



A BRIDGE BETWEEN DUBOVITSKIĬ- FEDERER THEOREMS AND THE COAREA FORMULA

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Abstract

The Morse–Sard theorem requires that a mapping $v: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is of class C^k , $k > \max(n - m, 0)$. In 1957 Dubovitskiĭ generalized this result by proving that almost all level sets for a C^k mapping has \mathcal{H}^s -negligible intersection with its critical set, where $s = \max(n - m - k + 1, 0)$. Here the critical set, or m -critical set is defined as $Z_{v,m} = \{x \in \mathbb{R}^n : \text{rank } \nabla v(x) < m\}$. Another generalization was obtained independently by Dubovitskiĭ and Federer in 1966, namely for C^k mappings $v: \mathbb{R}^n \rightarrow \mathbb{R}^d$ and integers $m \leq d$ they proved that the set of m -critical values $v(Z_{v,m})$ is \mathcal{H}^{q_0} -negligible for $q_0 = m - 1 + \frac{n-m+1}{k}$. They also established the sharpness of these results within the C^k category.

Here we prove that Dubovitskiĭ’s theorem can be generalized to the case of continuous mappings of the Sobolev–Lorentz class $W_{p,1}^k(\mathbb{R}^n, \mathbb{R}^d)$, $p = \frac{n}{k}$ (these are minimal integrability assumptions that guarantees the continuity of mappings). In this situation the mappings need not to be everywhere differentiable and in order to handle the set of nondifferentiability points, we establish for such mappings an analog of the Luzin N -property with respect to lower dimensional Hausdorff content. Finally, we formulate and prove a *bridge theorem* that includes all the above results as particular cases. As a limiting case in this bridge theorem we also establish a new coarea type formula: if $E \subset \{x \in \mathbb{R}^n : \text{rank } \nabla v(x) \leq m\}$, then

$$\int_E J_m v(x) \, dx = \int_{\mathbb{R}^d} \mathcal{H}^{n-m}(E \cap v^{-1}(y)) \, d\mathcal{H}^m(y).$$

The mapping v is \mathbb{R}^d -valued, with arbitrary d , and the formula is obtained without any restrictions on the image $v(\mathbb{R}^n)$ (such as m -rectifiability or σ -finiteness with respect to the m -Hausdorff measure). These last results are new also for smooth mappings, but are presented here in the general Sobolev context.

The proofs of all these results are based on our previous joint papers with J. Bourgain (2013, 2015).

Key words: *Sobolev–Lorentz mappings, Luzin N -property, Morse–Sard theorem, Dubovitskiĭ theorems, Dubovitskiĭ–Federer theorem, Coarea formula*

1 Introduction

The Morse–Sard theorem in its classical form states that the image of the set of critical points of a C^{n-m+1} smooth mapping $v: \mathbb{R}^n \rightarrow \mathbb{R}^m$ has zero Lebesgue measure in \mathbb{R}^m . More precisely, assuming that $n \geq m$, the set of critical points for v is $Z_v = \{x \in \mathbb{R}^n : \text{rank } \nabla v(x) < m\}$ and the conclusion is that

$$\mathcal{L}^m(v(Z_v)) = 0. \quad (1.1)$$

The theorem was proved by Morse [42] in the case $m = 1$ and subsequently by Sard [47] in the general vector-valued case. The celebrated results of Whitney [51] show that the C^{n-m+1} smoothness assumption on the mapping v is sharp. However, the following result gives valuable information also for less smooth mappings.

Theorem A (Dubovitskiĭ 1957 [18]). *Let $n, m, k \in \mathbb{N}$, and let $v: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a C^k -smooth mapping. Put $s = n - m - k + 1$. Then*

$$\mathcal{H}^s(Z_v \cap v^{-1}(y)) = 0 \quad \text{for a.a. } y \in \mathbb{R}^m, \quad (1.2)$$

where \mathcal{H}^s denotes the s -dimensional Hausdorff measure and Z_v is the set of critical points of v .

Here and in the following we interpret \mathcal{H}^β as the counting measure when $\beta \leq 0$. Thus for $k \geq n - m + 1$ we have $s \leq 0$, and \mathcal{H}^s in (1.2) becomes simply the counting measure, so the Dubovitskiĭ theorem contains the Morse–Sard theorem as particular case¹.

A few years later and almost simultaneously, Dubovitskiĭ [19] in 1967 and Federer [23, Theorem 3.4.3] in 1969² published another important generalization of the Morse–Sard theorem.

Theorem B (Dubovitskiĭ–Federer). *For $n, k, d \in \mathbb{N}$ let $m \in \{1, \dots, \min(n, d)\}$ and $v: \mathbb{R}^n \rightarrow \mathbb{R}^d$ be a C^k -smooth mapping. Put $q_\circ = m + \frac{s}{k}$. Then*

$$\mathcal{H}^{q_\circ}(v(Z_{v,m})) = 0. \quad (1.3)$$

where, as above, $s = n - m - k + 1$ and $Z_{v,m}$ denotes the set of m -critical points of v defined as $Z_{v,m} = \{x \in \mathbb{R}^n : \text{rank } \nabla v(x) < m\}$.

In view of the wide range of applicability of the above results it is a natural and compelling problem to decide to what extent they admit extensions to classes of Sobolev mappings. The first Morse–Sard result in the Sobolev context that we are aware of is due to L. De Pascale [16] (though see also [34]). It states that (1.1) holds for mappings v of

¹It is interesting to note that because of the isolation of the former Soviet Union this first Dubovitskiĭ theorem was almost unknown to Western mathematicians; another proof was given in the recent paper [9] covering also some extensions to the case of Hölder spaces.

²Federer announced [22] his result in 1966, this announcement (without any proofs) was sent on 08.02.1966. For the historical details, Dubovitskiĭ sent his paper [19] (with complete proofs) a month earlier, on 10.01.1966.

class $W_{p,\text{loc}}^k(\mathbb{R}^n, \mathbb{R}^m)$ when $k \geq \max(n - m + 1, 2)$ and $p > n$. Note that by the Sobolev embedding theorem any mapping on \mathbb{R}^n which is locally of Sobolev class W_p^k for some $p > n$ is in particular C^{k-1} , so the critical set Z_v can be defined as usual.

In the recent paper [25] P. Hajłasz and S. Zimmermann proved Theorem A under the assumption that $v \in W_{p,\text{loc}}^k(\mathbb{R}^n, \mathbb{R}^m)$, $p > n$, which corresponds to that used by L. De Pascale [16].

In view of the existing counter-examples to Morse–Sard type results in the classical C^k context the issue *is not* the value of k , — that is, how many weak derivatives are needed. Instead the question is, what are the *minimal* integrability assumptions on the weak derivatives for Morse–Sard type results to be valid in the Sobolev case. Of course, it is natural here to restrict attention to continuous mappings, and so to require from the considered function spaces that the inclusion $v \in W_p^k(\mathbb{R}^n, \mathbb{R}^d)$ should guarantee at least the continuity of v (assuming always that the mappings are precisely represented). For values $k \in \{1, \dots, n-1\}$ it is well-known that $v \in W_p^k(\mathbb{R}^n, \mathbb{R}^d)$ is continuous for $p > \frac{n}{k}$ and could be discontinuous for $p \leq \frac{n}{k}$. So **the borderline case** is $p = p_o = \frac{n}{k}$. It is well-known (see for instance [29, 31]) that $v \in W_{p_o}^k(\mathbb{R}^n, \mathbb{R}^d)$ is continuous if the derivatives of k -th order belong to the Lorentz space $L_{p_o,1}$, we will denote the space of such mappings by $W_{p_o,1}^k(\mathbb{R}^n, \mathbb{R}^d)$. We refer to section 3 for relevant definitions and notation.

In [32] it was shown that mappings $v \in W_{p_o,1}^k(\mathbb{R}^n, \mathbb{R}^d)$ are differentiable (in the classical Fréchet–Peano sense) at each point outside some \mathcal{H}^{p_o} -negligible set A_v .³ Thus we define for integers $m \leq \min\{n, d\}$ the m -critical set as

$$Z_{v,m} = \{x \in \mathbb{R}^n \setminus A_v : \text{rank } \nabla v(x) < m\}. \quad (1.4)$$

In previous joint papers of two of the authors with J. Bourgain [12, 13] and in [31, 32] this definition of critical set was used and a corresponding Dubovitskiĭ–Federer Theorem B was established for mappings of Sobolev class $W_{p_o}^k(\mathbb{R}^n, \mathbb{R}^d)$. If, in addition, the highest derivative $\nabla^k v$ belongs to the Lorentz space $L_{p_o,1}$ (in particular, if $k = n$ since $L_{1,1} = L_1$), also the Luzin N -property with respect to the p_o -dimensional Hausdorff content was proven. It implies, in particular, that the image of the set A_v of nondifferentiability points has zero measure, and consequently, C^1 -smoothness of almost all level sets follows.

In this paper we prove the Dubovitskiĭ Theorem A for mappings of the same Sobolev–Lorentz class $W_{p_o,1}^k$ and with values in \mathbb{R}^d for arbitrary $d \geq m$.

Theorem 1.1. *Let $k, m \in \{1, \dots, n\}$, $d \geq m$ and $v \in W_{p_o,1}^k(\mathbb{R}^n, \mathbb{R}^d)$. Then the equality*

$$\mathcal{H}^s(Z_{v,m} \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^m\text{-a.a. } y \in \mathbb{R}^d \quad (1.5)$$

holds, where as above $s = n - m - k + 1$ and $Z_{v,m}$ denotes the set of m -critical points of v : $Z_{v,m} = \{x \in \mathbb{R}^n \setminus A_v : \text{rank } \nabla v(x) < m\}$.

³It was also proven that each point $x \in \mathbb{R}^n \setminus A_v$ is an L_{p_o} -Lebesgue point for the weak gradient ∇v . Note that for mappings of the classical Sobolev space $W_{p_o}^k(\mathbb{R}^n)$ the corresponding exceptional set U has small Bessel capacity $\mathcal{B}_{k-m,p}(U) < \varepsilon$, and, respectively, the gradients $\nabla^m v$ are well-defined in \mathbb{R}^n except for some exceptional set of zero Bessel capacity $\mathcal{B}_{k-m,p}$ (see, e.g., Chapter 3 in [53] or [9]).

To the best of our knowledge the result is new even when the mapping $v: \mathbb{R}^n \rightarrow \mathbb{R}^d$ is of class C^k since we allow here $m < d$ (compare with Theorem A). However, the main thrust of the result is the extension to the Sobolev–Lorentz context that we believe is essentially sharp. In this context we also wish to emphasize that the result is in harmony with our definition of critical set (recall that $\mathcal{H}^{p_0}(A_v) = 0$) and the following new analog of the Luzin N -property:

Theorem 1.2. *Let $k, m \in \{1, \dots, n\}$, $d \geq m$ and $v \in W_{p_0,1}^k(\mathbb{R}^n, \mathbb{R}^d)$. Then for any set A with $\mathcal{H}^{p_0}(A) = 0$ we have*

$$\mathcal{H}^s(A \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^m\text{-a.a. } y \in \mathbb{R}^d, \quad (1.6)$$

where again $s = n - m - k + 1$.

We end this section with remarks about the possibility to localize our results.

