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Abstract. Assuming a strong convexity condition, we give Hausdorff dimension bounds on the singular set of generalised minima of functionals of linear growth. These seem to be the first of their kind. Besides, the first Sobolev regularity results for generalised minima without the local boundedness hypothesis are established.

1. Introduction

It is a fundamental fact in elliptic regularity theory that minimisers of variational problems of the type

$$\text{to minimise } \mathfrak{F}[u,\Omega] := \int_\Omega f(\mathrm{D}u)\,\mathrm{d}x \text{ within a class } \mathscr{D} \ni u\colon \Omega \to \mathbb{R}^N, \tag{1.1}$$

where Ω is an open Lipschitz subset of $\mathbb{R}^n$, $N>1$ and $f\colon \mathbb{R}^{N\times n}\to\mathbb{R}_{\geqslant 0}$ is a convex C^2–function satisfying suitable growth and ellipticity conditions, are not necessarily locally Hölder continuous unless f has special structure. This phenomenon reveals a fundamental difference between the scalar ($N=1$) and the vectorial ($N>1$) case: In fact, if $N=1$, then standard De Giorgi–Nash–Moser theory predicts gradients of minimisers of suitably regular variational integrals to be locally Hölder continuous, whereas in the vectorial case the *singular set* $\Sigma := \bigcup_{0<\alpha<1}\mathrm{Sing}_\alpha(u)$, where

$$\mathrm{Sing}_\alpha(u) := \{x\in\Omega\colon\ \mathrm{D}u \text{ is not of class } \mathrm{C}^{0,\alpha} \text{ in any neighbourhood of } x\},\qquad 0<\alpha<1,$$

does not need to be empty. As a consequence, if $N>1$, when measuring regularity in terms of Hölder continuity, one may only expect *partial regularity* to hold true for minimisers, that is, $\Omega\setminus\Sigma$ is open together with $\mathscr{L}^n(\Omega\setminus\Sigma)=0$. In this situation, it is natural to attempt to quantify the size of Σ. In this direction, a suitable device is to derive estimates on the Hausdorff dimension of Σ, and the overall aim of singular set estimates is to verify that $\dim_{\mathcal{H}}(\Sigma)<n$ so that the Hausdorff dimension of Σ indeed is strictly smaller than that of the ambient space. The aim of this paper – which we shall describe in detail in the following lines – is to give such Hausdorff dimension bounds for the singular set for minima of multiple integrals which are of *linear growth* subject to a convenient strong convexity condition, which we introduce now:

Definition 1.1 (μ–ellipticity)**.** *Let $1<\mu<\infty$. A variational integrand $f\in\mathrm{C}^2(\mathbb{R}^{N\times n})$ is called μ–*elliptic *provided there exist two constants $0<\lambda\leqslant\Lambda<\infty$ such that*

$$\lambda\frac{|\mathbf{Z}|^2}{(1+|\mathbf{Y}|^2)^{\frac{\mu}{2}}}\leqslant\langle f''(\mathbf{Y})\mathbf{Z},\mathbf{Z}\rangle\leqslant\Lambda\frac{|\mathbf{Z}|^2}{1+|\mathbf{Y}|}\qquad \textit{for all}\ \mathbf{Y},\mathbf{Z}\in\mathbb{R}^{N\times n}. \tag{1.2}$$

In what follows, we shall work with a μ–elliptic integrand, $1<\mu<\infty$, $f\in\mathrm{C}^2(\mathbb{R}^{N\times n})$ so that f in particular is Lipschitz. Under these conditions, it is easy to show that there

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exists a constant $c = c(\mu) > 0$ such that

$$|f(\mathbf{Z})| \leqslant c(1+|\mathbf{Z}|) \qquad \text{for all } \mathbf{Z} \in \mathbb{R}^{N\times n}, \tag{1.3}$$

what precisely is what we understand by functionals of *linear growth*. Here, $\mu = 1$ is excluded since 1–elliptic integrands are of $L\log L$–growth and thus do not belong to those of linear growth. Being of linear growth, it is easy to see by non–reflexivity of $\mathrm{W}^{1,1}$ and the concomitant lack of weak compactness, that minimising sequences of $\mathfrak{F}$ do not need to possess subsequences that converge weakly in $\mathrm{W}^{1,1}$. It is thus reasonable to lift the variational principle (1.1) to the space BV of functions of bounded variation which, when endowed with weak*–convergence, enjoy fairly better compactness properties. Let us recall that a measurable function $u\colon \Omega \to \mathbb{R}^N$ belongs to $\mathrm{BV}(\Omega;\mathbb{R}^N)$ provided $u \in \mathrm{L}^1(\Omega;\mathbb{R}^N)$ and the distributional gradient $\mathrm{D}u$ can be represented by a $\mathbb{R}^{N\times n}$–valued Radon measure of finite total variation, in formulae $\mathrm{D}u \in \mathcal{M}(\Omega;\mathbb{R}^{N\times n})$. Now, given a minimising sequence (u_k) contained in some Dirichlet class $\mathscr{D}_{u_0} := u_0 + \mathrm{W}_0^{1,1}(\Omega;\mathbb{R}^N)$, all terms $\mathfrak{F}[u_k]$ are meaningful, however, this is not so a priori for a possible limit map $u \in \mathrm{BV}(\Omega;\mathbb{R}^N)$. Thus, the functional $\mathfrak{F}$ must be relaxed in order to be well–defined for $u \in \mathrm{BV}(\Omega;\mathbb{R}^N)$. A similar concept, which shall prove to be equivalent, is given by that of *generalised minima*:

Definition 1.2 (Generalised Minima)**.** *Let Ω be an open and bounded Lipschitz subset of $\mathbb{R}^n$ and fix a boundary datum $u_0 \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$. The set of* generalised minima *of $\mathfrak{F}$ given by (1.1) consists of all those $u \in \mathrm{BV}(\Omega;\mathbb{R}^N)$ for which there exists an $\mathfrak{F}$–minimising sequence $(u_k) \subset \mathscr{D}_{u_0} := u_0 + \mathrm{W}_0^{1,1}(\Omega;\mathbb{R}^N)$ such that $u_k \to u$ strongly in $\mathrm{L}^1(\Omega;\mathbb{R}^N)$ as $k \to \infty$. The set of all generalised minima is denoted* $\mathrm{GM}(\mathfrak{F})$*.*

The preceding definition is important as a function $u \in \mathrm{BV}(\Omega;\mathbb{R}^N)$ is a generalised minimiser of $\mathfrak{F}$ if and only if it is a minimiser of the relaxed functional

$$\overline{\mathfrak{F}}[u] := \int_\Omega f(\nabla u)\,\mathrm{d}x + \int_\Omega f^\infty\left(\frac{\mathrm{dD}u}{\mathrm{d}|\mathrm{D}^s u|}\right)\mathrm{d}|\mathrm{D}^s u| + \int_{\partial\Omega} f^\infty((\mathrm{Tr}(u)-u_0)\otimes \nu_{\partial\Omega})\,\mathrm{d}\mathcal{H}^{n-1}$$

over $\mathrm{BV}(\Omega;\mathbb{R}^N)$; here $\mathrm{D}u = \mathrm{D}^{ac}u + \mathrm{D}^s u = \nabla u \mathscr{L}^n + \frac{\mathrm{dD}^s u}{\mathrm{d}|\mathrm{D}^s u|}|\mathrm{D}^s u|$ is the Radon–Nikodým decomposition of the measure $\mathrm{D}u$ into its absolutely continuous and singular parts, respectively, f^∞ the recession function (see (4.8)) and Tr denotes the boundary trace operator on $\mathrm{BV}(\Omega;\mathbb{R}^N)$. The importance of generalised minima is that a function $u \in \mathrm{BV}(\Omega;\mathbb{R}^N)$ is a generalised minimiser of $\mathfrak{F}$ if and only if it is a minimiser of $\overline{\mathfrak{F}}$ over $\mathrm{BV}(\Omega;\mathbb{R}^N)$. In this case, we have

$$\overline{\mathfrak{F}}[u] = \inf_{\mathscr{D}_{u_0}} \mathfrak{F} = \min_{\mathrm{BV}(\Omega;\mathbb{R}^N)} \overline{\mathfrak{F}}. \tag{1.4}$$

