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L1–ESTIMATES AND A–WEAKLY DIFFERENTIABLE FUNCTIONS

by

Bogdan Raita
University of Oxford

Oxford Centre for Nonlinear PDE
Mathematical Institute
University of Oxford
Andrew Wiles Building
ROQ, Woodstock Road
Oxford, UK
OX2 6GG

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BOGDAN RAITA

Abstract. The present note serves as a technical overview of recent work done in the study of Sobolev and pointwise regularity for elliptic systems

$$\mathbb{A}u = f,$$

where the source term f is an L^1–map or a measure. Here $\mathbb{A}$ denotes a linear, homogeneous, differential operator with constant coefficients. We are specifically interested in the vectorial case, i.e., u and f are vector fields. We connect the results to both elliptic systems and $\mathrm{W}^{\mathbb{A},p}$–spaces, i.e., spaces of maps $u \in \mathrm{L}^p$ such that $\mathbb{A}u \in \mathrm{L}^p$. Some new remarks are presented in the final section.

1. Introduction

The by now classical Calderón–Zygmund theory revolves around describing quantitative properties of solutions of linear, homogeneous, elliptic systems

$$\mathbb{A}u = f \text{ in } \mathbb{R}^n, \tag{1.1}$$

for $f \in \mathrm{L}^p$, for which the prototypical example is Laplace's equation. By taking the Fourier transform, the differential system is reduced to a polynomial system, which leads to the definition of the *characteristic polynomial* (also termed *Fourier symbol map*). In the simple case when both the solution and the source term are scalar fields, solvability of the algebraic equation is equivalent to the characteristic polynomial having no real roots. It is not difficult to formulate computable invertibility conditions for the possibly vector–valued characteristic polynomial, which constitute the formal definition of *ellipticity* of a linear differential operator. Assuming that the algebraic system is solved, the inverse Fourier transform gives the solution as a convolution of the source term with a negatively homogeneous kernel.

More technically, the reasoning above implies existence of a solution of (1.1) which is a tempered distribution. Under a loose smallness condition at infinity, feasible in physical models, ellipticity also ensures uniqueness of a solution. Since f is assumed to be an L^p–map, the natural quantitative property that we expect of u, which is also desirable in numerical approximations, is that of integrability of its weak derivatives. In particular, if $\mathbb{A}$ is of order k, the best possible result in this direction is that the k–th distributional derivatives of u are also L^p–maps. If $1 < p < \infty$, this can be proved in the positive by an immediate application of boundedness of singular integrals on L^p, established by Calderón and Zygmund in the seminal paper [21].

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Instances of their result were known before, such as boundedness of the Hilbert transform between L^p–spaces for $1<p<\infty$, which corresponds to the scalar case $n=1$ and is due to RIESZ. A vectorial example is Korn's inequality, which gives coercive estimates for problems in linear elasticity and can be stated as

$$\|\nabla u\|_{\mathrm{L}^2} \leqslant c\|\mathcal{E}u\|_{\mathrm{L}^2}$$

for compactly supported, smooth maps $u\colon \mathbb{R}^n \to \mathbb{R}^n$. Here $\mathcal{E}u$ denotes the infinitesimal linear strain arising from the deformation u, i.e., the symmetric part of the $n\times n$ matrix ∇u. To express the generalization concisely, we introduce the homogeneous spaces $\dot{\mathrm{W}}^{\mathbb{A},p}$ as the closure of compactly supported, smooth maps in the (semi–)norm $u\mapsto \|\mathbb{A}u\|_{\mathrm{L}^p}$. Boundedness of singular integrals implies that

$$\dot{\mathrm{W}}^{\mathbb{A},p} \simeq \dot{\mathrm{W}}^{k,p}, \tag{1.2}$$

provided that $\mathbb{A}$ is elliptic and $1<p<\infty$.