Remark 1.1. We have formulated the results in the context of mappings $v: \mathbb{R}^n \rightarrow \mathbb{R}^d$ for mere convenience. However, the reader can easily check that the essence of our results is at the local level and so they also apply to mappings $v: N \rightarrow D$ that are locally of class $W_{p_0,1}^k$ between a second countable n -dimensional smooth manifold N and a d -dimensional smooth manifold D .

Remark 1.2. Since for an open set $U \subset \mathbb{R}^n$ of finite measure the estimate $\|1_U \cdot f\|_{L_{p_0,1}} \leq C_U \|f\|_{L_p(U)}$ holds for $p > p_0$ (see, e.g., [37, Theorem 3.8]), and, consequently,

$$W_p^k(U) \subset W_{p_0,1}^k(U) \subset W_{p_0}^k(U),$$

the results of the above theorems 1.1–1.2 are in particular valid for mappings v that are locally of class W_p^k with $p > p_0 = \frac{n}{k}$.

2 A Bridge between the theorems of Dubovitskiĭ and Federer

Originally, the purpose of the present paper was very concrete: to extend the Dubovitskiĭ Theorem A to the Sobolev context (since the Federer–Dubovitskiĭ Theorem B had been extended before in [31, 32], see Introduction and Subsection 4). But when our paper was finished and ready for submission, the very natural question arose. Theorem A asserts that \mathcal{H}^m -almost all preimages are small (with respect to \mathcal{H}^s -measure), and Theorem B claims that \mathcal{H}^{q_0} -almost all preimages are empty. Could we connect these results? More precisely, could we say something about \mathcal{H}^q -almost all preimages for other values of q , say, for $q \in [m-1, q_0]$? The affirmative answer is contained in the next theorem.

Theorem 2.1. *$k, m \in \{1, \dots, n\}$, $d \geq m$ and $v \in W_{p_0,1}^k(\mathbb{R}^n, \mathbb{R}^d)$. Then for any $q \in (m-1, \infty)$ the equality*

$$\mathcal{H}^{\mu_q}(Z_{v,m} \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^q\text{-a.a. } y \in \mathbb{R}^d \quad (2.1)$$

holds, where

$$\mu_q := s + k(m - q), \quad s = n - m - k + 1, \quad (2.2)$$

and $Z_{v,m}$ again denotes the set of m -critical points of v : $Z_{v,m} = \{x \in \mathbb{R}^n \setminus A_v : \text{rank } \nabla v(x) \leq m - 1\}$.

Let us note, that the behavior of the function μ_q is very natural:

$$\begin{aligned} \mu_q &= 0 && \text{for } q = q_\circ = m - 1 + \frac{n-m+1}{k}; \\ \mu_q &< 0 && \text{for } q > q_\circ; \\ \mu_q &= s && \text{for } q = m; \\ \mu_q &= n - m + 1 && \text{for } q = m - 1. \end{aligned} \quad (2.3)$$

The last value cannot be improved in view of the trivial example of a linear mapping $L: \mathbb{R}^n \rightarrow \mathbb{R}^d$ of rank $m - 1$.

Evidently, Theorem 2.1 contains all the previous theorems (Morse–Sard, A, B, 1.1 and 4.2) as particular cases and it is new even for the smooth case.

We emphasize the fact that in stating Theorem 2.1 we skipped the borderline case $q = m - 1$, $\mu_q = n - m + 1$. Of course, for this case we cannot assert that \mathcal{H}^{m-1} -almost all preimages in the m -critical set $Z_{v,m}$ have zero \mathcal{H}^{n-m+1} -measure as the above mentioned counterexample with a linear mapping $L: \mathbb{R}^n \rightarrow \mathbb{R}^d$ of rank $m - 1$ shows. But for this borderline case we obtain instead the following analog of the classical coarea formula:

Theorem 2.2. *Let $n, d \in \mathbb{N}$, $m \in \{0, \dots, \min(n, d)\}$, and $v \in W_{n,1}^1(\mathbb{R}^n, \mathbb{R}^d)$. Then for any Lebesgue measurable subset E of $Z_{v,m+1} = \{x \in \mathbb{R}^n \setminus A_v : \text{rank } \nabla v(x) \leq m\}$ we have*

$$\int_E J_m v(x) \, dx = \int_{\mathbb{R}^d} \mathcal{H}^{n-m}(E \cap v^{-1}(y)) \, d\mathcal{H}^m(y), \quad (2.4)$$

where $J_m v(x)$ denotes the m -Jacobian of v defined as the product of the m largest singular values of the matrix $\nabla v(x)$.

The proof relies crucially on the results of [44] and [27] that give criteria for the validity of the coarea formula for Lipschitz mappings between metric spaces, see also [6] and [38, 39].

Thus, to study the level sets for the borderline case $q = m - 1$ in Theorem 2.1, one must take $m' = m - 1$ instead of m in Theorem 2.2.

Remark 2.1. Note that for the case $m = n$ the formula (2.4) corresponds to the *area formula* whose validity for Sobolev mappings supporting the N -property is well-known (see, e.g., [35] and [29], where the N -property was established for mappings of class $W_{n,1}^1$).

But for $m < n$ the result is new even for smooth mappings, since usually the formula (2.4) is proved under the assumption $d = m$ (see, e.g., [39] for Sobolev functions $W_p^1(\mathbb{R}^n, \mathbb{R}^m)$) or, when $m < d$, under the assumption that the image $v(E)$ is a \mathcal{H}^m - σ -finite set (e.g., [44], [27], see also Theorem 5.1 of the present paper).

From the Coarea formula (2.4) it follows directly, that the set of $y \in \mathbb{R}^d$ where the integrand in the right-hand side of (2.4) is positive, is \mathcal{H}^m - σ -finite. Indeed, from Theorem 2.2 and [27, Theorem 1.3] we obtain immediately the following more precise statement:

Corollary 2.1. *Let $m \in \{0, \dots, \min(d, n)\}$ and $v \in W_{n,1}^1(\mathbb{R}^n, \mathbb{R}^d)$. Then the set*

$$\left\{ y \in \mathbb{R}^d : \mathcal{H}^{n-m}(Z_{v,m+1} \cap v^{-1}(y)) > 0 \right\}$$

is \mathcal{H}^m -rectifiable, i.e., it is a union of a set of \mathcal{H}^m -measure zero and a countable family of images $g_i(S_i)$ of Lipschitz mappings $g_i: S_i \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^d$. Here again $Z_{v,m+1} = \{x \in \mathbb{R}^n \setminus A_v : \text{rank } \nabla v(x) \leq m\}$.

Remark 2.2. In view of the embedding $W_{p_o,1}^k(\mathbb{R}^n) \hookrightarrow W_{n,1}^1(\mathbb{R}^n)$ for $k \in \{1, \dots, n\}$, $p_o = \frac{n}{k}$ (see, e.g., [37, §8]), the assertions of Theorem 2.2 and Corollary 2.1 are in particular valid for the mappings $v \in W_{p_o,1}^k(\mathbb{R}^n, \mathbb{R}^d)$, i.e., under the conditions of Theorem 2.1.

Again Theorems 2.1 and 2.2 are in harmony with our definition of critical set (recall that $\mathcal{H}^{p_o}(A_v) = 0$) because of the following analog of the Luzin N -property:

Theorem 2.3. *Let $k \in \{1, \dots, n\}$, $p_o = n/k$ and $v \in W_{p_o,1}^k(\mathbb{R}^n, \mathbb{R}^d)$. Then for every $p \in [p_o, n]$, $q \in [0, p]$ and for any set $E \subset \mathbb{R}^n$ with $H^p(E) = 0$ we have*

$$\mathcal{H}^{p-q}(E \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^q\text{-a.a. } y \in \mathbb{R}^d. \quad (2.5)$$

In particular,

$$\mathcal{H}^p(v(E)) = 0 \quad \text{whenever } \mathcal{H}^p(E) = 0, \quad p \in [p_o, n]. \quad (2.6)$$

By simple calculation we have for $q \in [0, q_o]$ that

$$\mu_q = n - m - k + 1 + k(m - q) = (p_o - q)k + (m - 1)(k - 1) \geq \max(p_o - q, 0). \quad (2.7)$$

Theorem 2.3 then yields

Corollary 2.2. *Let $k, m \in \{1, \dots, n\}$ and $v \in W_{p_o,1}^k(\mathbb{R}^n, \mathbb{R}^d)$. Then for every $q \in [0, +\infty)$ and for any set E with $\mathcal{H}^{p_o}(E) = 0$ we have*

$$\mathcal{H}^{\mu_q}(E \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^q\text{-a.a. } y \in \mathbb{R}^d. \quad (2.8)$$

Consequently, for every $q \in [0, +\infty)$

$$\mathcal{H}^{\mu_q}(A_v \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^q\text{-a.a. } y \in \mathbb{R}^d, \quad (2.9)$$

where we recall that A_v is the set of nondifferentiability points of v (cf. with (2.1)).

Finally, applying the N -property (Theorem 2.3) for $p = n$, $q = m \leq n$, we obtain

Corollary 2.3. *Let $n, d \in \mathbb{N}$, $m \in [0, n]$, and $v \in W_{n,1}^1(\mathbb{R}^n, \mathbb{R}^d)$. Then for any set E of zero n -Lebesgue measure $\mathcal{L}^n(E) = 0$ the identity*

$$\mathcal{H}^{n-m}(E \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^m\text{-a.a. } y \in \mathbb{R}^d \quad (2.10)$$

holds.

Thus the sets of n -Lebesgue measure zero (in particular, the set of nondifferentiability points A_v) are negligible in the Coarea formula (2.4).

Finally, let us comment briefly on the proofs that merge ideas from our previous papers [13], [31, 32] and [25]. In particular, the joint papers [12, 13] by two of the authors with J. Bourgain contain many of the key ideas that allow us to consider nondifferentiable Sobolev mappings. For the implementation of these ideas one relies on estimates for the Hardy–Littlewood maximal function in terms of Choquet type integrals with respect to Hausdorff capacity. In order to take full advantage of the Lorentz context we exploit the recent estimates from [32] (recalled in Theorem 3.1 below, see also [1] for the case $p = 1$). As in [13] (and subsequently in [31]) we also crucially use Y. Yomdin’s (see [52]) entropy estimates of near critical values for polynomials (recalled in Theorem 3.2 below).

In addition to the above mentioned papers there is a growing number of papers on the topic, including [5, 7, 8, 9, 14, 24, 26, 41, 43, 45, 49, 50].

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3 Preliminaries

By an n -dimensional interval we mean a closed cube in \mathbb{R}^n with sides parallel to the coordinate axes. If Q is an n -dimensional cubic interval then we write $\ell(Q)$ for its sidelength.

For a subset S of \mathbb{R}^n we write $\mathcal{L}^n(S)$ for its outer Lebesgue measure. The m -dimensional Hausdorff measure is denoted by \mathcal{H}^m and the m -dimensional Hausdorff content by \mathcal{H}_α^m . Recall that for any subset S of \mathbb{R}^n we have by definition

$$\mathcal{H}^m(S) = \lim_{\alpha \searrow 0} \mathcal{H}_\alpha^m(S) = \sup_{\alpha > 0} \mathcal{H}_\alpha^m(S),$$

where for each $0 < \alpha \leq \infty$,

$$\mathcal{H}_\alpha^m(S) = \inf \left\{ \sum_{i=1}^{\infty} (\text{diam } S_i)^m : \text{diam } S_i \leq \alpha, \quad S \subset \bigcup_{i=1}^{\infty} S_i \right\}.$$

It is well known that $\mathcal{H}^n(S) = \mathcal{H}_\infty^n(S) \sim \mathcal{L}^n(S)$ for sets $S \subset \mathbb{R}^n$.