The fundamental background result we rely on is given by the following

Proposition 1.3 (Anzellotti & Giaquinta, [2], Thm. 1.1)**.** *Let $f \in \mathrm{C}^2(\mathbb{R}^{N\times n})$ be a convex function of linear growth such that $D^2 f(z)$ is positive definite for every $z \in \mathbb{R}^{N\times n}$. Let $u \in \mathrm{BV}(\Omega;\mathbb{R}^N)$ be a generalised minimiser of $\mathfrak{F}$. If $(x,z) \in \Omega \times \mathbb{R}^{N\times n}$ satisfies*

$$\lim_{\rho\searrow 0}\left[\fint_{\mathrm{B}(x,\rho)} |\nabla u - z|\,\mathrm{d}\mathscr{L}^n + \frac{|\mathrm{D}^s u|(\mathrm{B}(x,\rho))}{\mathscr{L}^n(\mathrm{B}(x,\rho))}\right] = 0, \tag{1.5}$$

then u is of class $\mathrm{C}^{1,\alpha}$ in a neighbourhood of x for any $0 < \alpha < 1$.

It needs to be stressed that, when working with μ–elliptic variational integrands, then the positive definiteness hypothesis is automatically satisfied and hence the preceding proposition applies to all of what follows. Moreover, we wish to mention that the result itself applies to autonomous integrands only; indeed, the non–autonomous case seems to require a higher integrability result in the spirit of Gehring's lemma which, however, is hard to achieve in the linear growth setting. We can now state the main results of the present note:

Theorem 1.4 ($\mathrm{GM}(\mathfrak{F})\cap \mathrm{L}^{\infty}_{\mathrm{loc}}$)**.** *Let $f\in(\mathrm{C}^2\cap\mathrm{C}^{0,1})(\mathbb{R}^{N\times n})$ be a μ–elliptic integrand with $1\leqslant\mu<3$ and suppose Ω is connected. Then the following holds*[1]*:*

(a) **Regularity.** *There holds*

$$\mathrm{GM}(\mathfrak{F})\cap\mathrm{L}^{\infty}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)\subset\mathrm{W}^{1,4-\max\{2,\mu\}}_{\mathrm{loc}}(\Omega;\mathbb{R}^N). \tag{1.6}$$

and, if $\mu\leqslant 2$,

$$\mathrm{GM}(\mathfrak{F})\cap\mathrm{L}^{\infty}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)\subset\mathrm{W}^{1,4-\mu}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)\cap\mathrm{BV}_{2,\mathrm{loc}}(\Omega;\mathbb{R}^N), \tag{1.7}$$

where BV_2 is the space of L^1–function whose second distributional derivatives are finite Radon measures on Ω.

(b) **Uniqueness.** *If Ω is connected, then any two elements of $\mathrm{GM}(\mathfrak{F})\cap\mathrm{L}^{\infty}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ only differ by a constant.*

(c) **Dimension Bound.** *If $1\leqslant\mu\leqslant 2$, then*

$$\dim_{\mathcal{H}}(\Sigma_u)\leqslant n-1 \qquad \text{for all } u\in\mathrm{GM}(\mathfrak{F})\cap\mathrm{L}^{\infty}_{\mathrm{loc}}(\Omega;\mathbb{R}^N). \tag{1.8}$$

The dimension bounds given by (c) rely on the classical measure density lemma, Lemma 2.2 below, and extend the dimension bounds for the singular set available in the literature; see [7, 16, 17, 14, 15] for a comprehensive overview and more background information. If $\mu<1+\frac{2}{n}$, the quick proof we provide for Theorem 1.4 also overcomes the so–called *local boundedness assumption* which seems to be a substantial ingredient for the Sobolev regularity results available in the literature; see [5, 9]:

Theorem 1.5. *Let $f\in(\mathrm{C}^2\cap\mathrm{C}^{0,1})(\mathbb{R}^{N\times n})$ be a μ–elliptic integrand with $1\leqslant\mu<1+\frac{2}{n}$. Then the following holds:*

(a) **Regularity.** *There holds*

$$\begin{aligned}\mathrm{GM}(\mathfrak{F})&\subset\mathrm{W}^{1,(2-\mu)n/(n-2)}_{\mathrm{loc}}(\Omega;\mathbb{R}^N) &&\text{if } n\geqslant 2,\\ \mathrm{GM}(\mathfrak{F})&\subset\mathrm{W}^{1,\mathrm{BMO}}_{\mathrm{loc}}(\Omega;\mathbb{R}^N) &&\text{if } n=2.\end{aligned} \tag{1.9}$$

(b) **Uniqueness.** *If Ω is connected, then any two elements of $\mathrm{GM}(\mathfrak{F})$ only differ by a constant.*

(c) **Dimension Bound.**
  (a) *If $n\geqslant 3$ and $1\leqslant\mu\leqslant\frac{n}{n-1}$, then $\dim_{\mathcal{H}}(\Sigma_u)\leqslant n-1$.*
  (b) *If $n=2$ and $1\leqslant\mu<2$, then $\dim_{\mathcal{H}}(\Sigma_u)\leqslant n-1$.*

In proving the previous theorem, we shall tacitly assume that the Dirichlet data satisfy $u_0\in\mathrm{W}^{1,2}(\Omega;\mathbb{R}^N)$ for technical simplicity; the statement for general boundary data follows as outlined in [3, Chpt. 4]. As an interesting sidefact, we obtain that the local boundedness assumption for generalised minima is obsolete provided $n=2$ and $\mu\leqslant\frac{n}{n-1}$ *without any assumption on radial structure*, we even have $\mathrm{GM}(\mathfrak{F})\subset\mathrm{C}^{0,s}(\Omega;\mathbb{R}^N)$ for any $0<s<1$. Finally, a word on the structure of the paper: Section 2 gathers some preliminary definitions and facts, Theorem 1.4 is established in section 3 and Theorem 1.5 in the consecutive section 4. The latter two sections and the approaches to the theorems outlined above differ in that we work with different regularisation procedures; in fact, working from the local boundedness hypotheses of Theorem 1.4, we may use the uniqueness theory from [9] which is not so for all generalised minima.

2. Preliminaries

In this section we briefly set up the notation used throughout and record various definitions and background facts.

[1]As proved in [5, 9], $\mathrm{GM}(\mathfrak{F})\cap\mathrm{L}^{\infty}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)\subset\mathrm{W}^{1,L\log L}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ so (a) and (b) of the Theorem remain valid, too. However, it not clear to us how to establish even $\dim_{\mathcal{H}}(\Sigma_v)<n$ for $\mu>2$.