The purpose of this note is to follow later developments of the program initiated by CALDERÓN and ZYGMUND. At this stage, we see that the analysis naturally splits into either considering systems on bounded domains, or the limiting cases $p\in\{1,\infty\}$. We will not give precise statements in the Introduction; instead, in Section 2, we will list all results to be discussed later. Despite the author's best efforts, the list is far from exhaustive. It is rather a case–by–case streamline leading to the possible new developments described in Section 5. In each case covered, we aim to give a similar treatment to the above discussion of elliptic systems (1.1) with $1<p<\infty$. More precisely, we establish implications, if not equivalences, between

(a) Regularity results for $\mathbb{A}u=f$ (e.g., $\mathrm{D}^k u\in \mathrm{L}^p$).
(b) Embeddings of $\mathrm{W}^{\mathbb{A},p}$ (e.g., $\dot{\mathrm{W}}^{\mathbb{A},p}\simeq \dot{\mathrm{W}}^{k,p}$).
(c) Simplified properties of systems (e.g., if $\mathbb{A}u=0$ and u is small at infinity, then $u=0$).
(d) Computable algebraic conditions in frequency space (e.g., the characteristic polynomial is invertible away from zero).

The rule of thumb for the theorems presented is that the correspondence (a)–(b) is often obvious; to link (c)–(d) is a matter of Fourier analysis and algebra. The heart of the matter is connecting (c) to (b) or (a). To this end, we briefly introduce the the notions of type (c) that we will use and roughly explain their implications.

The first property we discuss is existence of a fundamental solution for $\mathbb{A}$, by which we mean that there exists a tempered distribution Φ such that

$$\mathbb{A}\Phi = \delta_0 w$$

for some non–zero vector w. If w is a scalar, it has been noticed for a long time that one can use fundamental solutions to retrieve solutions of (1.1) by convolution. For example, the fundamental solutions of the Laplacian operator can be computed explicitly in all dimensions. In fact, if $\mathbb{A}$ is any non–zero scalar operator, the Malgrange–Ehrenpreis Theorem guarantees existence of fundamental solutions of $\mathbb{A}$. In contrast, the gradient of scalar fields defined on the plane does not have a fundamental solution, as $\delta_0 v$ is not curl–free unless $v=0$. In fact, it may even seem difficult to construct a vectorial operator that has fundamental solutions. A canonical, first order example is given by the elliptic operator $(\operatorname{div},\operatorname{curl})$ on $\mathbb{R}^n$, which has the fundamental solution $\Phi(x):=x|x|^{-n}$. Should Φ be a map of locally bounded variation (as would be the case, should (1.2) hold for $p=1$), by the Gagliardo–Nirenberg–Sobolev inequality, Φ would be locally $\mathrm{L}^{n/(n-1)}$–integrable.

This, however, is not the case for $n > 1$, which shows that classical Calderón–Zygmund theory fails in the limiting case $p = 1$. Of course, $(\mathrm{div}, \mathrm{curl})$ is not the only example of this failure: $\Delta u \in \mathrm{L}^1$ does not imply $\mathrm{D}^2 u \in \mathrm{L}^1_{\mathrm{loc}}$, and, more generally, Ornstein showed in [53] that L^1–Calderón–Zygmund estimates fail unless $\mathbb{A}$ is a linear (algebraic) modification of D^k.

On the other hand, non–existence of fundamental solutions is a sufficient condition for estimates on lower order terms. The first instance of this observation goes back to the work of Bourgain and Brezis [13, Thm. 25], where an algebraic condition sufficient for the inequality

$$\|u\|_{\mathrm{L}^{n/(n-1)}} \leqslant c\|\mathbb{A}u\|_{\mathrm{L}^1}, \quad u \in \mathrm{C}_c^\infty$$

to hold is formulated for first order operators $\mathbb{A}$. An extensive theory establishing the correlation between estimates on weaker derivatives and non–existence of fundamental solutions is developed in Van Schaftingen's work [80], which we discuss in some detail in Section 3.2. Here we only relate it to a historical fact. The Sobolev inequality was proved for $1 < p < n$ by Sobolev [62], whereas the case $p = 1$ was covered much later, independently, by Gagliardo [35] and Nirenberg [52]. Van Schaftingen's work precisely pins down the phenomenon due to which a different approach is needed in the limiting case $p = 1$.

More recently, the author identified the existence of fundamental solutions with critical pointwise estimates for solutions of (1.1) in the spirit of [22, 28]. These developments are discussed in Sections 2.3, 4.2.