To simplify the notation, we write $\|f\|_{L_p}$ instead of $\|f\|_{L_p(\mathbb{R}^n)}$, etc.

The Sobolev space $W_p^k(\mathbb{R}^n, \mathbb{R}^d)$ is as usual defined as consisting of those \mathbb{R}^d -valued functions $f \in L_p(\mathbb{R}^n)$ whose distributional partial derivatives of orders $l \leq k$ belong to

$L_p(\mathbb{R}^n)$ (for detailed definitions and differentiability properties of such functions see, e.g., [20], [40], [53], [17]). Denote by $\nabla^k f$ the vector-valued function consisting of all k -th order partial derivatives of f arranged in some fixed order. However, for the case of first order derivatives $k = 1$ we shall often think of $\nabla f(x)$ as the Jacobi matrix of f at x , thus the $d \times n$ matrix whose r -th row is the vector of partial derivatives of the r -th coordinate function.

We use the norm

$$\|f\|_{W_p^k} = \|f\|_{L_p} + \|\nabla f\|_{L_p} + \cdots + \|\nabla^k f\|_{L_p},$$

and unless otherwise specified all norms on the spaces \mathbb{R}^s ($s \in \mathbb{N}$) will be the usual euclidean norms.

Working with locally integrable functions, we always assume that the precise representatives are chosen. If $w \in L_{1,\text{loc}}(\Omega)$, then the precise representative w^* is defined for all $x \in \Omega$ by

$$w^*(x) = \begin{cases} \lim_{r \searrow 0} \oint_{B(x,r)} w(z) \, dz, & \text{if the limit exists and is finite,} \\ 0 & \text{otherwise,} \end{cases} \quad (3.1)$$

where the dashed integral as usual denotes the integral mean,

$$\oint_{B(x,r)} w(z) \, dz = \frac{1}{\mathcal{L}^n(B(x,r))} \int_{B(x,r)} w(z) \, dz,$$

and $B(x, r) = \{y : |y - x| < r\}$ is the open ball of radius r centered at x . Henceforth we omit special notation for the precise representative writing simply $w^* = w$.

We will say that x is an L_p -Lebesgue point of w (and simply a Lebesgue point when $p = 1$), if

$$\oint_{B(x,r)} |w(z) - w(x)|^p \, dz \rightarrow 0 \quad \text{as } r \searrow 0.$$

If $k < n$, then it is well-known that functions from Sobolev spaces $W_p^k(\mathbb{R}^n)$ are continuous for $p > \frac{n}{k}$ and could be discontinuous for $p \leq p_o = \frac{n}{k}$ (see, e.g., [40, 53]). The Sobolev–Lorentz space $W_{p_o,1}^k(\mathbb{R}^n) \subset W_{p_o}^k(\mathbb{R}^n)$ is a refinement of the corresponding Sobolev space that for our purposes turns out to be convenient. Among other things functions that are locally in $W_{p_o,1}^k$ on \mathbb{R}^n are in particular continuous.

Here we shall mainly be concerned with the Lorentz space $L_{p,1}$, and in this case one may rewrite the norm as (see for instance [37, Proposition 3.6])

$$\|f\|_{p,1} = \int_0^{+\infty} [\mathcal{L}^n(\{x \in \mathbb{R}^n : |f(x)| > t\})]^{\frac{1}{p}} \, dt. \quad (3.2)$$

We record the following subadditivity property of the Lorentz norm for later use.

Lemma 3.1 (see, e.g., [46] or [37]). Suppose that $1 \leq p < \infty$ and $E = \bigcup_{j \in \mathbb{N}} E_j$, where E_j are measurable and mutually disjoint subsets of \mathbb{R}^n . Then for all $f \in L_{p,1}$ we have

$$\sum_j \|f \cdot 1_{E_j}\|_{L_{p,1}}^p \leq \|f \cdot 1_E\|_{L_{p,1}}^p,$$

where 1_E denotes the indicator function of the set E .

Denote by $W_{p,1}^k(\mathbb{R}^n)$ the space of all functions $v \in W_p^k(\mathbb{R}^n)$ such that in addition the Lorentz norm $\|\nabla^k v\|_{L_{p,1}}$ is finite.

For a mapping $u \in L_1(Q, \mathbb{R}^d)$, $Q \subset \mathbb{R}^n$, $m \in \mathbb{N}$, define the polynomial $P_{Q,m}[u]$ of degree at most m by the following rule:

$$\int_Q y^\alpha (u(y) - P_{Q,m}[u](y)) \, dy = 0 \quad (3.3)$$

for any multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ of length $|\alpha| = \alpha_1 + \dots + \alpha_n \leq m$. Denote $P_Q[u] = P_{Q,k-1}[u]$.

The following well-known bound will be used on several occasions.

Lemma 3.2 (see, e.g., [32]). Suppose $v \in W_{p_0,1}^k(\mathbb{R}^n, \mathbb{R}^d)$ with $k \in \{1, \dots, n\}$. Then v is a continuous mapping and for any n -dimensional cubic interval $Q \subset \mathbb{R}^n$ the estimate

$$\sup_{y \in Q} |v(y) - P_Q[v](y)| \leq C \|1_Q \cdot \nabla^k v\|_{L_{p_0,1}} \quad (3.4)$$

holds, where C is a constant depending on n, d only. Moreover, the mapping $v_Q(y) = v(y) - P_Q[v](y)$, $y \in Q$, can be extended from Q to the whole of \mathbb{R}^n such that the extension (denoted again) $v_Q \in W_{p_0,1}^k(\mathbb{R}^n, \mathbb{R}^d)$ and

$$\|\nabla^k v_Q\|_{L_{p_0,1}(\mathbb{R}^n)} \leq C_0 \|1_Q \cdot \nabla^k v\|_{L_{p_0,1}}, \quad (3.5)$$

where C_0 also depends on n, d only.

Corollary 3.1 (see, e.g., [31]). Suppose $v \in W_{p_0,1}^k(\mathbb{R}^n, \mathbb{R}^d)$ with $k \in \{1, \dots, n\}$. Then v is a continuous mapping and for any n -dimensional cubic interval $Q \subset \mathbb{R}^n$ the estimate

$$\text{diam } v(Q) \leq C \left(\frac{\|\nabla v\|_{L_{p_0}(Q)}}{\ell(Q)^{k-1}} + \|1_Q \cdot \nabla^k v\|_{L_{p_0,1}} \right) \leq C \left(\frac{\|\nabla v\|_{L_p(Q)}}{\ell(Q)^{\frac{n}{p}-1}} + \|1_Q \cdot \nabla^k v\|_{L_{p_0,1}} \right) \quad (3.6)$$

holds for every $p \in [p_0, n]$.

The above results can easily be adapted to give that $v \in C_0(\mathbb{R}^n)$, the space of continuous functions on \mathbb{R}^n that vanish at infinity (see for instance [37, Theorem 5.5]).

Let \mathcal{M}^β be the space of all nonnegative Borel measures μ on \mathbb{R}^n such that

$$\|\mu\|_\beta = \sup_{I \subset \mathbb{R}^n} \ell(I)^{-\beta} \mu(I) < \infty, \quad (3.7)$$

where the supremum is taken over all n -dimensional cubic intervals $I \subset \mathbb{R}^n$ and $\ell(I)$ denotes side-length of I . We need the following important strong-type estimates for maximal functions (it was proved in [32] based on classic results of D.R. Adams [1] and some new analog of the trace theorem for Riesz potentials of Lorentz functions for the limiting case $q = p$, see Theorems 0.2–0.4 and Corollary 2.1 in [32]).

Theorem 3.1 ([32]). *Let $p \in (1, \infty)$, $k, l \in \{1, \dots, n\}$, $l \leq k$, $(k - l)p < n$. Then for any function $f \in W_{p,1}^k(\mathbb{R}^n)$ the estimates*

$$\|\nabla^l f\|_{L_p(\mu)}^p \leq C \|\mu\|_\beta \|\nabla^k f\|_{L_{p,1}}^p \quad \forall \mu \in \mathcal{M}^\beta, \quad (3.8)$$

$$\int_0^\infty \mathcal{H}_\infty^\beta(\{x \in \mathbb{R}^n : \mathcal{M}(|\nabla^l f|^p)(x) \geq t\}) dt \leq C \|\nabla^k f\|_{L_{p,1}}^p \quad (3.9)$$

hold, where $\beta = n - (k - l)p$, the constant C depends on n, k, p only, and

$$\mathcal{M}f(x) = \sup_{r>0} \int_{B(x,r)} |f(y)| dy$$

is the usual Hardy–Littlewood maximal function of f .

The result is true also for $p = 1$, $k > l$ and is in this case due to D. Adams [1].

For a subset A of \mathbb{R}^m and $\varepsilon > 0$ the ε -entropy of A , denoted by $\text{Ent}(\varepsilon, A)$, is the minimal number of closed balls of radius ε covering A . Further, for a linear map $L: \mathbb{R}^n \rightarrow \mathbb{R}^d$ we denote by $\lambda_j(L)$, $j = 1, \dots, d$, its singular values arranged in decreasing order: $\lambda_1(L) \geq \lambda_2(L) \geq \dots \geq \lambda_d(L)$. Geometrically the singular values are the lengths of the semiaxes of the ellipsoid $L(\partial B(0, 1))$. We recall that the singular values of L coincide with the eigenvalues repeated according to multiplicity of the symmetric nonnegative linear map $\sqrt{LL^*}: \mathbb{R}^d \rightarrow \mathbb{R}^d$. Also for a mapping $f: \mathbb{R}^n \rightarrow \mathbb{R}^d$ that is approximately differentiable at $x \in \mathbb{R}^n$ put $\lambda_j(f, x) = \lambda_j(d_x f)$, where by $d_x f$ we denote the approximate differential of f at x . The next result is the second basic ingredient of our proof.

Theorem 3.2 ([52]). *For any polynomial $P: \mathbb{R}^n \rightarrow \mathbb{R}^d$ of degree at most k , for each ball $B \subset \mathbb{R}^n$ of radius $r > 0$, and any number $\varepsilon > 0$ we have that*

$$\begin{aligned} \text{Ent}(\varepsilon r, \{P(x) : x \in B, \lambda_1 \leq 1 + \varepsilon, \dots, \lambda_{m-1} \leq 1 + \varepsilon, \lambda_m \leq \varepsilon, \dots, \lambda_d \leq \varepsilon\}) \\ \leq C_Y (1 + \varepsilon^{1-m}), \end{aligned}$$

where the constant C_Y depends on n, d, k, m only and for brevity we wrote $\lambda_j = \lambda_j(P, x)$.

The application of Theorem 3.1 is facilitated through the following simple estimate (see for instance Lemma 2 in [17], cf. with [11]).

Lemma 3.3. *Let $u \in W_1^1(\mathbb{R}^n, \mathbb{R}^d)$. Then for any ball $B \subset \mathbb{R}^n$ of radius $r > 0$ and for any number $\varepsilon > 0$ the estimate*

$$\text{diam}(\{u(x) : x \in B, (\mathcal{M}\nabla u)(x) \leq \varepsilon\}) \leq C_M \varepsilon r$$

holds, where C_M is a constant depending on n, d only.

We need also the following approximation result.