2.1. **Notation.** Given $a\in\mathbb{R}^N$, $b\in\mathbb{R}^n$, we use $a\otimes b:=ab^{\mathsf{T}}$ for the usual tensor product. Moreover, $\mathscr{L}^n$ and $\mathcal{H}^{n-1}$ denote the n–dimensional Lebesgue and $(n-1)$–dimensional Hausdorff measures, respectively. For a locally integrable function $v\colon\mathbb{R}^n\to\mathbb{R}^m$, and an open set $U\subset\mathbb{R}^n$ we use the equivalent notation

$$(u)_U:=\fint_U v\,\mathrm{d}x=\frac{1}{\mathscr{L}^n(U)}\int_U v\,\mathrm{d}x.$$

2.2. **Functions of Bounded Variation.** Most importantly, given an open subset Ω of $\mathbb{R}^n$, the space $\mathrm{BV}(\Omega;\mathbb{R}^N)$ is defined as the collection of all $v\in\mathrm{L}^1(\Omega;\mathbb{R}^N)$ whose distributional gradients are $\mathbb{R}^{N\times n}$–valued Radon measures of finite total variation on Ω. By the Riesz representation theorem, the latter amounts to requiring

$$|\mathrm{D}v|(\Omega):=\sup\left\{\int_\Omega\langle v,\mathrm{div}(\varphi)\rangle\,\mathrm{d}x\colon\;\varphi\in\mathrm{C}^1_c(\Omega;\mathbb{R}^{N\times n}),\;|\varphi|\leqslant 1\right\}<\infty.$$

Let us say that a sequence $(v_k)\subset\mathrm{BV}(\Omega;\mathbb{R}^N)$ converges to $v\in\mathrm{BV}(\Omega;\mathbb{R}^N)$ in the *weak*–sense* if and only if $v_k\to v$ strongly in $\mathrm{L}^1(\Omega;\mathbb{R}^N)$ and $\mathrm{D}v_k\overset{*}{\rightharpoonup}\mathrm{D}v$ in the usual weak*–sense of matrix–valued Radon measures on Ω as $k\to\infty$; in this sense, we also use the notion of *weak*–convergence in* BV. Combined with the Banach–Alaoglu Theorem, the Rellich–Kondrachov compactness theorem gives the following compactness result: If Ω is an open and bounded Lipschitz subset of $\mathbb{R}^n$ and (v_k) is uniformly bounded in $\mathrm{BV}(\Omega;\mathbb{R}^N)$, that is, $\sup_{k\in\mathbb{N}}\|v_k\|_{\mathrm{BV}}:=\sup_{k\in\mathbb{N}}\|v_k\|_{\mathrm{L}^1}+|\mathrm{D}v_k|(\Omega)<\infty$, then there exists a subsequence $(v_{k(j)})\subset(v_k)$ and $v\in\mathrm{BV}(\Omega;\mathbb{R}^N)$ such that $v_{k(j)}\overset{*}{\rightharpoonup}v$ in $\mathrm{BV}(\Omega;\mathbb{R}^N)$ as $j\to\infty$.

For more information on BV–spaces, the reader is referred to [1, 10].

2.3. **Miscalleneous.** As a direct consequence of the arguments given in [9, Prop. 2.6] and [13, Lem. 2.10, 5.2], we note

Lemma 2.1. *Let $v\in\mathrm{L}^1(\mathbb{R}^n)$ and let $\Delta^{\pm}_{s,h}v(x):=(v(x+he_s)-v(x))/h$ for $|h|>0$ and $s=1,...,n$. Then the following hold:*

(a) $\|\Delta^{\pm}_{s,h}v\|_{(\mathrm{W}^{1,\infty}_0)^*}\leqslant\|v\|_{\mathrm{L}^1}$.

(b) *Let $f\colon\mathbb{R}^{N\times n}\to\mathbb{R}_{\geqslant 0}$ be a convex function such that $f-\alpha|\cdot|^2$ is a positive function on $\mathbb{R}^{N\times n}$ for some $\alpha>0$ and suppose $u_0\in\mathrm{W}^{1,2}(\Omega;\mathbb{R}^N)$ is fixed. Then the functional $\mathcal{F}\colon(\mathrm{W}^{1,\infty}_0)^*(\Omega;\mathbb{R}^N)\to\mathbb{R}$ given by*

$$\mathcal{F}[w]:=\begin{cases}\mathfrak{F}[w] & \text{if } w\in u_0+\mathrm{W}^{1,p}_0(\Omega;\mathbb{R}^N),\\ +\infty & \text{if } w\in(\mathrm{W}^{1,\infty}_0(\Omega;\mathbb{R}^N)^*\setminus(u_0+\mathrm{W}^{1,p}_0(\Omega;\mathbb{R}^N))\end{cases}$$

is lower semicontinuous with respect to strong convergence in $(\mathrm{W}^{1,\infty}_0(\Omega;\mathbb{R}^N))^*$.

We further record an important result due to Giusti which is crucial for the dimension reduction:

Lemma 2.2 (Measure Density Lemma, [11], Proposition 2.7)**.** *Let $E\subset\mathbb{R}^n$ be an open subset and μ be a finite Radon measure on E. For any $0<\alpha<n$ we then have $\dim_{\mathcal{H}}(E_\alpha)\leqslant\alpha$, where*

$$E_\alpha:=\big\{x_0\in E\colon\;\limsup_{\rho\searrow 0}\rho^{-\alpha}\mu(\mathrm{B}(x,\rho))>0\big\}.$$

3. Proof of Theorem 1.4

The proof we give for Theorem 1.4 relies on various auxiliary facts on which we report now, most notably the regularity proof of Bildhauer [3] and the uniqueness theorem due to Beck & Schmidt [9]. Firstly, recalling our assumption $u_0\in\mathrm{W}^{1,2}(\Omega;\mathbb{R}^N)$ which, following the arguments outlined in [3, Rem. 2.5] can be suitably weakened to $u_0\in\mathrm{W}^{1,1}(\Omega;\mathbb{R}^N)$ without any further efforts, we consider for $k\in\mathbb{N}$ the minimisation problems

$$(P_k)\qquad\text{to minimise }\;\mathfrak{F}_k[v]:=\mathfrak{F}[v]+\frac{1}{2k}\int_\Omega|\mathrm{D}v|^2\,\mathrm{d}x\;\text{ over }\widetilde{\mathcal{D}}:=u_0+\mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^N).$$

By strict convexity of $\mathfrak{F}$, each problem (P_k) possesses a unique minimiser $v_k \in \widetilde{\mathcal{D}}$. Now, by (1.3) and $\mathfrak{F}_k[v_k] \leqslant \mathfrak{F}_1[u_0]$, it is easily verified that (v_k) is a minimising sequence for $\mathfrak{F}$. Invoking the c usual weak*–compactness result on BV, this demonstrates both

$$v_k \overset{*}{\rightharpoonup} v \text{ weak* in } \mathrm{BV}(\Omega;\mathbb{R}^N) \qquad \text{and} \qquad \frac{1}{k}v_k \to 0 \text{ strongly in } \mathrm{L}^2(\Omega;\mathbb{R}^N) \tag{3.1}$$

for a non–relabelled subsequence and some limit map $v \in \mathrm{BV}(\Omega;\mathbb{R}^N)$. By [3, Lem. 2.6], $(v_k) \subset \widetilde{\mathcal{D}} \subset u_0 + \mathrm{W}^{1,1}_0(\Omega;\mathbb{R}^N)$ is $\mathfrak{F}$–minimising, and passing to a suitable subsequence, we may assume that $v_k \to v$ strongly in $\mathrm{L}^1(\Omega;\mathbb{R}^N)$ as $k \to \infty$; in consequence, $v \in \mathrm{GM}(\mathfrak{F})$. Due to the presence of the dominating quadratic and radially symmetric leading term in the definition of $\mathfrak{F}_k$, it follows as a consequence of [21, Thm.] that $v_k \in (\mathrm{W}^{1,\infty}_{\mathrm{loc}} \cap \mathrm{W}^{2,2}_{\mathrm{loc}})(\Omega;\mathbb{R}^N)$ and hence, it follows from [3, Lem. 4.19], where the Euler–Lagrange equation of v_k is suitably tested by $\varphi := \partial_j(\rho^2\partial_j v_k)$ for $j = 1, ..., n$, we deduce that for every relatively compact subset U of Ω there exists $c = c(U, \mu)$ such that if f is μ–elliptic, then there holds with $\Gamma_k := 1 + |\,\mathrm{D}v_k|^2$

$$\sup_{k\in\mathbb{N}} \int_U \frac{|\partial_j \mathrm{D}v_k|^2}{(1+|\,\mathrm{D}v_k|^2)^{\frac{\mu}{2}}}\Gamma_k^s \,\mathrm{d}x \leqslant c \sup_{k\in\mathbb{N}} \int_U \frac{|\,\mathrm{D}u_k|^2}{1+|\,\mathrm{D}v_k|}\Gamma_k^s \,\mathrm{d}x, \tag{3.2}$$

where $s \geqslant 0$ is allowed to be an arbitrary non–negative number given f is radially symmetric and $s = 0$ otherwise; see section 4 for the derivation of a similar bound with a slightly different regularisation.

