. Let

$$V = \frac{v}{w}, \qquad B_i = 2\sum_{j=1,2} a_{ij}\frac{w_{x_j}}{w} + b_i.$$

By calculation, we have

$$\sum_{i,j=1,2} (a_{ij}V_{x_i})_{x_j} + \sum_{i=1,2} B_i V_{x_i} + \frac{Lw}{w}V = \frac{Lv}{w}. \tag{A.2}$$

By assumption, we know that $\frac{Lw}{w} \leq 0$ and $\frac{Lv}{w} \geq 0$. Therefore, by the weak maximum principle, Theorem 8.1 in [16], we conclude (A.1). □

Theorem A.2. *Suppose that $v, w \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy*

(i) *$Lv \geq Lw$ and $Lw < 0$ in Ω;*
(ii) *$w > 0$ in $\bar{\Omega}$.*

Then $\frac{v}{w}$ achieves its positive maximum on the boundary or no greater than 1 *in* Ω*:*

$$\sup_{\Omega}\Big(\frac{v}{w}\Big) \le \max\Big\{\sup_{\partial\Omega}\Big(\frac{v^+}{w}\Big), 1\Big\}. \tag{A.3}$$

Proof. Equation (A.2) implies

$$\begin{aligned}\sum_{i,j=1,2}(a_{ij}V_{x_i})_{x_j} + \sum_{i=1,2} B_i V_{x_i} &= \frac{Lw}{w}(1-V) + \frac{Lv - Lw}{w}\\ &\ge \frac{Lw}{w}(1-V).\end{aligned} \tag{A.4}$$

Assume that V achieves the maximum value $M > 1$ at some interior point $\mathbf{x}_0 \in \Omega$. Then, by continuity of V, there exists a ball $B_M \equiv B_r(\mathbf{x}_0) \subset \Omega$ such that

$$\sup_{B_M} V = \sup_{\Omega} V = M > 1,$$
$$V > 1 \qquad \text{in } B_M.$$

Therefore,

$$\frac{Lw}{w}(1-V) > 0 \qquad \text{in } B_M,$$

and (A.4) implies

$$\sum_{i,j=1,2}(a_{ij}V_{x_i})_{x_j} + \sum_{i=1,2} B_i V_{x_i} > 0 \qquad \text{in } B_M. \tag{A.5}$$

By the strong maximum principle, Theorem 8.19 in [16], we conclude

$$V \equiv M \qquad \text{in } B_M.$$

This implies that

$$\sum_{i,j=1,2}(a_{ij}V_{x_i})_{x_j} + \sum_{i=1,2} B_i V_{x_i} = 0 \qquad \text{in } B_M,$$

which contradicts (A.5). This completes the proof. □

Appendix B. **The Shock Polar.** We consider the uniform constant transonic flows with horizontal incoming supersonic flows. We now employ the Rankine-Hugoniot conditions (3.6)–(3.9) to derive a criterion for different arcs $\widehat{TS}$ and $\widehat{TH}$ on the shock polar.

Assume that U^- and U are constant supersonic and subsonic states, respectively. The shock-front is a straight line: $y_1 = sy_2$. Let $k = \frac{u_2}{u_1}$ and $k^- = \frac{u_2{}^-}{u_1{}^-} = 0$. Then the Rankine-Hugoniot conditions (3.6)–(3.9) give rise to

$$\Big[\frac{1}{\rho u_1}\Big] = -ks, \tag{B.1}$$

$$\Big[u_1 + \frac{p}{\rho u_1}\Big] = -pks, \tag{B.2}$$

$$u_1 k = [p]s, \tag{B.3}$$

$$\Big[\frac{1}{2}u_1^2(1+k^2) + \frac{\gamma p}{(\gamma-1)\rho}\Big] = 0. \tag{B.4}$$

From (B.3), $s = \frac{u_1 k}{[p]}$. Replacing s in (B.1) and (B.2), we obtain

$$\Big[\frac{1}{\rho u_1}\Big][\,p\,] + u_1 k^2 = 0, \tag{B.5}$$

$$\Big[u_1 + \frac{p}{\rho u_1}\Big][\,p\,] + u_1 p k^2 = 0. \tag{B.6}$$

From equations (B.4)–(B.6), we can solve ρ, u_1, and k in terms of p. Regarding (ρ, u_1, k) as functions of p, we differentiate (B.4)–(B.6) with respect to p to obtain

$$BX = f, \tag{B.7}$$

where

$$B = \begin{pmatrix} -\frac{[p]}{\rho^2 u_1}, & k^2 - \frac{[p]}{\rho u_1^2}, & 2u_1 k \\ -p\frac{[p]}{\rho^2 u_1}, & pk^2 - \frac{p[p]}{\rho u_1^2} + [p], & 2pu_1 k \\ \frac{\gamma p}{(\gamma-1)\rho^2}, & -u_1(k^2+1), & -u_1^2 k \end{pmatrix},$$

$$X = (\rho_p, (u_1)_p, k_p)^\top,$$

$$f = (-[\frac{1}{\rho u_1}], -[u_1 + \frac{p}{\rho u_1}] - \frac{[\,p\,]}{\rho u_1} + u_1 k^2, \frac{\gamma}{(\gamma-1)\rho})^\top.$$

We solve equation (B.7) for k_p to obtain

$$k_p = -\frac{\rho\, C_p}{C_0}, \tag{B.8}$$

where

$$C_p = [p]\Big(c^2 + (\gamma-1)q^2 - \gamma u_1^2\Big) + (\gamma-1)\rho q^2 u_2^2 + \Big[\frac{1}{\rho u_1}\Big]\rho^2 c^2 u_1 q^2, \tag{B.9}$$

$$C_0 = u_1^3 u_2 \rho^2\Big((\gamma+1)p + (\gamma-1)p^-\Big). \tag{B.10}$$

Notice that $C_0 > 0$. Then, when state U belongs to $\widehat{TS}$, we find that $k_p > 0$, which is equivalent to

$$C_p < 0. \tag{B.11}$$

On $\widehat{TH}$, we obtain that $C_p > 0$.

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