Theorem 3.3 (see Theorem 2.1 in [32]). *Let $p \in (1, \infty)$, $k, l \in \{1, \dots, n\}$, $l \leq k$, $(k - l)p < n$. Then for any $f \in W_{p,1}^k(\mathbb{R}^n)$ and for each $\varepsilon > 0$ there exist an open set $U \subset \mathbb{R}^n$ and a function $g \in C^l(\mathbb{R}^n)$ such that*

- (i) $\mathcal{H}_\infty^{n-(k-l)p}(U) < \varepsilon$;
- (ii) each point $x \in \mathbb{R}^n \setminus U$ is an L_p -Lebesgue point for $\nabla^j f$, $j = 0, \dots, l$;
- (iii) $f \equiv g$, $\nabla^j f \equiv \nabla^j g$ on $\mathbb{R}^n \setminus U$ for $j = 1, \dots, l$.

Note that in the analogous theorem for the case of Sobolev mappings $f \in W_p^k(\mathbb{R}^n)$ the assertion (i) should be reformulated as follows:

$$(i') \mathcal{B}_{k-l,p}(U) < \varepsilon \text{ if } l < k,$$

where $\mathcal{B}_{\alpha,p}(U)$ denotes the Bessel capacity of the set U (see, e.g., Chapter 3 in [53] or [9]).

Recall that for $1 < p < \infty$ and $0 < n - \alpha p < n$ the smallness of $\mathcal{H}_\infty^{n-\alpha p}(U)$ implies the smallness of $\mathcal{B}_{\alpha,p}(U)$, but that the opposite is false since $\mathcal{B}_{\alpha,p}(U) = 0$ whenever $\mathcal{H}_\infty^{n-\alpha p}(U) < \infty$. On the other hand, for $1 < p < \infty$ and $0 < n - \alpha p < \beta \leq n$ the smallness of $\mathcal{B}_{\alpha,p}(U)$ implies the smallness of $\mathcal{H}_\infty^\beta(U)$ (see, e.g., [4]). So the usual assertion (i') is essentially weaker than (i).

4 Luzin N - and Morse–Sard properties for Sobolev–Lorentz mappings

In this section we briefly recall some theorems from [31, 32] which we need. The following result is an analog of the Luzin N -property with respect to the Hausdorff content.

Theorem 4.1 ([31, 32]). *Let $k \in \{1, \dots, n\}$, $q \in [p_\circ, n]$, and $v \in W_{p_\circ,1}^k(\mathbb{R}^n, \mathbb{R}^d)$. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any set $E \subset \mathbb{R}^n$ if $\mathcal{H}_\infty^q(E) < \delta$, then $\mathcal{H}_\infty^q(v(E)) < \varepsilon$. In particular, $\mathcal{H}^q(v(E)) = 0$ whenever $\mathcal{H}^q(E) = 0$.*

The next assertion is the precise analog of the Dubovitskiĭ–Federer theorem B (see Introduction 1) which includes the Morse–Sard result.

Theorem 4.2 ([31, 32]). *If $k, m \in \{1, \dots, n\}$, Ω is an open subset of \mathbb{R}^n , and $v \in W_{p_\circ,1,\text{loc}}^k(\Omega, \mathbb{R}^d)$, then $\mathcal{H}^{q_\circ}(v(Z_{v,m})) = 0$.*

Recall that in our notation

$$p_\circ = \frac{n}{k}, \quad s = n - m - k + 1, \quad q_\circ = m + \frac{s}{k} = p_\circ + (m - 1)(1 - k^{-1}), \quad (4.1)$$

and $Z_{v,m} = \{x \in \Omega : \text{rank } \nabla v(x) < m\}$.

Finally, here we recall some differentiability properties of Sobolev–Lorentz functions.

Theorem 4.3 ([31, 32]). *Let $k \in \{1, \dots, n\}$ and $v \in W_{p_o, 1}^k(\mathbb{R}^n, \mathbb{R}^d)$. Then there exists a Borel set $A_v \subset \mathbb{R}^n$ such that $\mathcal{H}^{p_o}(A_v) = 0$ and for any $x \in \mathbb{R}^n \setminus A_v$ the function v is differentiable (in the classical Fréchet sense) at x , furthermore, the classical derivative coincides with $\nabla v(x)$ (x is a L_{p_o} -Lebesgue point for ∇v).*

The case $k = 1$, $p_o = n$ is a classical result due to Stein [48] (see also [29]), and for $k = n$, $p_o = 1$ the result is due to Dorronsoro [17].

Theorem 4.3 admits the following generalization.

Theorem 4.4 ([31, 32]). *Let $k, l \in \{1, \dots, n\}$, $l \leq k$, and $v \in W_{p_o, 1}^k(\mathbb{R}^n, \mathbb{R}^d)$. Then there exists a Borel set $A_{v,l} \subset \mathbb{R}^n$ such that $\mathcal{H}^{l p_o}(A_{v,l}) = 0$ and for any $x \in \mathbb{R}^n \setminus A_{v,l}$ the function v is l -times differentiable (in the classical Fréchet–Peano sense) at x , i.e.,*

$$\lim_{r \searrow 0} \sup_{y \in B(x, r) \setminus \{x\}} \frac{|v(y) - T_{v,l,x}(y)|}{|x - y|^l} = 0,$$

where $T_{v,l,x}(y)$ is the Taylor polynomial of order l for v centered at x (which is well defined $\mathcal{H}^{l p_o}$ -a.e. by Theorem 4.3).

5 Proofs of the main results

5.1 Proof of the Luzin type N -property

In this subsection we are going to prove Theorem 2.3 and as a consequence Theorem 1.2. Now fix $n \in \mathbb{N}$, $k \in \{1, \dots, n\}$, $p \in [p_o, n]$ and $q \in [0, p]$. Denote in this subsection

$$\mu = p - q. \tag{5.1}$$

Fix also a mapping $v \in W_{p_o, 1}^k(\mathbb{R}^n, \mathbb{R}^d)$. For a set $E \subset \mathbb{R}^n$ define the set function

$$\Phi(E) = \inf_{E \subset \bigcup_{\alpha} D_{\alpha}} \sum_{\alpha} (\text{diam } D_{\alpha})^{\mu} [\text{diam } v(D_{\alpha})]^q, \tag{5.2}$$

where the infimum is taken over all countable families of compact sets $\{D_{\alpha}\}_{\alpha \in \mathbb{N}}$ such that $E \subset \bigcup_{\alpha} D_{\alpha}$. By Theorem 6.1 (see Appendix), $\Phi(\cdot)$ is a countably subadditive set-function with the property

$$\Phi(E) = 0 \Rightarrow \left[\mathcal{H}^{\mu}(E \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^q\text{-almost all } y \in \mathbb{R}^d \right]. \tag{5.3}$$

Thus the assertion of Theorem 2.3 amounts to

$$\Phi(E) = 0 \quad \text{whenever } \mathcal{H}^p(E) = 0. \tag{5.4}$$

The proof of this follows the ideas of [31].

By a **dyadic interval** we understand a cubic interval of the form $[\frac{k_1}{2^l}, \frac{k_1+1}{2^l}] \times \dots \times [\frac{k_n}{2^l}, \frac{k_n+1}{2^l}]$, where k_i, l are integers. The following assertion is straightforward, and hence we omit its proof here.

Lemma 5.1. *For any n -dimensional cubic interval $J \subset \mathbb{R}^n$ there exist dyadic intervals Q_1, \dots, Q_{2^n} such that $J \subset Q_1 \cup \dots \cup Q_{2^n}$ and $\ell(Q_1) = \dots = \ell(Q_{2^n}) \leq 2\ell(J)$.*

Let $\{Q_\alpha\}_{\alpha \in A}$ be a family of n -dimensional dyadic intervals. We say that the family $\{Q_\alpha\}$ is **regular**, if for any n -dimensional dyadic interval Q the estimate

$$\ell(Q)^p \geq \sum_{\alpha: Q_\alpha \subset Q} \ell(Q_\alpha)^p \quad (5.5)$$

holds. Since dyadic intervals are either nonoverlapping or contained in one another, (5.5) implies that any regular family $\{Q_\alpha\}$ must in particular consist of nonoverlapping intervals.

Lemma 5.2 (see Lemma 2.3 in [13]). *Let $\{Q_\alpha\}$ be a family of n -dimensional dyadic intervals. Then there exists a regular family $\{J_\beta\}$ of n -dimensional dyadic intervals such that $\bigcup_\alpha Q_\alpha \subset \bigcup_\beta J_\beta$ and*

$$\sum_\beta \ell(J_\beta)^p \leq \sum_\alpha \ell(Q_\alpha)^p.$$

Lemma 5.3 (see Lemma 3.4 in [32] and Lemma 2.4 in [31]). *Let $v \in W_{p_0,1}^k(\mathbb{R}^n, \mathbb{R}^d)$. For each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, v) > 0$ such that for any regular family $\{Q_\alpha\}$ of n -dimensional dyadic intervals we have if*

$$\sum_\alpha \ell(Q_\alpha)^p < \delta, \quad (5.6)$$

then

$$\sum_\alpha \left[\|1_{Q_\alpha} \cdot \nabla^k v\|_{L_{p_0,1}}^p + \frac{1}{\ell(Q_\alpha)^{n-p}} \int_{Q_\alpha} |\nabla v|^p \right] < \varepsilon. \quad (5.7)$$

Proof of Theorem 2.3. Let $\mathcal{H}^p(E) = 0$. Take $\varepsilon > 0$ and $\delta = \delta(\varepsilon, v) < 1$ from Lemma 5.3. Take also the regular family $\{Q_\alpha\}$ of n -dimensional dyadic intervals such that $E \subset \bigcup_\alpha Q_\alpha$ and

$$\sum_\alpha \ell(Q_\alpha)^p < \delta \quad (5.8)$$

where the existence of such family follows directly from Lemmas 5.1 and 5.2. Then by Lemma 5.3 the estimate (5.7) holds. Denote $r_\alpha = \ell(Q_\alpha)$. By estimate (3.6),

$$[\text{diam } v(Q_\alpha)]^q \leq C \left(\frac{\|\nabla v\|_{L_p(Q_\alpha)}^q}{r_\alpha^{\frac{(n-1)q}{p}}} + \|1_{Q_\alpha} \cdot \nabla^k v\|_{L_{p_0,1}}^q \right). \quad (5.9)$$

Therefore, by definition of $\Phi(E)$ (see (5.2)), we have

$$\begin{aligned}
\Phi(E) &\leq C \sum_{\alpha} r_{\alpha}^{\mu} \left(\frac{\|\nabla v\|_{L_p(Q_{\alpha})}^q}{r_{\alpha}^{\frac{(n-p-1)q}{p}}} + \|1_{Q_{\alpha}} \cdot \nabla^k v\|_{L_{p_0,1}}^q \right) \\
&\stackrel{\text{H\"older ineq.}}{\leq} c \left(\sum_{\alpha} r_{\alpha}^{\frac{\mu p}{p-q}} \right)^{\frac{p-q}{p}} \cdot \left[\sum_{\alpha} \left(\frac{1}{\ell(Q_{\alpha})^{n-p}} \int_{Q_{\alpha}} |\nabla v|^p + \|1_{Q_{\alpha}} \cdot \nabla^k v\|_{L_{p_0,1}}^p \right) \right]^{\frac{q}{p}} \\
&\stackrel{(5.1), (5.7)}{\leq} c \left(\sum_{\alpha} r_{\alpha}^p \right)^{\frac{p-q}{p}} \cdot \varepsilon^{\frac{q}{p}} \\
&\stackrel{(5.8)}{\leq} c \delta^{\frac{p-q}{p}} \cdot \varepsilon^{\frac{q}{p}}.
\end{aligned} \tag{5.10}$$

Since $\varepsilon > 0$ and $\delta > 0$ are arbitrary small, (5.10) turns to the equality $\Phi(E) = 0$ and by (5.3) the required assertion is proved. \square

Remark 5.1. Note that the regularity assumptions in the last theorem are sharp: for example, the Luzin N -property fails in general for continuous mappings $v \in W_n^1(\mathbb{R}^n, \mathbb{R}^n)$ (here $k = 1$, $p_0 = p = n = q$, $\mu = 0$), see, e.g., [36]. The sharpness of our assumptions for general order Sobolev spaces, though not on the Sobolev–Lorentz scale, is also a consequence of the recent and interesting results in [26]. See also [28] for earlier results in this direction.