It is the previous inequality (3.2) for $s > 0$ where the Uhlenbeck structure enters the argument of Bildhauer. Whereas it is explicitely asserted in [3, Thm 4.25] that a radially symmetric $\mu < 3$–elliptic integrand produces at least one generalised minimiser of class $\mathrm{W}^{1,p}_{\mathrm{loc}}$ for all $1 \leqslant p < \infty$, it is only mentioned that in absence of radial symmetry of the integrands, integrability can be improved up to some $p = p(\mu) > 1$ without determining this value explicitely; see [3, Rem. 4.27]. Unfortunately, it is a bit tiresome to extract the precise higher integrability out of the proof, and hence we briefly revisit the argument for the convenience of the reader to prove

$$\mathrm{D}v \in \mathrm{L}^{4-\max\{2,\mu\}}_{\mathrm{loc}}(\Omega;\mathbb{R}^{N\times n}) \tag{3.3}$$

Proof of (3.3). Let $1 \leqslant \mu_1 \leqslant \mu_2$; then any μ_1–elliptic integrand is μ_2–elliptic. Given $1 < \mu < 3$, we put

$$\widetilde{\mu} := \max\{2, \mu\}.$$

so that every μ–elliptic integrand is $\widetilde{\mu}$–elliptic. Following [5] and letting $\mu < 3$, we test the Euler–Lagrange equation satisfied by v_k with $\varphi := \rho^2\Gamma_k^{(3-\widetilde{\mu})/2}v_k$, where $\rho \in \mathrm{C}^1_c(\mathrm{B}_R;[0,1])$ is a cut–off function with $\mathbb{1}_{\mathrm{B}_{R/2}} \leqslant \rho \leqslant \mathbb{1}_{\mathrm{B}_R}$ for a given ball $\mathrm{B}_R := \mathrm{B}(x_0, R) \Subset \Omega$. Note that φ is an admissible choice as a test function because $v_k \in (\mathrm{W}^{1,\infty}_{\mathrm{loc}} \cap \mathrm{W}^{2,2}_{\mathrm{loc}})(\Omega;\mathbb{R}^N)$. We consequently obtain by use of $|f'| \leqslant C$ after rearranging terms

$$\begin{aligned}
\mathbf{I}_k := \int_\Omega \langle f'(\mathrm{D}v_k), \rho^2\Gamma_k^{(3-\widetilde{\mu})/2}\,\mathrm{D}v_k\rangle \,\mathrm{d}x + \frac{1}{k}\int_\Omega \langle \mathrm{D}v_k, \rho^2\Gamma_k^{(3-\widetilde{\mu})/2}\,\mathrm{D}v_k\rangle \,\mathrm{d}x \\
\lesssim \int_{\mathrm{B}_R} |\rho\nabla\rho|\,\Gamma_k^{(3-\widetilde{\mu})/2}\,\mathrm{d}x + \frac{1}{k}\int_\Omega |\rho\nabla\rho|\,\Gamma_k^{\frac{4-\widetilde{\mu}}{2}}\,\mathrm{d}x \\
+ \int_\Omega \rho^2\Gamma_k^{\frac{2-\widetilde{\mu}}{2}}|\,\mathrm{D}^2v_k|\,\mathrm{d}x + \frac{1}{k}\int_\Omega \rho^2\Gamma_k^{\frac{3-\widetilde{\mu}}{2}}|\,\mathrm{D}^2v_k|\,\mathrm{d}x \\
=: \mathbf{II}_k^{(1)} + ... + \mathbf{II}_k^{(4)},
\end{aligned}$$

where the constants implicit in '$\lesssim$' do not depend on k. It is at the last estimate where the local boundedness hypotheses enters in a crucial way; note that we have used $\sup_{k\in\mathbb{N}} \|v_k\|_{\mathrm{L}^\infty(\mathrm{spt}(\rho);\mathbb{R}^N)} < \infty$. By [3, Rem. 4.2], there exists $c > 0$ such that

$\langle f'(\xi),\xi\rangle \geqslant c(1+|\xi|^2)^{\frac{1}{2}}-c$ for all $\xi\in\mathbb{R}^{N\times n}$, and thus $\mathbf{I}_k$ can be estimated from below by

$$\int_\Omega \rho^2\Gamma_k^{\frac{4-\widetilde{\mu}}{2}}\,\mathrm{d}x-\int_\Omega \rho^2\Gamma_k^{\frac{3-\widetilde{\mu}}{2}}\,\mathrm{d}x+\frac{1}{k}\int_\Omega \rho^2\Gamma_k^{\frac{5-\widetilde{\mu}}{2}}\,\mathrm{d}x-\frac{1}{k}\int_\Omega \rho^2\Gamma_k^{\frac{3-\widetilde{\mu}}{2}}\,\mathrm{d}x\lesssim \mathbf{I}_k.$$

On the other side, to estimate $\mathbf{II}_k$ conveniently, we use Young's inequality with free parameter $\theta>0$ to deduce (with constant implicit in '$\lesssim$' now depending on θ)

$$\begin{aligned}\mathbf{II}_k \overset{3-\widetilde{\mu}\leqslant 1}{\lesssim} 1+\frac{1}{\theta k}\int_\Omega |\rho|^2\,\Gamma_k^{\frac{5-\widetilde{\mu}}{2}}\,\mathrm{d}x+\frac{\theta}{k}\int_\Omega \Gamma_k^{(3-\widetilde{\mu})/2}\,\mathrm{d}x+\theta\int_\Omega |\rho|^2|\,\mathrm{D}^2v_k|^2\Gamma_k^{-\frac{\mu}{2}}\,\mathrm{d}x\\ +\frac{C}{\theta}\int_\Omega |\rho|^2\Gamma_k^{\frac{4-\widetilde{\mu}}{2}}\,\mathrm{d}x+\frac{1}{2k}\int_\Omega |\rho|^2|\,\mathrm{D}^2v_k|^2\,\mathrm{d}x+\frac{1}{2k}\int_\Omega |\rho|^2\Gamma_k^{3-\widetilde{\mu}}\,\mathrm{d}x\end{aligned}$$

so that, for sufficiently large θ we obtain after absorbance and rearranging

$$\begin{aligned}\int_\Omega \rho^2\Gamma_k^{\frac{4-\widetilde{\mu}}{2}}\,\mathrm{d}x \lesssim 1+\Big(1+\frac{\theta}{k}\Big)\int_\Omega \Gamma_k^{(3-\widetilde{\mu})/2}\,\mathrm{d}x+\theta\int_\Omega |\rho|^2|\,\mathrm{D}^2v_k|^2\Gamma_k^{-\frac{\mu}{2}}\,\mathrm{d}x\\ +\frac{1}{2k}\int_\Omega |\rho|^2|\,\mathrm{D}^2v_k|^2\,\mathrm{d}x+\frac{1}{2k}\int_\Omega |\rho|^2\Gamma_k^{3-\widetilde{\mu}}\,\mathrm{d}x.\end{aligned}$$

Now note that since $\widetilde{\mu}\geqslant 2$, $(3-\widetilde{\mu})/2\leqslant 1/2$, and so the second term on the right is uniformly bounded in k. Moreover, by (3.2), the third and fourth term is also bounded uniformly in k, and so the uniform boundedness of the left side with respect to k follows from uniform boundedness of the last term. In fact, since $3-\widetilde{\mu}\leqslant 1$, the last term can be majorised by $(C/k)+(C/k)\int_\Omega|\,\mathrm{D}v_k|^2\,\mathrm{d}x$ which is also uniformly bounded.