5.2 Dubovitskiĭ theorem for Sobolev mappings

Fix integers $k, m \in \{1, \dots, n\}$, $d \geq m$ and a mapping $v \in W_{p_0,1}^k(\mathbb{R}^n, \mathbb{R}^d)$. Then, by Theorem 4.3, there exists a Borel set A_v such that $\mathcal{H}^{p_0}(A_v) = 0$ and all points of the complement $\mathbb{R}^n \setminus A_v$ are L_{p_0} -Lebesgue points for the weak gradient ∇v . Moreover, we can arrange that v is differentiable (in the classical Fréchet sense) at every point $x \in \mathbb{R}^n \setminus A_v$ with derivative $\nabla v(x)$ (so the classical derivative coincides with the precise representative of the weak gradient at x).

Denote $Z_{v,m} = \{x \in \Omega \setminus A_v : \text{rank } \nabla v(x) < m\}$. Fix a number

$$q \in [m-1, q_0).$$

Denote in this subsection

$$\mu = \mu_q = n - m - k + 1 + (m - q)k. \tag{5.11}$$

Since $q < q_0 = m - 1 + \frac{n-m+1}{k}$, we have $\mu > 0$.

The purpose here is to prove the assertion of the *bridge Dubovitskiĭ–Federer Theorem 2.1* which is equivalent (by virtue of Theorem 6.1) to

$$\Phi(Z_{v,m}) = 0 \quad \text{if } q > m - 1, \tag{5.12}$$

where for each fixed $q \in [m-1, q_\circ)$ we denoted

$$\Phi(E) = \inf_{E \subset \bigcup_{\alpha} D_{\alpha}} \sum_{\alpha} (\text{diam } D_{\alpha})^{\mu} [\text{diam } v(D_{\alpha})]^q. \quad (5.13)$$

As indicated the infimum is taken over all countable families of compact sets $\{D_{\alpha}\}_{\alpha \in \mathbb{N}}$ such that $E \subset \bigcup_{\alpha} D_{\alpha}$. Note that the case $q = q_\circ$, $\mu_q = 0$ was considered in [31, 32] (see also Subsection 4), so we shall omit it here.

Before embarking on the detailed proof we make some preliminary observations that allow us to make a few simplifying assumptions. In view of our definition of critical set we have that

$$Z_{v,m} = \bigcup_{j \in \mathbb{N}} \{x \in Z_{v,m} : |\nabla v(x)| \leq j\}.$$

Consequently we only need to prove that $\Phi(Z'_v) = 0$ for $q \in (m-1, q_\circ)$, where

$$Z'_v = \{x \in Z_{v,m} : |\nabla v(x)| \leq 1\}.$$

For convenience, below we use the notation $\|f\|_{L_{p_\circ,1}(I)}$ instead of $\|1_I \cdot f\|_{L_{p_\circ,1}}$. The following lemma contains the main step in the proof of Theorems 2.1 and 2.2.

Lemma 5.4. *Let $q \in [m-1, q_\circ)$. Then for any n -dimensional dyadic interval $I \subset \mathbb{R}^n$ the estimate*

$$\Phi(Z'_v \cap I) \leq C(\ell(I)^\mu \|\nabla^k v\|_{L_{p_\circ,1}(I)}^q + \ell(I)^{\mu+m-1} \|\nabla^k v\|_{L_{p_\circ,1}(I)}^{q-m+1}) \quad (5.14)$$

holds, where the constant C depends on n, m, k, d only.

Proof. By virtue of (3.5) it suffices to prove that

$$\Phi(Z'_v \cap I) \leq C(\ell(I)^\mu \|\nabla^k v_I\|_{L_{p_\circ,1}(\mathbb{R}^n)}^q + \ell(I)^{\mu+m-1} \|\nabla^k v_I\|_{L_{p_\circ,1}(\mathbb{R}^n)}^{q-m+1}) \quad (5.15)$$

for the mapping v_I defined in Lemma 3.2, where $C = C(n, m, k, d)$ is a constant.

Fix an n -dimensional dyadic interval $I \subset \mathbb{R}^n$ and recall that $v_I(x) = v(x) - P_I(x)$ for all $x \in I$. Denote

$$\sigma = \|\nabla^k v_I\|_{L_{p_\circ,1}}, \quad r = \ell(I),$$

and for each $j \in \mathbb{Z}$

$$E_j = \{x \in I : (\mathcal{M}|\nabla v_I|^{p_\circ})(x) \in (2^{j-1}, 2^j]\} \quad \text{and} \quad \delta_j = \mathcal{H}_\infty^{p_\circ}(E_j).$$

Then by Theorem 3.1 (applied for the case $p = p_\circ = \frac{n}{k}$, $l = 1$, $\beta = p_\circ$),

$$\sum_{j=-\infty}^{\infty} \delta_j 2^j \leq C \sigma^{p_\circ} \quad (5.16)$$

for a constant C depending on n, m, k, d only. By the definition of the Hausdorff measure, for each $j \in \mathbb{Z}$ there exists a family of balls $B_{ij} \subset \mathbb{R}^n$ of radii r_{ij} such that

$$E_j \subset \bigcup_{i=1}^{\infty} B_{ij} \quad \text{and} \quad \sum_{i=1}^{\infty} r_{ij}^{p_0} \leq c\delta_j. \quad (5.17)$$

Of course, using standard covering lemmas we can assume without loss of generality that the concentric balls \tilde{B}_{ij} with radii $\frac{1}{5}r_{ij}$ are disjoint, hereby follows in particular that

$$\sum_{i \in \mathbb{N}, j \in \mathbb{Z}} r_{ij}^n \leq Cr^n \quad \text{and} \quad \sum_{i \in \mathbb{N}, j \in \mathbb{Z}} r_{ij}^{\lambda n} \leq Cr^{\lambda n} \quad \forall \lambda \geq 1. \quad (5.18)$$

Denote

$$Z_j = Z'_v \cap E_j \quad \text{and} \quad Z_{ij} = Z_j \cap B_{ij}.$$

By construction $Z'_v \cap I = \bigcup_j Z_j$ and $Z_j = \bigcup_i Z_{ij}$. Put

$$\varepsilon_* = \frac{1}{r} \|\nabla^k v_I\|_{L_{p_0,1}} = \frac{\sigma}{r},$$

and let j_* be the integer satisfying $\varepsilon_*^{p_0} \in (2^{j_*-1}, 2^{j_*}]$. Denote $Z_* = \bigcup_{j < j_*} Z_j$, $Z_{**} = \bigcup_{j \geq j_*} Z_j$. Then by construction

$$Z'_v \cap I = Z_* \cup Z_{**}, \quad Z_* \subset \{x \in Z'_v \cap I : (\mathcal{M}|\nabla v_I|^{p_0})(x) < \varepsilon_*^{p_0}\}.$$

Since $\nabla P_I(x) = \nabla v(x) - \nabla v_I(x)$, $|\nabla v_I(x)| \leq 2^{j/p_0}$, $|\nabla v(x)| \leq 1$, and $\lambda_m(v, x) = 0$ for $x \in Z_{ij}$, we have⁴

$$Z_{ij} \subset \{x \in B_{ij} : \lambda_1(P_I, x) \leq 1 + 2^{j/p_0}, \dots, \lambda_{m-1}(P_I, x) \leq 1 + 2^{j/p_0}, \lambda_m(P_I, x) \leq 2^{j/p_0}\}.$$

Applying Theorem 3.2 and Lemma 3.3 to mappings P_I , v_I , respectively, with $B = B_{ij}$ and $\varepsilon = \varepsilon_j = 2^{j/p_0}$, we find a finite family of balls $T_s \subset \mathbb{R}^d$, $s = 1, \dots, s_j$ with $s_j \leq C_Y(1 + \varepsilon_j^{1-m})$, each of radius $(1 + C_M)\varepsilon_j r_{ij}$, such that

$$\bigcup_{s=1}^{s_j} T_s \supset v(Z_{ij}).$$

Therefore, for every $j \geq j_*$ we have

$$\Phi(Z_{ij}) \stackrel{(5.13)}{\leq} C_1 s_j \varepsilon_j^q r_{ij}^{q+\mu} = C_2 (1 + \varepsilon_j^{1-m}) 2^{\frac{jq}{p_0}} r_{ij}^{q+\mu} \leq C_2 (1 + \varepsilon_*^{1-m}) 2^{\frac{jq}{p_0}} r_{ij}^{q+\mu}, \quad (5.19)$$

⁴Here we use the following elementary fact: for any linear maps $L_1: \mathbb{R}^n \rightarrow \mathbb{R}^d$ and $L_2: \mathbb{R}^n \rightarrow \mathbb{R}^d$ the estimates $\lambda_l(L_2 + L_1) \leq \lambda_l(L_1) + \|L_2\|$ hold for all $l = 1, \dots, m$, see, e.g., [52, Proposition 2.5 (ii)].

where all the constants C_α above depend on n, m, k, d only. By the same reasons, but this time applying Theorem 3.2 and Lemma 3.3 with $\varepsilon = \varepsilon_*$ and instead of the balls B_{ij} we take a ball $B \supset I$ with radius \sqrt{nr} , we have

$$\begin{aligned} \Phi(Z_*) &\leq C_3(1 + \varepsilon_*^{1-m})\varepsilon_*^q r^{q+\mu} \stackrel{\text{definitions}}{=} C_3(1 + \sigma^{1-m} r^{m-1})\sigma^q r^\mu \\ &= C_3(r^\mu \sigma^q + r^{\mu+m-1} \sigma^{q-m+1}). \end{aligned} \quad (5.20)$$

From (5.19) we get immediately

$$\Phi(Z_{**}) \leq C_2(1 + \varepsilon_*^{1-m}) \sum_{j \geq j_*} \sum_i 2^{\frac{jq}{p_o}} r_{ij}^{q+\mu}. \quad (5.21)$$

Further estimates splits into the two possibilities.

Case I. $q \geq p_o$. Then

$$\begin{aligned} \Phi(Z_{**}) &\stackrel{(5.21)}{\leq} C_2(1 + \varepsilon_*^{1-m}) \left(\sum_{j \geq j_*} \sum_i 2^j r_{ij}^{(q+\mu)\frac{p_o}{q}} \right)^{\frac{q}{p_o}} \\ &\leq C_2(1 + \varepsilon_*^{1-m}) r^\mu \left(\sum_{j \geq j_*} \sum_i 2^j r_{ij}^{p_o} \right)^{\frac{q}{p_o}} \\ &\stackrel{(5.17)}{\leq} C_4(1 + \varepsilon_*^{1-m}) r^\mu \left(\sum_{j \geq j_*} 2^j \delta_j \right)^{\frac{q}{p_o}} \\ &\stackrel{(5.16)}{\leq} C_5(1 + \varepsilon_*^{1-m}) r^\mu \sigma^q = C_5(r^\mu \sigma^q + r^{\mu+m-1} \sigma^{q-m+1}) \end{aligned} \quad (5.22)$$

Case II. $q < p_o$. Recalling (5.11) we get by an elementary calculation

$$\begin{aligned} \frac{\mu p_o}{p_o - q} &= n \cdot \frac{n - qk + [mk - m - k + 1]}{n - qk} \\ &= n \cdot \frac{n - qk + (m-1)(k-1)}{n - qk} \geq n, \end{aligned} \quad (5.23)$$

therefore,

$$\begin{aligned} \Phi(Z_{**}) &\stackrel{(5.21), \text{H\"older ineq.}}{\leq} C_2(1 + \varepsilon_*^{1-m}) \left(\sum_{j \geq j_*} \sum_i 2^j r_{ij}^{p_o} \right)^{\frac{q}{p_o}} \cdot \left(\sum_{j \geq j_*} \sum_i r_{ij}^{\frac{\mu p_o}{p_o - q}} \right)^{\frac{p_o - q}{p_o}} \\ &\stackrel{(5.17), (5.16)}{\leq} C_6(1 + \varepsilon_*^{1-m}) \sigma^q \left(\sum_{j \geq j_*} \sum_i r_{ij}^{\frac{\mu p_o}{p_o - q}} \right)^{\frac{p_o - q}{p_o}} \\ &\stackrel{(5.18), (5.23)}{\leq} C_6(1 + \varepsilon_*^{1-m}) \sigma^q r^\mu \\ &= C_6(r^\mu \sigma^q + r^{\mu+m-1} \sigma^{q-m+1}). \end{aligned} \quad (5.24)$$

Now for both cases (I) and (II) we have by (5.22), (5.24) that $\Phi(Z_{**}) \leq C(r^\mu \sigma^q + r^{\mu+m-1} \sigma^{q-m+1})$, and, by virtue of the earlier estimate (5.20), we conclude that

$$\Phi(Z'_v \cap I) = \Phi(Z_* \cup Z_{**}) \leq \Phi(Z_*) + \Phi(Z_{**}) \leq C(r^\mu \sigma^q + r^{\mu+m-1} \sigma^{q-m+1}).$$

The lemma is proved. \square

Corollary 5.1. *Let $q \in [m-1, q_\circ)$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any subset E of \mathbb{R}^n we have $\Phi(Z'_v \cap E) \leq \varepsilon$ provided $\mathcal{L}^n(E) \leq \delta$. In particular, $\Phi(Z_{v,m} \cap E) = 0$ whenever $\mathcal{L}^n(E) = 0$.*

Proof. We start by recording the following elementary identity (see (5.11)):

$$\frac{(\mu + m - 1)p_\circ}{p_\circ - q + m - 1} = n. \quad (5.25)$$

Let $\mathcal{L}^n(E) \leq \delta$, then we can find a family of nonoverlapping n -dimensional dyadic intervals I_α such that $E \subset \bigcup_\alpha I_\alpha$ and $\sum_\alpha \ell^n(I_\alpha) < C\delta$. Of course, for sufficiently small δ the estimates

$$\|\nabla^k v\|_{L_{p_\circ,1}(I_\alpha)} < 1, \quad \ell(I_\alpha) \leq \delta^{\frac{1}{n}} \quad (5.26)$$

are fulfilled for every α . Denote

$$r_\alpha = \ell(I_\alpha), \quad \sigma_\alpha = \|\nabla^k v\|_{L_{p_\circ,1}(I_\alpha)}, \quad \sigma = \|\nabla^k v\|_{L_{p_\circ,1}}. \quad (5.27)$$

In view of Lemma 5.4 we have

$$\Phi(E) \leq C \sum_\alpha r_\alpha^{\mu+m-1} \sigma_\alpha^{q-m+1} + C \sum_\alpha r_\alpha^\mu \sigma_\alpha^q. \quad (5.28)$$

Now let us estimate the first sum. Since by our assumptions

$$q < q_\circ = m - 1 + \frac{n - m + 1}{k} \leq m - 1 + p_\circ \quad \text{hence} \quad p_\circ > q - m + 1$$

we have

$$\begin{aligned} \sum_\alpha r_\alpha^{\mu+m-1} \sigma_\alpha^{q-m+1} &\stackrel{\text{H\"older ineq.}}{\leq} C \left(\sum_\alpha \sigma_\alpha^{p_\circ} \right)^{\frac{q-m+1}{p_\circ}} \cdot \left(\sum_\alpha r_\alpha^{\frac{(\mu+m-1)p_\circ}{p_\circ-q+m-1}} \right)^{\frac{p_\circ-q+m-1}{p_\circ}} \\ &\stackrel{(5.25), \text{ Lemma 3.1}}{\leq} C' \sigma^{q-m+1} \cdot \left(\mathcal{L}^n(E) \right)^{\frac{p_\circ-q+m-1}{p_\circ}}. \end{aligned} \quad (5.29)$$

The estimates of the second sum are again handled by consideration of two separate cases.

Case I. $q \geq p_\circ$. Then

$$\sum_\alpha r_\alpha^\mu \sigma_\alpha^q \stackrel{(5.26)}{\leq} \delta_n^{\frac{\mu}{n}} \sum_\alpha \sigma_\alpha^{p_\circ} \stackrel{\text{Lemma 3.1}}{\leq} \sigma^{p_\circ} \cdot \delta_n^{\frac{\mu}{n}}. \quad (5.30)$$

Case II. $q < p_\circ$. Then

$$\begin{aligned} \sum_{\alpha} r_{\alpha}^{\mu} \sigma_{\alpha}^q &\stackrel{\text{H\"older ineq.}}{\leq} \left(\sum_{\alpha} \sigma_{\alpha}^{p_\circ} \right)^{\frac{q}{p_\circ}} \cdot \left(\sum_{\alpha} r_{\alpha}^{\frac{\mu p_\circ}{p_\circ - q}} \right)^{\frac{p_\circ - q}{p_\circ}} \\ &\stackrel{\text{Lemma 3.1, (5.23)}}{\leq} \sigma^q \delta^{\mu}. \end{aligned} \quad (5.31)$$

Now for both cases (I) and (II) we have by (5.28)–(5.31) that $\Phi(E) \leq h(\delta)$, where the function $h(\delta)$ satisfies the condition $h(\delta) \searrow 0$ as $\delta \searrow 0$. The lemma is proved. \square

By Theorem 3.3 (iii) (applied to the case $k = l$), our mapping v coincides with a mapping $g \in C^k(\mathbb{R}^n, \mathbb{R}^d)$ off an exceptional set of small n -dimensional Lebesgue measure. This fact, together with Corollary 5.1 and Dubovitskiĭ Theorem A, finishes the proof of Theorem 1.1 for the case $d = m$. But since Theorem 2.1 was not proved for C^k -smooth mappings⁵, we have to do this step now.

Lemma 5.5. *Let $q \in (m - 1, q_\circ)$ and $g \in C^k(\mathbb{R}^n, \mathbb{R}^d)$. Then*

$$\Phi_g(Z_{g,m}) = 0, \quad (5.32)$$

where Φ_g is calculated by the same formula (5.13) with g instead of v and $Z_{g,m} = \{x \in \mathbb{R}^n : \text{rank } \nabla g(x) < m\}$.

Proof. We can assume without loss of generality that g has compact support and that $|\nabla g(x)| \leq 1$ for all $x \in \mathbb{R}^n$. We then clearly have that $g \in W_{p_\circ,1}^k(\mathbb{R}^n, \mathbb{R}^d)$, hence we can in particular apply the above results to g . The following assertion plays the key role:

(*) *For any n -dimensional dyadic interval $I \subset \mathbb{R}^n$ the estimate*

$$\Phi(Z_{g,m} \cap I) \leq C(\ell(I)^\mu \|\nabla^k \bar{g}_I\|_{L_{p_\circ,1}(I)}^q + \ell(I)^{\mu+m-1} \|\nabla^k \bar{g}_I\|_{L_{p_\circ,1}(I)}^{q-m+1})$$

holds, where the constant C depends on n, m, k, d only, and we denoted

$$\nabla^k \bar{g}_I(x) = \nabla^k g(x) - \int_I \nabla^k g(y) \, dy.$$

The proof of (*) is almost the same as that of Lemma 5.4, with evident modifications (we need to take the approximation polynomial $P_I(x)$ of degree k instead of $k - 1$, etc.).

By elementary facts of the Lebesgue integration theory, for an arbitrary family of nonoverlapping n -dimensional dyadic intervals I_α one has

$$\sum_{\alpha} \|\nabla^k \bar{g}_{I_\alpha}\|_{L_{p_\circ,1}(I_\alpha)}^{p_\circ} \rightarrow 0 \quad \text{as} \quad \sup_{\alpha} \ell(I_\alpha) \rightarrow 0 \quad (5.33)$$

The proof of this estimate is really elementary since now $\nabla^k g$ is continuous and compactly supported function, and, consequently, is uniformly continuous and bounded.

⁵Even Theorem A was not proved for \mathbb{R}^d -valued mappings.

From (*) and (5.33), repeating the arguments of Corollary 5.1, using the assumptions on g and taking

$$\sigma_\alpha = \|\nabla^k \bar{g}_{I_\alpha}\|_{L_{p_\circ,1}(I_\alpha)}, \quad \sigma = \sum_\alpha \sigma_\alpha^{p_\circ}$$

in definitions (5.27), we obtain that $\Phi_g(Z_{g,m}) < \varepsilon$ for any $\varepsilon > 0$, hence the sought conclusion (5.32) follows. \square

By Theorem 3.3 (iii) (applied to the case $k = l$), the investigated mapping v equals a mapping $g \in C^k(\mathbb{R}^n, \mathbb{R}^d)$ off an exceptional set of small n -dimensional Lebesgue measure. This fact together with Lemma 5.5 readily implies

Corollary 5.2 (cp. with [16]). *Let $q \in (m-1, q_\circ)$. Then there exists a set \tilde{Z}_v of n -dimensional Lebesgue measure zero such that $\Phi(Z'_v \setminus \tilde{Z}_v) = 0$. In particular, $\Phi(Z'_v) = \Phi(\tilde{Z}_v)$.*

From Corollaries 5.1 and 5.2 we conclude that $\Phi(Z'_v) = 0$, and this concludes the proof of Theorem 2.1.