Since $\widetilde{\mu}<3$, $(4-\widetilde{\mu})/2>\frac{1}{2}$ and hence for each $U\Subset\Omega$, $(\mathrm{D}v_k)$ is uniformly bounded in the reflexive space $\mathrm{L}^{4-\widetilde{\mu}}(U;\mathbb{R}^{N\times n})$ and so, by passing to a a non–relabelled subsequence, we may not only assume that $v_k\overset{*}{\rightharpoonup} v$ in $\mathrm{BV}(\Omega;\mathbb{R}^N)$ but $\mathrm{D}v_k\rightharpoonup w\in\mathrm{L}^{4-\widetilde{\mu}}_{\mathrm{loc}}(\Omega;\mathbb{R}^{N\times n})$ too. Letting $\varphi\in\mathrm{C}_c(\Omega;\mathbb{R}^N)$ be arbitrary, we see by

$$\langle \mathrm{D}v,\varphi\rangle_{\mathcal{M}\times\mathrm{C}}=\lim_{k\to\infty}\langle \mathrm{D}v_k,\varphi\rangle_{\mathcal{M}\times\mathrm{C}}=\lim_{k\to\infty}\int_\Omega\langle \mathrm{D}v_k,\varphi\rangle\,\mathrm{d}x=\int_\Omega\langle w,\varphi\rangle\,\mathrm{d}x$$

that $\mathrm{D}v=w$ in Ω indeed. The proof is complete. □

Remark 3.1 (Additional Iterations). As demonstrated in [3, Thm. 4.25], it is possible to iterate the above argument to deduce $\mathrm{W}^{1,p}_{\mathrm{loc}}$–regularity of generalised minima given f is of Uhlenbeck structure. We believe that the exponent $p=4-\max\{2,\mu\}$ for $\mu<3$ is optimal in the absence of Uhlenbeck structure. Indeed, in the setting of the proof of [3, Thm. 4.25], we would know $\Gamma_k^{(1+\alpha_0)/2}\in\mathrm{L}^1$ locally uniformly in k for $\alpha_0:=3-\widetilde{\mu}$ and thus the proof would continue with $\alpha:=\alpha_0+3-\widetilde{\mu}=6-2\widetilde{\mu}$. However, in this situation, [3, Eq. (32), Chapt. 4.2] produces a term of the form $\int_\Omega\rho^2\Gamma_k^{(\alpha-3)/2}|\,\mathrm{D}^2v_k|^2\,\mathrm{d}x$. In order to control this term, we have to use (3.2) with $s=0$, which is possible if and only if $3-\alpha\geqslant\widetilde{\mu}$, that is, $\widetilde{\mu}\geqslant 3$, thereby contradicting our assumption of $\mu<3$.

We can now complete the

Proof of Theorem 1.4. Starting with (b), we remark that by [9, Thm. 1.10], generalised minima of $\mu=3$–elliptic variational integrals are unique up to constants on connected domains. Since every μ–elliptic variational integrand with $\mu<3$ is 3–elliptic, we thereby deduce uniqueness of generalised minima up to constants. This settles (b), and as a consequence we deduce from (a) that every locally bounded generalised minimiser of a μ–elliptic variational integral shares the same regularity as the particular generalised minimiser constructed above, i.e, $\mathrm{GM}(\mathfrak{F})\cap\mathrm{L}^\infty_{\mathrm{loc}}(\Omega;\mathbb{R}^N)\subset\mathrm{W}^{1,4-\max\{2,\mu\}}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ by (3.3). To conclude the proof, we turn to (c) and note that since $(\mathrm{D}v_k)$ is uniformly bounded in $\mathrm{L}^{4-\max\{2,\mu\}}(U;\mathbb{R}^{N\times n})$ for all $U\Subset\Omega$, we obtain by Young's inequality

$$\int_U|\,\mathrm{D}^2v_k|\,\mathrm{d}x\leqslant c\int_U\frac{|\,\mathrm{D}^2v_k|^2}{(1+|\,\mathrm{D}v_k|^2)^{\frac{\mu}{2}}}\,\mathrm{d}x+c\int_U(1+|\,\mathrm{D}v_k|^2)^{\frac{\mu}{2}}=:\mathbf{I}_k+\mathbf{II}_k. \tag{3.4}$$

The terms $\mathbf{I}_k$ are uniformly bounded in k due to (3.2) with $s = 0$ (note that the right side of (3.2) can be uniformly estimated against a multiple of $1 + |\mathrm{D}v|(\Omega)$). On the other hand, $\mathbf{II}_k$ is certainly bounded if $\mu \leqslant 4 - \max\{2, \mu\}$ which in turn is the case if and only if $\mu \leqslant 2$. We hence deduce by (3.4) that $(\mathrm{D}v_k)$ is uniformly bounded in $\mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^{N\times n})$. Hence, again invoking the weak*–compactness theorem in BV, we deduce that there exists a non–relabelled subsequence and $w \in \mathrm{BV}_{\mathrm{loc}}(\Omega; \mathbb{R}^{N\times n})$ such that $\mathrm{D}v_k \llcorner U \overset{*}{\rightharpoonup} w$. In conclusion, $v \in \mathrm{BV}_{2,\mathrm{loc}}(\Omega; \mathbb{R}^N)$. Since $\mathrm{D}^s v \equiv 0$ in Ω by (a) and (b), the characterisation of the regular set, cf. (1.5), that if $x_0 \in \Sigma_v$, then by Poincaré's Inequality in BV (cf. [10, Thm. 5.6.1])

$$\limsup_{r\searrow 0} \frac{|\mathrm{D}^2 v|(\mathrm{B}(x_0, r))}{\mathscr{L}^n(\mathrm{B}(x_0, r))^{\frac{n-1}{n}}} \geqslant \limsup_{r\searrow 0} \fint_{\mathrm{B}(x_0,r)} |\mathrm{D}v - (\mathrm{D}v)_{x_0,r}| \,\mathrm{d}x > 0 \tag{3.5}$$

and thus by Lemma 2.2, $\dim_{\mathcal{H}}(\Sigma_v) \leqslant n - 1$. The proof is complete. □

4. Proof of Theorem 1.5

4.1. **Regularisation.** The regularisation we invoke here is a minor modification of that invoked by Beck & Schmidt [9]. Let $v \in \mathrm{GM}(\mathfrak{F})$ be arbitrary. By area–strict approximation, we find $(u_k) \subset \mathrm{W}^{1,1}(\Omega; \mathbb{R}^N)$ such that $u_k \to v$ area–strictly as $k \to \infty$ and $\mathrm{Tr}(u_k) = \mathrm{Tr}(v)$ $\mathcal{H}^{n-1}$–a.e. on $\partial\Omega$ for all $k \in \mathbb{N}$. We hence obtain as a merger of Reshetnyak's theorem, Lemma 4.4, $\mathfrak{F}[u_k] \to \overline{\mathfrak{F}}[v] = \inf_{\mathscr{D}} \mathfrak{F}$ so that (u_k) is a minimising sequence for $\mathfrak{F}$ indeed and thus, passing to a suitable subsequence if necessary, we can assume without loss of generality that $\mathfrak{F}[u_k] \leqslant \inf_{\mathscr{D}} \mathfrak{F} + 1/(2Lk^2)$ for all $k \in \mathbb{N}$; here, $L > 0$ denotes the Lipschitz constant of f (recall that being convex and of linear growth, f is Lipschitz). We infer

$$\mathfrak{F}[v] \leqslant \mathfrak{F}[w] + L\|\mathrm{D}(v - w)\|_{\mathrm{L}^1(\Omega;\mathbb{R}^{N\times n})} \qquad \text{for all } v, w \in \mathrm{W}^{1,1}(\Omega; \mathbb{R}^N). \tag{4.1}$$