5.3 The proof of the Coarea formula

Fix $v \in W_{n,1}^1(\mathbb{R}^n, \mathbb{R}^d)$. Applying Lemma 5.4 for $k = 1$, $p_\circ = n$, $\mu = n - m + 1$ and $q = m - 1$, and afterwards making the shift of indices $(m-1) \rightarrow m$, we obtain the following key estimate:

Let $m \in \{0, \dots, n-1\}$. Then for any n -dimensional dyadic interval $I \subset \mathbb{R}^n$ the estimate

$$\Phi(Z'_v \cap I) \leq C(\ell(I)^{n-m} \|\nabla^k v\|_{L_{p_\circ,1}(I)}^m + \ell(I)^n) \quad (5.34)$$

holds, where $Z'_v = \{x \in \Omega \setminus A_v : \text{rank } \nabla v(x) \leq m, \quad |\nabla v(x)| \leq 1\}$, the constant C depends on n, m, d only, and

$$\Phi(E) = \inf_{E \subset \bigcup_\alpha D_\alpha} \sum_\alpha (\text{diam } D_\alpha)^{n-m} [\text{diam } v(D_\alpha)]^m. \quad (5.35)$$

This implies (by the same arguments as in the proof of Corollary 5.1) that for any measurable set $E \subset \mathbb{R}^n$ with $\mathcal{L}^n(E) < \infty$ the inequality

$$\Psi(Z'_v \cap E) < \infty \quad (5.36)$$

holds, where $\Psi(E)$ is defined as

$$\Psi(E) = \lim_{\delta \rightarrow 0} \inf_{\substack{E \subset \bigcup_\alpha D_\alpha, \\ \text{diam } D_\alpha \leq \delta}} \sum_\alpha (\text{diam } D_\alpha)^{n-m} [\text{diam } v(D_\alpha)]^m, \quad (5.37)$$

here the infimum is taken over all countable families of compact sets $\{D_\alpha\}_{\alpha \in \mathbb{N}}$ such that $E \subset \bigcup_\alpha D_\alpha$ and $\text{diam } D_\alpha \leq \delta$ for all α .

By Theorem 6.2, the bound (5.36) implies the validity of the following assertion:

$$\text{the set } \left\{ y \in \mathbb{R}^d : \mathcal{H}^{n-m}(E \cap Z'_v \cap f^{-1}(y)) > 0 \right\} \text{ is } \mathcal{H}^m \text{ } \sigma\text{-finite.} \quad (5.38)$$

Since

$$Z_{v,m+1} = \{x \in \Omega \setminus A_v : \text{rank } \nabla v(x) \leq m\} = \bigcup_j \left\{ x \in Z_{v,m+1} : |\nabla v(x)| \leq j \right\},$$

we infer from (5.38) that in fact

$$\text{the set } \left\{ y \in \mathbb{R}^d : \mathcal{H}^{n-m}(Z_{v,m+1} \cap f^{-1}(y)) > 0 \right\} \text{ is } \mathcal{H}^m \text{ } \sigma\text{-finite.} \quad (5.39)$$

Next we prove that the sets where $\text{rank } \nabla v \leq m-1$ are negligible in the coarea formula.

Lemma 5.6. *The equality*

$$\mathcal{H}^{n-m}(Z_{v,m} \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^m\text{-almost all } y \in \mathbb{R}^d \quad (5.40)$$

holds, where $Z_{v,m} = \{x \in \mathbb{R}^n \setminus A_v : \text{rank } \nabla v(x) \leq m-1\}$ is the set of m -critical points.

Proof. We apply Theorem 2.1 with the parameters $q = m$, $k = 1$, $p_o = n$. Then by (2.1)

$$\mathcal{H}^{\mu_q}(Z_{v,m} \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^m\text{-almost all } y \in \mathbb{R}^d, \quad (5.41)$$

where $\mu_q = n - m - k + 1 + (q - m)k = n - m$. The last identity taken together with (5.41) concludes the proof. \square

In the papers [44, 27] the authors identified criteria for the validity of the Coarea formula for Lipschitz mappings. The following result is particularly useful to us.

Theorem 5.1 (see, e.g., Theorem 1.4 in [27]). *Let $m \in \{0, 1, \dots, n\}$, and $g \in C^1(\mathbb{R}^n, \mathbb{R}^d)$. Suppose that the set $E \subset \mathbb{R}^n$ is measurable and $\text{rank } \nabla g(x) \equiv m$ for all $x \in E$. Assume also that the set $g(E)$ is \mathcal{H}^m - σ -finite. Then the coarea formula*

$$\int_E J_m g(x) \, dx = \int_{\mathbb{R}^d} \mathcal{H}^{n-m}(E \cap g^{-1}(y)) \, d\mathcal{H}^m(y) \quad (5.42)$$

holds, where $J_m g(x)$ denotes the m -Jacobian of g .

Of course, (5.39) and (5.40) are in particular valid also for C^k -smooth mappings. So from Theorem 5.1 and properties (5.39)–(5.40) we obtain the following result which surprisingly is new even in this smooth case.

Theorem 5.2. *Let $m \in \{0, \dots, n\}$ and $g \in C^1(\mathbb{R}^n, \mathbb{R}^d)$. Then for any measurable set $E \subset Z_{g,m+1} = \{x \in \mathbb{R}^n : \text{rank } \nabla g(x) \leq m\}$ the coarea formula*

$$\int_E J_m g(x) \, dx = \int_{\mathbb{R}^d} \mathcal{H}^{n-m}(E \cap g^{-1}(y)) \, d\mathcal{H}^m(y) \quad (5.43)$$

holds, where $J_{g,m}(x)$ again denotes the m -Jacobian of g .

By Theorem 3.3 (iii) (applied to the case $k = l = 1$), the investigated mapping $v \in W_{n,1}^1(\mathbb{R}^n, \mathbb{R}^d)$ coincides with a smooth mapping $g \in C^1(\mathbb{R}^n, \mathbb{R}^d)$ off a set of small n -dimensional Lebesgue measure. This fact together with Theorems 5.2, 6.1 and Corollary 5.1 easily imply the required assertion of Theorem 2.2.

6 Appendix

Fix numbers $n, d \in \mathbb{N}$, $\mu \in (0, n]$, $q \in (0, d]$, and a continuous function $f: \mathbb{R}^n \rightarrow \mathbb{R}^d$. For a set $E \subset \mathbb{R}^n$ define the set function

$$\Phi(E) = \inf_{E \subset \bigcup_{\alpha} D_{\alpha}} \sum_{\alpha} (\text{diam } D_{\alpha})^{\mu} [\text{diam } v(D_{\alpha})]^q, \quad (6.1)$$

where the infimum is taken over all countable families of compact sets $\{D_{\alpha}\}_{\alpha \in \mathbb{N}}$ such that $E \subset \bigcup_{\alpha} D_{\alpha}$.

This section is devoted to the proof of following assertion:

Theorem 6.1. *The above defined set function $\Phi(\cdot)$ is countably subadditive and*

$$\Phi(E) = 0 \Rightarrow \left[\mathcal{H}^{\mu}(E \cap f^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^q\text{-almost all } y \in \mathbb{R}^d \right]. \quad (6.2)$$

We start by recalling the following technical fact from [15]:

Lemma 6.1. *For any set $E \subset \mathbb{R}^n$, if $E = \bigcup_{i=1}^{\infty} E_i$ and $E_i \subset E_{i+1}$ for all $i \in \mathbb{N}$, then*

$$\mathcal{H}_{\infty}^{\mu}(E) = \lim_{i \rightarrow \infty} \mathcal{H}_{\infty}^{\mu}(E_i). \quad (6.3)$$

Proof of Theorem 6.1. The first assertion is evident. Let us prove the second one, i.e., the implication (6.2). Without loss of generality we can assume that f is compactly supported, and more specifically that $f^{-1}(y)$ is a compact subset of the closed unit ball $\overline{B(0, 1)}$ for every $y \in \mathbb{R}^d \setminus \{0\}$.

Let $E \subset \mathbb{R}^n$ and assume that $\Phi(E) = 0$. Without loss of generality we can assume that $0 \notin f(E)$ and

$$E = \bigcap_{j=1}^{\infty} \bigcup_{i=1}^{\infty} D_{ij},$$

where D_{ij} are compact sets in \mathbb{R}^n and

$$\sum_{i=1}^{\infty} (\text{diam } D_{ij})^{\mu} [\text{diam } f(D_{ij})]^q \xrightarrow{j \rightarrow \infty} 0. \quad (6.4)$$

Of course, then E is a Borel set. Suppose that the assertion (6.2) is false, then we can assume without loss of generality that there exists a set $\mathcal{F} \subset f(E)$ such that

$$\mathcal{H}^q(\mathcal{F}) > 0 \quad \text{and} \quad \mathcal{H}_{\infty}^{\mu}(E \cap f^{-1}(y)) \geq \frac{5}{2} \quad \text{for all } y \in \mathcal{F}. \quad (6.5)$$

Unfortunately, we can not assume right now that the set \mathcal{F} is Borel, so we need some careful preparations.

Denote $E_{kj} = \bigcup_{i=1}^k D_{ij}$, $E_j = \bigcup_{i=1}^{\infty} D_{ij}$. In this notation $E = \bigcap_{j=1}^{\infty} E_j$. Evidently, all these sets are Borel. By Lemma 6.1,

$$\mathcal{H}_{\infty}^{\mu}(E_j \cap f^{-1}(y)) = \lim_{k \rightarrow \infty} \mathcal{H}_{\infty}^{\mu}(E_{kj} \cap f^{-1}(y)) \quad \text{for each } y \in f(E_j). \quad (6.6)$$

Denote further $F_{kj} = f(E_{kj})$. Fix an arbitrary point y with the property

$$\mathcal{H}^{\mu}(E_{kj} \cap f^{-1}(y)) \leq 1.$$

Since E_{kj} is a compact set, the set $E_{kj} \cap f^{-1}(y)$ is compact as well. Then it follows by elementary means that the sets $E_{kj} \cap f^{-1}(z)$ lie in the ε -neighborhood of the set $E_{kj} \cap f^{-1}(y)$, where $\varepsilon \searrow 0$ as $z \rightarrow y$, $z \in f(E_{kj})$. Therefore, there exists $\delta = \delta(y) > 0$ such that

$$\mathcal{H}_{\infty}^{\mu}(E_{kj} \cap f^{-1}(z)) \leq 2 \quad \text{if } |z - y| < \delta. \quad (6.7)$$

Hence, there exists a relatively open set $\tilde{F}_{kj} \subset F_{kj}$ (i.e., \tilde{F}_{kj} is open in the induced topology of the set F_{kj}) such that

$$\{y \in \mathbb{R}^d : \mathcal{H}_{\infty}^{\mu}(E_{kj} \cap f^{-1}(y)) \leq 1\} \subset \tilde{F}_{kj} \subset \{y \in \mathbb{R}^d : \mathcal{H}_{\infty}^{\mu}(E_{kj} \cap f^{-1}(y)) \leq 2\}. \quad (6.8)$$

Since by construction F_{kj} is a compact set and \tilde{F}_{kj} is relatively open in F_{kj} , we conclude that **the set \tilde{F}_{kj} is Borel** (this fact plays an important role here). Further, since $E_{kj} \subset E_j$, we have for each $k \in \mathbb{N}$,

$$\{y \in \mathbb{R}^d : \mathcal{H}_{\infty}^{\mu}(E_j \cap f^{-1}(y)) \leq 1\} \subset \{y \in \mathbb{R}^d : \mathcal{H}_{\infty}^{\mu}(E_{kj} \cap f^{-1}(y)) \leq 1\} \subset \tilde{F}_{kj}$$

and therefore,

$$\{y \in \mathbb{R}^d : \mathcal{H}_{\infty}^{\mu}(E_j \cap f^{-1}(y)) \leq 1\} \subset \tilde{F}_j, \quad (6.9)$$

where we denote $\tilde{F}_j = \bigcap_{k=1}^{\infty} \tilde{F}_{kj}$. On other hand, (6.6) and the second inclusion in (6.8) imply $\tilde{F}_j \subset \{y \in \mathbb{R}^d : \mathcal{H}_{\infty}^{\mu}(E_j \cap f^{-1}(y)) \leq 2\}$, so we have

$$\{y \in \mathbb{R}^d : \mathcal{H}_{\infty}^{\mu}(E_j \cap f^{-1}(y)) \leq 1\} \subset \tilde{F}_j \subset \{y \in \mathbb{R}^d : \mathcal{H}_{\infty}^{\mu}(E_j \cap f^{-1}(y)) \leq 2\}. \quad (6.10)$$

Denote now $\tilde{G}_j = f(E_j) \setminus \tilde{F}_j$. Then we can rewrite (6.10) as

$$\{y \in \mathbb{R}^d : \mathcal{H}_{\infty}^{\mu}(E_j \cap f^{-1}(y)) > 2\} \subset \tilde{G}_j \subset \{y \in \mathbb{R}^d : \mathcal{H}_{\infty}^{\mu}(E_j \cap f^{-1}(y)) > 1\}. \quad (6.11)$$

Since $E \subset E_j$, we have from (6.5) that $\mathcal{F} \subset \{y \in \mathbb{R}^d : \mathcal{H}_{\infty}^{\mu}(E_j \cap f^{-1}(y)) > 2\} \subset \tilde{G}_j$ for all $j \in \mathbb{N}$, therefore

$$\mathcal{F} \subset \tilde{G}, \quad (6.12)$$

where we denote $\tilde{G} = \bigcap_{j=1}^{\infty} \tilde{G}_j$. On the other hand, the second inclusion in (6.11) yields

$$\tilde{G} \subset \{y \in \mathbb{R}^d : \mathcal{H}_{\infty}^{\mu}(E_j \cap f^{-1}(y)) > 1\} \quad (6.13)$$

for each $j \in \mathbb{N}$. Since \tilde{G} is a **Borel set** and by (6.12), (6.5) the inequalities $\mathcal{H}^q(\tilde{G}) \geq \mathcal{H}^q(\mathcal{F}) > 0$ hold, by [21, Corollary 4.12] there exists a Borel set $G \subset \tilde{G}$ and a positive constant $b \in \mathbb{R}$ such that $0 < \mathcal{H}^q(G) < \infty$ and

$$\mathcal{H}^q(G \cap B(y, r)) \leq b r^q \quad (6.14)$$

for any ball $B(y, r) = \{z \in \mathbb{R}^d : |z - y| < r\}$ with the center $y \in G$. Of course, by (6.13)

$$G \subset \{y \in \mathbb{R}^d : \mathcal{H}_{\infty}^{\mu}(E_j \cap f^{-1}(y)) > 1\} \quad (6.15)$$

for all $j \in \mathbb{N}$. For $S \subset \mathbb{R}^n$ consider the set function

$$\tilde{\Phi}(S) = \int_G^* \mathcal{H}_{\infty}^{\mu}(S \cap f^{-1}(y)) \, d\mathcal{H}^q(y), \quad (6.16)$$

where \int^* means the *upper integral*⁶. By standard facts of Lebesgue integration theory, $\tilde{\Phi}(\cdot)$ is a countably subadditive set-function (see, e.g., [20], [25]).

From (6.4) and (6.14) it follows that

$$\begin{aligned} \sum_{i=1}^{\infty} (\text{diam } D_{ij})^{\mu} [\text{diam } f(D_{ij})]^q &\geq c \sum_{i=1}^{\infty} (\text{diam } D_{ij})^{\mu} \mathcal{H}^q[G \cap f(D_{ij})] \\ &\geq C \sum_{i=1}^{\infty} \tilde{\Phi}(D_{ij}) \geq C \tilde{\Phi}(E_j). \end{aligned}$$

⁶We use the notion of the upper integral since we do not know whether or not the function $y \mapsto \mathcal{H}_{\infty}^{\mu}(S \cap f^{-1}(y))$ is measurable.

Consequently, $\tilde{\Phi}(E_j) \rightarrow 0$ as $j \rightarrow \infty$. On the other hand, from (6.15) and (6.16) we conclude

$$\tilde{\Phi}(E_j) \geq \int_G^* d\mathcal{H}^q(y) = \mathcal{H}^q(G) > 0,$$

which is the desired contradiction. The proof of the Theorem 6.1 is finished. \square

6.1 \mathcal{H}^q - σ -finiteness of the image

Now again fix numbers $n, d \in \mathbb{N}$, $\mu \in (0, n]$, $q \in (0, d]$ and a continuous mapping $f: \mathbb{R}^n \rightarrow \mathbb{R}^d$. We define the set function by letting for a set $E \subset \mathbb{R}^n$,

$$\Psi(E) = \lim_{\delta \rightarrow 0} \inf_{\substack{E \subset \bigcup_{\alpha} D_{\alpha}, \\ \text{diam } D_{\alpha} \leq \delta}} \sum_{\alpha} (\text{diam } D_{\alpha})^{\mu} [\text{diam } f(D_{\alpha})]^q, \quad (6.17)$$

where the infimum is taken over all countable families of compact sets $\{D_{\alpha}\}_{\alpha \in \mathbb{N}}$ such that $E \subset \bigcup_{\alpha} D_{\alpha}$ and $\text{diam } D_{\alpha} \leq \delta$ for all α .

This subsection is devoted to the following assertion:

Theorem 6.2. *The above defined $\Psi(\cdot)$ is a countably subadditive set-function and for any $\lambda > 0$ the estimate*

$$\mathcal{H}^q(\{y \in \mathbb{R}^d : \mathcal{H}^{\mu}(E \cap f^{-1}(y)) \geq \lambda\}) \leq 5 \frac{\Psi(E)}{\lambda} \quad (6.18)$$

holds.

Proof. The first assertion is evident and we focus on proving the estimate (6.18). Without loss of generality we can assume that $f^{-1}(y)$ is a compact subset of the closed unit ball $\overline{B(0, 1)}$ for every $y \in \mathbb{R}^d \setminus \{0\}$. Let $E \subset \mathbb{R}^n$ and

$$\Psi(E) = \sigma < \infty.$$

Without loss of generality assume also that $0 \notin f(E)$ and

$$E = \bigcap_{j=1}^{\infty} \bigcup_{i=1}^{\infty} D_{ij},$$

where D_{ij} are compact sets in \mathbb{R}^n satisfying

$$\sum_{i=1}^{\infty} (\text{diam } D_{ij})^{\mu} [\text{diam } f(D_{ij})]^q \xrightarrow{j \rightarrow \infty} \sigma, \quad (6.19)$$

and

$$\text{diam } D_{ij} + \text{diam } f(D_{ij}) \leq \frac{1}{j}. \quad (6.20)$$

Of course, E is a Borel set. Fix $\lambda > 0$ and take a set $\mathcal{F} \subset f(E)$ such that

$$\mathcal{H}_\infty^\mu(E \cap f^{-1}(y)) \geq \frac{5}{2}\lambda \quad \text{for all } y \in \mathcal{F}. \quad (6.21)$$

Further we assume that

$$\mathcal{H}^q(\mathcal{F}) > 0 \quad (6.22)$$

since if $\mathcal{H}^q(\mathcal{F}) = 0$, there is nothing to prove. Denote $E_j = \bigcup_{i=1}^{\infty} D_{ij}$. Repeating almost verbatim the arguments from the proof of the previous Theorem 6.1, we can construct a Borel set $\tilde{G} \subset \mathbb{R}^d$ such that

$$\mathcal{F} \subset \tilde{G} \subset \{y \in \mathbb{R}^d : \mathcal{H}_\infty^\mu(E_j \cap f^{-1}(y)) > \lambda\} \quad (6.23)$$

for each $j \in \mathbb{N}$. Since \tilde{G} is a **Borel set** and since, by (6.23) and (6.22), the inequalities $\mathcal{H}^q(\tilde{G}) \geq \mathcal{H}^q(\mathcal{F}) > 0$ hold, we deduce by [21, Theorem 4.10] the existence of a Borel set $G \subset \tilde{G}$ such that $0 < \mathcal{H}^q(G) < \infty$. Put

$$G_l = \left\{ x \in G : \mathcal{H}^q(G \cap B(x, r)) \leq 2r^q \quad \forall r \in (0, 1/l) \right\}. \quad (6.24)$$

Then by construction all the sets G_l are Borel, $G_l \subset G_{l+1}$, moreover, by [20, Theorem 2 of §2.3] we have

$$\mathcal{H}^q \left[G \setminus \left(\bigcup_{l=1}^{\infty} G_l \right) \right] = 0$$

and consequently,

$$\mathcal{H}^q(G) = \lim_{l \rightarrow \infty} \mathcal{H}^q(G_l). \quad (6.25)$$

For $S \subset \mathbb{R}^n$ consider the set function

$$\Psi_l(S) = \int_{G_l}^* \mathcal{H}_\infty^\mu(S \cap f^{-1}(y)) \, d\mathcal{H}^q(y), \quad (6.26)$$

where \int^* means the *upper integral*⁷. By routine arguments of Lebesgue integration theory it follows that $\Psi(\cdot)$ is a countably subadditive set-function (see, e.g., [20], [25]).

From (6.19), (6.20) and (6.24) it follows for $j > l$ that

$$\begin{aligned} \sum_{i=1}^{\infty} (\text{diam } D_{ij})^\mu [\text{diam } f(D_{ij})]^q &\geq \frac{1}{2} \sum_{i=1}^{\infty} (\text{diam } D_{ij})^\mu \mathcal{H}^q[G_l \cap f(D_{ij})] \\ &\geq \frac{1}{2} \sum_{i=1}^{\infty} \Psi_l(D_{ij}) \geq \frac{1}{2} \Psi_l(E_j). \end{aligned} \quad (6.27)$$

⁷We use the notion of upper integral as it is unclear whether the function $y \mapsto \mathcal{H}_\infty^\mu(S \cap f^{-1}(y))$ is measurable.

On the other hand, the second inclusion in (6.23) implies

$$\Psi_l(E_j) \geq \lambda \int_{G_l}^* d\mathcal{H}^q(y) = \lambda \mathcal{H}^q(G_l). \quad (6.28)$$

From (6.27), (6.28), (6.19) we infer

$$\mathcal{H}^q(G_l) \leq \frac{2\sigma}{\lambda}, \quad (6.29)$$

and therefore, by (6.25),

$$\mathcal{H}^q(G) \leq \frac{2\sigma}{\lambda}. \quad (6.30)$$

Since this estimate is true for **any** Borel set $G \subset \tilde{G}$ with $\mathcal{H}^q(G) < \infty$, and since \tilde{G} is Borel as well, we infer from [21, Theorem 4.10] that

$$\mathcal{H}^q(\tilde{G}) \leq \frac{2\sigma}{\lambda}. \quad (6.31)$$

In particular, by the inclusion $\mathcal{F} \subset \tilde{G}$, this implies

$$\mathcal{H}^q(\mathcal{F}) \leq \frac{2\sigma}{\lambda}, \quad (6.32)$$

or in other words,

$$\mathcal{H}^q(y \in \mathbb{R}^d : \mathcal{H}^\mu(E \cap f^{-1}(y)) \geq \frac{5}{2}\lambda) \leq 2 \frac{\Psi(E)}{\lambda}. \quad (6.33)$$

The proof of Theorem 6.2 is complete. \square

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