By denseness of $u_0 + \mathrm{W}^{1,2}_0(\Omega; \mathbb{R}^N)$ in $\mathscr{D}$ for the L^1–gradient metric, there holds

$$\inf_{\mathscr{D}} \mathfrak{F} = \inf_{u_0 + \mathrm{W}^{1,2}_0(\Omega;\mathbb{R}^N)} \mathfrak{F}$$

and hence we find $\widetilde{v}_k \in u_0 + \mathrm{W}^{1,2}_0(\Omega; \mathbb{R}^N)$ with $\|\mathrm{D}(v_k - u_k)\|_{\mathrm{L}^1} \leqslant 1/(2Lk^2)$ for any $k \in \mathbb{N}$. Hence, by (4.1), $\mathfrak{F}[\widetilde{v}_k] \leqslant \mathfrak{F}[u_k] + L\|\mathrm{D}(\widetilde{v}_k - u_k)\|_{\mathrm{L}^1} \leqslant (...) \leqslant \inf_{\mathscr{D}} \mathfrak{F} + 1/k^2$. Let us now put for $w \in \mathrm{W}^{1,2}(\Omega; \mathbb{R}^N)$

$$\mathfrak{F}_k[w] := \mathfrak{F}[w] + \frac{1}{\beta_k} \int_\Omega |w|^2 \,\mathrm{d}x, \qquad \text{with } \beta_k := 2k^2 \int_\Omega 1 + |\mathrm{D}u_k|^2 \,\mathrm{d}x$$

and consequently define on $(\mathrm{W}^{1,\infty}_0(\Omega; \mathbb{R}^N))^*$ functionals $\mathfrak{F}_k$ by

$$\mathfrak{F}_k[w] := \begin{cases} \mathfrak{F}[w] & \text{if } w \in u_0 + \mathrm{W}^{1,2}_0(\Omega; \mathbb{R}^N), \\ \infty & \text{if } w \in (\mathrm{W}^{1,\infty}_0(\Omega; \mathbb{R}^N))^* \setminus (u_0 + \mathrm{W}^{1,2}_0(\Omega; \mathbb{R}^N)). \end{cases}$$

By Lemma 2.1(b), each $\mathscr{F}_k$ is lower semicontinuous with respect to norm convergence in $(\mathrm{W}^{1,\infty}_0(\Omega; \mathbb{R}^N))^*$ and thus we are in position to apply Ekeland's variational principle [11, Thm. 5.6] to each $\mathfrak{F}_k$, hence providing us with $(v_k) \subset (\mathrm{W}^{1,\infty}_0(\Omega; \mathbb{R}^N))^*$ such that for all $k \in \mathbb{N}$ and all $w \in (\mathrm{W}^{1,\infty}_0(\Omega; \mathbb{R}^N))^*$ we have

$$\|v_k - \widetilde{v}_k\|_{(\mathrm{W}^{1,\infty}_0(\Omega;\mathbb{R}^N))^*} \leqslant \frac{1}{k} \text{ and } \mathfrak{F}_k[v_k] \leqslant \mathfrak{F}_k[w] + \frac{1}{k}\|w - v_k\|_{(\mathrm{W}^{1,\infty}_0(\Omega;\mathbb{R}^N))^*}. \tag{4.2}$$

Testing the second inequality of (4.2) with $w = \widetilde{v}_k$, we easily deduce by definition of $\mathfrak{F}_k[v_k]$ that $v_k \in u_0 + \mathrm{W}^{1,2}_0(\Omega; \mathbb{R}^N)$ and hence, testing the second inequality of (4.2) with $w := v_k \pm \varepsilon\varphi$ for arbitrary $\varepsilon > 0$ and $\varphi \in \mathrm{W}^{1,2}_0(\Omega; \mathbb{R}^N)$, we obtain after a routine estimation and sending $\varepsilon \searrow 0$

$$\left| \int_\Omega \langle f'_k(\mathrm{D}v_k), \mathrm{D}\varphi \rangle \,\mathrm{d}x \right| \leqslant \frac{1}{k}\|\varphi\|_{(\mathrm{W}^{1,\infty}_0(\Omega;\mathbb{R}^{N\times n}))^*} \qquad \text{for all } \varphi \in \mathrm{W}^{1,2}_0(\Omega; \mathbb{R}^N). \tag{4.3}$$

for all $k \in \mathbb{N}$. Now we have

Lemma 4.1. *Let* $U \Subset \Omega$ *with* $\mathrm{dist}(U, \partial\Omega) > 0$. *If* $f \in \mathrm{C}^2(\mathbb{R}^{N\times n})$ *is a* μ*–elliptic integrand with* $\mu \leqslant 3$, *then there exists a constant* $C = C(U,\mu) > 0$ *such that the sequence* $(v_k) \subset \mathrm{W}^{1,2}(\Omega;\mathbb{R}^N)$ *as constructed above satisfies* $(v_k) \subset \mathrm{W}^{2,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ *together with*

$$\int_U \frac{|\Delta_{s,h}\,\mathrm{D}v_k|^2}{(1+|\,\mathrm{D}v_k|^2)^{\frac{\mu}{2}}}\,\mathrm{d}x \leqslant C \qquad \textit{for all } k \in \mathbb{N}. \tag{4.4}$$

Proof. Recall that $(v_k) \subset \mathrm{W}^{1,2}(\Omega;\mathbb{R}^N)$. Let $\mathrm{B} := \mathrm{B}(x_0,R) \Subset \Omega$ be a ball, let $0 < r < R$ and pick $\rho \in \mathrm{C}^1_c(\mathrm{B};[0,1])$ with $\mathbb{1}_{\mathrm{B}(x_0,r)} \leqslant \rho \leqslant \mathbb{1}_{\mathrm{B}}$. For $s = 1,...,n$ and $0 < |h| < \mathrm{dist}(\partial\Omega;\mathrm{B})$, $\varphi := \Delta^-_{s,h}(\rho^2\Delta_{s,h}v_k)$ is admissible in (4.3). Inserting φ into (4.3) and rearranging terms gives

$$\begin{aligned}\int_\Omega \langle \Delta_{s,h}f'(\mathrm{D}v_k), \rho^2\Delta_{s,h}\,\mathrm{D}v_k\rangle\,\mathrm{d}x + \frac{1}{\beta_k}\int_\Omega |\rho\Delta_{s,h}\,\mathrm{D}v_k|^2\,\mathrm{d}x \leqslant \frac{1}{k}\|\Delta^-_{s,h}(\rho^2\Delta_{s,h}v_k)\|_{(\mathrm{W}^{1,\infty}_0)^*} \\ + \int_\Omega \langle \Delta_{s,h}f'(\mathrm{D}v_k), 2\rho\nabla\rho\otimes\Delta_{s,h}v_k\rangle\,\mathrm{d}x + \frac{1}{\beta_k}\int_\Omega \langle\Delta_{s,h}\,\mathrm{D}v_k, 2\rho\nabla\rho\otimes\Delta_{s,h}v_k\rangle\,\mathrm{d}x\end{aligned}$$

Let us now put

$$\mathscr{B}^{(k)}_{x,h}[\xi,\eta] := \int_0^1 f''(\mathrm{D}v_k(x) + th\Delta_{s,h}\,\mathrm{D}v_k(x))[\xi,\eta]\,\mathrm{d}t, \qquad \xi,\eta \in \mathbb{R}^{N\times n}$$

so that for all x,h,k, $\mathscr{B}^{(k)}_{x,h}$ is a positive definite bilinear form on $\mathbb{R}^{N\times n}$. In consequence, applying Young's inequality to these bilinear forms and suitably absorbing terms, the previous inequality translates yields

$$\begin{aligned}\int_\Omega \mathscr{B}^{(k)}_{x,h}[\rho\Delta_{s,h}\,\mathrm{D}v_k, \rho\Delta_{s,h}\,\mathrm{D}v_k]\,\mathrm{d}x + \frac{1}{\beta_k}\int_\Omega |\rho\Delta_{s,h}\,\mathrm{D}v_k|^2\,\mathrm{d}x \lesssim \frac{1}{k}\|\Delta^-_{s,h}(\rho^2\Delta_{s,h}v_k)\|_{(\mathrm{W}^{1,\infty}_0)^*} \\ + \int_\Omega \mathscr{B}^{(k)}_{x,h}[\rho\nabla\rho\otimes\Delta_{s,h}v_k, \rho\nabla\rho\otimes\Delta_{s,h}v_k]\,\mathrm{d}x + \frac{1}{\beta_k}\int_\Omega |\rho\nabla\rho\otimes\Delta_{s,h}v_k|^2\,\mathrm{d}x,\end{aligned}$$

with the constants implicit in '$\lesssim$' independent of k. Now, a routine estimation using the definition of μ–ellipticity, Definition 1.1, gives

$$\begin{aligned}\int_\Omega \frac{|\rho\Delta_{s,h}\,\mathrm{D}v_k|^2}{(1+|\,\mathrm{D}v_k|^2)^{\frac{\mu}{2}}}\,\mathrm{d}x + \frac{1}{\beta_k}\int_\Omega |\rho\Delta_{s,h}\,\mathrm{D}v_k|^2\,\mathrm{d}x \lesssim \frac{1}{k}\|\Delta^-_{s,h}(\rho^2\Delta_{s,h}v_k)\|_{(\mathrm{W}^{1,\infty}_0)^*} \\ + \int_\Omega \frac{|\rho\nabla\rho\otimes\Delta_{s,h}v_k|^2}{1+|\,\mathrm{D}v_k|}\,\mathrm{d}x + \frac{1}{\beta_k}\int_\Omega |\rho\nabla\rho\otimes\Delta_{s,h}v_k|^2\,\mathrm{d}x \lesssim 1 + \frac{1}{k}\|\rho^2\Delta_{s,h}v_k\|_{\mathrm{L}^1(\Omega;\mathbb{R}^N)} \lesssim 1,\end{aligned}$$

with all constants implicit in '$\lesssim$' do not depend on k (also recall that $\sup_{k\in\mathbb{N}}\|\,\mathrm{D}v_k\|_{\mathrm{L}^1} < \infty$). For fixed k we therefore obtain $v_k \in \mathrm{W}^{2,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^{N\times n})$, and using positivity of the term $(1/\beta_k)\int_\Omega |\rho\Delta_{s,h}\,\mathrm{D}v_k|^2\,\mathrm{d}x$, estimate (4.4) follows at once. □

4.2. **Conclusion.** Based on the Ekeland–type regularisation, we finally come to the

Proof of Theorem 1.5. Let us firstly assume that $n \geqslant 3$. We define the following variant of a V–function, $1 < \mu < 2$:

$$V_\mu(\xi) := (1+|\xi|^2)^{\frac{2-\mu}{4}}, \qquad \xi \in \mathbb{R}^{N\times n}.$$

Given $w \in \mathrm{W}^{2,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$, we consequently obtain for $k \in \{1,...,n\}$

$$\frac{\partial}{\partial x_k}V_\mu(\mathrm{D}w) = \frac{2-\mu}{2}(1+|\,\mathrm{D}w|^2)^{\frac{-\mu-2}{4}}\,\mathrm{D}w\frac{\partial}{\partial x_k}\,\mathrm{D}w \tag{4.5}$$

and thus

$$\left|\frac{\partial}{\partial x_k}V_\mu(\mathrm{D}w)\right|^2 \leqslant c(\mu)(1+|\,\mathrm{D}w|^2)^{\frac{-\mu-2}{2}}|\,\mathrm{D}w|^2\left|\frac{\partial}{\partial x_k}\,\mathrm{D}w\right|^2 \leqslant c(\mu)(1+|\,\mathrm{D}w|^2)^{-\frac{\mu}{2}}\left|\frac{\partial}{\partial x_k}\,\mathrm{D}w\right|^2$$

which is non–vacuous provided $1 < \mu < 2$. As a consequence, we obtain by Lemma 4.1 that $(V_\mu(\mathrm{D}v_k))$ is uniformly bounded in $\mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega)$. By the Sobolev embedding $\mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega) \hookrightarrow$

$\mathrm{L}^{\frac{2n}{n-2}}_{\mathrm{loc}}(\Omega)$, we thus find that $V_\mu(\mathrm{D}v_k)^{\frac{2n}{n-2}}$ is locally uniformly bounded and hence obtain for any ball $\mathrm{B} \Subset \Omega$

$$\int_{\mathrm{B}} |\,\mathrm{D}v_k|^{\frac{(2-\mu)n}{n-2}}\,\mathrm{d}x \leqslant \int_{\mathrm{B}} V_\mu(\mathrm{D}v_k)^{\frac{2n}{n-2}}\,\mathrm{d}x \leqslant c(\mathrm{B}) < \infty.$$

Since $\mu < 1+\frac{2}{n}$ by assumption, $(2-\mu)n/(n-2) > 1$ and hence we conclude similarly as in the proof of Theorem 1.4 that the gradient satisfies $\mathrm{D}v \in \mathrm{L}^{(2-\mu)n/(n-2)}_{\mathrm{loc}}(\Omega;\mathbb{R}^{N\times n})$. If $n=2$, then $\mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^N) \hookrightarrow \mathrm{BMO}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$, where a measurable function $v\colon \Omega \to \mathbb{R}^N$ belongs to $\mathrm{BMO}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ if and only if

$$\sup_{x\in K}\ \sup_{r>0\colon\ \mathrm{B}(x,r)\subset\Omega} \fint_{\mathrm{B}(x,r)} |v-(v)_{x,r}|\,\mathrm{d}y < \infty.$$

for every compact subset K of Ω. By arbitrariness of the generalised minimiser v as assumed in the beginning of this section, Theorem 1.5(a) follows. The uniqueness part (b) is established in a similar vein as that of [9, Thm. 1.10]; namely, if $u,v \in \mathrm{GM}(\mathfrak{F})$ satisfy $\nabla u \neq \nabla v$ on a set of positive Lebesgue measure, then we obtain by strict convexity of f and convexity of v that $\overline{\mathfrak{F}}[(u+v)/2] < \frac{1}{2}(\overline{\mathfrak{F}}[u]+\overline{\mathfrak{F}}[v]) = \min_{\mathrm{BV}(\Omega;\mathbb{R}^N)}\overline{\mathfrak{F}}$, a contradiction. Therefore $\mathrm{D}u = \mathrm{D}v$ and by the regularity part (a) as established before, $\mathrm{D}^s u = \mathrm{D}^s v \equiv 0$. By connectedness of Ω, this gives $u = v+c$ for some $c \in \mathbb{R}^N$, and (b) is proven. To establish (c), note that

$$\int_{\mathrm{B}} |\,\mathrm{D}^2 v_k|\,\mathrm{d}x \leqslant C\int_{\mathrm{B}} \frac{|\,\mathrm{D}^2 v_k|^2}{(1+|\,\mathrm{D}v_k|)^\mu}\,\mathrm{d}x + \int_{\mathrm{B}} (1+|\,\mathrm{D}v_k|)^\mu\,\mathrm{d}x. \tag{4.6}$$

By (a), the second term on the right side is bounded uniformly in k provided

$$\mu \leqslant \frac{2-\mu}{2}\frac{2n}{n-2}, \quad \text{i.e.,}\quad \mu \leqslant \frac{n}{n-1}. \tag{4.7}$$

If $n=2$, then by the John–Nirenberg Lemma (see [11, Cor. 2.2]), $\mathrm{BMO}_{\mathrm{loc}} \hookrightarrow \mathrm{L}^q_{\mathrm{loc}}$ for every $1 \leqslant q < \infty$ and we obtain that the last term in (4.6) is bounded uniformly in k provided

$$\mu \leqslant \frac{2q}{2+q}$$

for arbitrary $1 \leqslant q < \infty$. Sending $q \nearrow \infty$, we obtain that $(\mathrm{D}^2 v_k)$ is bounded in L^1 uniformly in k provided $\mu < 2$. Now we may argue as in the proof of Theorem 1.4(c) to conclude the claim. The proof is complete. □

Remark 4.2 (1–elliptic integrands)**.** Suppose f is a 1–elliptic integrand, thus typically of $\mathrm{L}\log\mathrm{L}$–growth. In this situation, by the classical De Valleé–Poussin Lemma, minimisers belong to $\mathrm{W}^{1,\mathrm{L}\log\mathrm{L}}(\Omega;\mathbb{R}^N)$. Albeit not of linear growth, Theorem 1.5(a) applies to such integrands too and consequently yields, e.g., for $n=3$ that minima automatically belong to $\mathrm{W}^{1,3}_{\mathrm{loc}}(\Omega;\mathbb{R}^N) \hookrightarrow \mathrm{BMO}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$.

Remark 4.3 (Non–autonomous Integrands)**.** As a variation of the theme of the present paper, we might also study regularity, uniqueness and dimension bounds for the singular set of minima of variational integrals of the form (1.1), where f additionally depends on x, $f = f(x,\cdot)$. Given that partial regularity holds for generalised minima and the dependence on x is Hölder regular, then a merger of the above arguments with the fundamental work of Mingione [18] is very likely to yield higher fractional differentiability of gradients, hence implying dimension bounds in a similar vein as Theorems 1.4 and 1.5; note, however, that here the singular set would possibly have a slightly different characterisation, see, e.g., [17, 18]. However, as outlined in [12], the non–availability of a suitable reverse Hölder inequality unfortunately complicates matters in that until now no partial regularity result is available for generalised minima of such variational integrals but hopefully can be tackled in a future publication.

To conclude with, let us remark that both Theorems 1.4 and 1.4 stick to a parameter range $\mu \leqslant 2$ with respect to higher differentiability and it is not clear to us how to extend it to $\mu > 2$. In particular, as the usual area–integrand $f(\xi) := \sqrt{1+|\xi|^2}$ is 3–elliptic, it would be desirable to extend both Theorems to $\mu \leqslant 3$ (without radial structure). To the best of our knowledge, there is no higher integrability result for generalised minima provided $\mu > 3$ and hence a higher differentiablity estimate seems equally hopeless in this case. Also, very recently SCHMIDT [20] investigated degenerated problems of linear growth which the assumptions of the partial regularity theorem, Prop. 1.3, do not apply to and could establish a partial regularity result. However, as our results are restricted to strongly convex integrands that automatically match the conditions of Prop. 1.3, a generalisation to degenerate problems as studied in [20] requires different tools.

APPENDIX

In this appendix we complement the main part by recalling lower semicontinuity properties of the relaxed functional. Let $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a convex C^2–variational integrand such that $\lambda|\mathbf{Z}| \leqslant f(\mathbf{Z}) \leqslant \Lambda(1+|\mathbf{Z}|)$ holds for all $\mathbf{Z} \in \mathbb{R}^{N\times n}$ and two fixed constants $0 < \lambda \leqslant \Lambda < \infty$. In this situation, it is easy to verify that the *recession function*

$$f^{\infty}(\mathbf{Z}) := \lim_{r\searrow 0} rf\left(\frac{\mathbf{Z}}{r}\right), \qquad \mathbf{Z} \in \mathbb{R}^{N\times n} \tag{4.8}$$

exists and is both continuous and convex. It is also easy to see that f^{∞} is 1–homogeneous. Therefore, given a measure $\mu \in \mathcal{M}(\Omega;\mathbb{R}^{N\times n})$ for an open set $\Omega \subset \mathbb{R}^n$ and denoting its Radon–Nikodým decomposition with respect to Lebesgue measure $\mu = \mu^{ac} + \mu^s = \frac{\mathrm{d}\mu}{\mathrm{d}\mathscr{L}^n}\mathscr{L}^n + \frac{\mathrm{d}\mu}{\mathrm{d}|\mu^s|}|\mu^s|$, we define a functional $\mathfrak{F}[\mu]$ by

$$\mathfrak{F}[\mu] := \int_{\Omega} f\left(\frac{\mathrm{d}\mu}{\mathrm{d}\mathscr{L}^n}\right)\mathrm{d}\mathscr{L}^n + \int_{\Omega} f^{\infty}\left(\frac{\mathrm{d}\mu}{\mathrm{d}|\mu^s|}\right)\mathrm{d}|\mu^s|.$$

We note that this is a well–posed definition indeed due to homogeneity of f^{∞}. The following (semi–)continuity theorem is due to Reshetnyak [19]:

Lemma 4.4 (Reshetnyak (Semi–)Continuity Theorem)**.** *Let $m \in \mathbb{N}$ and let Ω be an open and bounded subset of $\mathbb{R}^n$ and let $(\mu_k) \subset \mathcal{M}(\Omega;\mathbb{R}^m)$ be a sequence that converges to some $\mu \in \mathcal{M}(\Omega;\mathbb{R}^m)$ in the weak*–sense as $k \to \infty$. Moreover, asssume that alll $\mu, \mu_1, \mu_2, \ldots$ take values in some closed convex cone $K \subset \mathbb{R}^m$. Then the following hold:*

(a) (Lower Semicontinuity Part.) *If $\overline{f}\colon K \to [0,\infty]$ is a lower semicontinuous, convex and 1–homogeneous function, then there holds*
$$\int_{\Omega} \overline{f}\Big(\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}\Big)\,\mathrm{d}|\mu| \leqslant \liminf_{k\to\infty} \int_{\Omega} \overline{f}\Big(\frac{\mathrm{d}\mu_k}{\mathrm{d}|\mu_k|}\Big)\,\mathrm{d}|\mu_k|.$$

(b) (Continuity Part.) *If (μ_k) converges strictly to μ in the sense that $\mu_k \overset{*}{\rightharpoonup} \mu$ and $|\mu_k|(\Omega) \to |\mu|(\Omega)$ as $k \to \infty$ and $\overline{f}\colon K \to [0,\infty)$ is continuous and 1–homogeneous, then there holds*
$$\int_{\Omega} \overline{f}\Big(\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}\Big)\,\mathrm{d}|\mu| = \lim_{k\to\infty} \int_{\Omega} \overline{f}\Big(\frac{\mathrm{d}\mu_k}{\mathrm{d}|\mu_k|}\Big)\,\mathrm{d}|\mu_k|.$$

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