The other notion of type (c) that we will discuss is that of the operator $\mathbb{A}$ having a finite dimensional null–space, i.e., the solution space of

$$\mathbb{A}u = 0 \text{ in } \mathbb{R}^n$$

is finite dimensional when u runs over distributions on $\mathbb{R}^n$. In some contrast to the fact that lack of fundamental solutions implies local critical integrability, we will see that the finite dimensional null–space property is equivalent to estimates up to the boundary and trace theorems. To the author's knowledge, this fact was first noticed by Smith, in connection to generalizations of Korn's inequality on domains. In fact, he introduced an algebraic condition that turns out to be equivalent to the finite dimensional null–space property. This equivalence was recently connected to boundary properties of integrable maps u satisfying

$$\mathbb{A}u = f \text{ in } \Omega$$

for bounded domains Ω and $f \in \mathrm{L}^1$. Namely, u attains traces in L^1 [19] and u has global higher integrability [38].

This document is organized as follows: Section 2 is a technical extension of the Introduction, where we introduce notation and list all results to be discussed. In Section 3, we discuss Smith's Theorem 2.8 and Van Schaftingen's Theorem 2.9. In Section 4, we discuss the author's contributions, Theorem 2.10 (joint with Gmeineder) and Theorem 2.11. Finally, in Section 5, we make some new remarks and briefly discuss possible developments concerning inequalities in the absence of ellipticity and properties of measures satisfying differential constraints.

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2. Old and new results

We take a moment to clarify notation. Above, $\mathbb{A}$ is a linear, k–th order, homogeneous differential operator with constant coefficients on $\mathbb{R}^n$ from V to W, where V, W are finite dimensional Euclidean spaces, i.e.,

$$\mathbb{A}u := \sum_{|\alpha|=k} A_\alpha \partial^\alpha u,$$

defined for maps $u\colon \mathbb{R}^n \to V$. The maps A_α are linear from V to W and fixed for each multi–index α. We will often prefer the notation

$$\mathbb{A}u := A(\mathrm{D}^k u), \tag{2.1}$$

where $A \in \mathrm{Lin}(V \odot^k \mathbb{R}^n, W)$ is fixed and $V \odot^k \mathbb{R}^n$ denotes the space of symmetric, V–valued, k–linear maps on $\mathbb{R}^n$, i.e., the space of k-th gradients.

The class of Schwartz functions and the space of tempered distributions are denoted by $\mathscr{S}$, $\mathscr{S}'$, respectively. For $u \in \mathscr{S}$, the Fourier transform is given by

$$\mathscr{F}u \equiv \hat{u}\colon \xi \in \mathbb{R}^n \mapsto \int_{\mathbb{R}^n} u(x) \mathrm{e}^{\mathrm{i}\,\xi\cdot x}\,\mathrm{d}x,$$

extended by duality to tempered distributions, contained in the space of distributions $\mathscr{D}'$. The space of bounded measures will be denoted by $\mathcal{M}$.

Ellipticity is defined as one–sided invertibility in frequency space; define the symbol map (characteristic polynomial) $\mathbb{A}[\xi] \in \mathrm{Lin}(V, W)$ of $\mathbb{A}$ by

$$\mathbb{A}[\xi]\hat{u}(\xi) := (-\,\mathrm{i})^k \widehat{\mathbb{A}u}(\xi) \tag{2.2}$$

for all $u \in \mathrm{C}_c^\infty(\mathbb{R}^n, V)$ and $\xi \in \mathbb{R}^n \setminus \{0\}$. We say that $\mathbb{A}$ is *elliptic* if and only if

$$\det\left(\mathbb{A}^*[\xi]\mathbb{A}[\xi]\right) \neq 0 \quad \text{for all} \quad \xi \in \mathbb{R}^n \setminus \{0\}. \tag{2.3}$$

Here $\mathbb{A}^*[\xi] \in \mathrm{Lin}(W, V)$ denotes the adjoint of $\mathbb{A}[\xi]$.

We will write $\mathscr{L}^n$ and $\mathscr{H}^d$ for Lebesgue and Hausdorff measure, respectively; in particular, if $d = 0$, we write δ_0 for the Dirac mass at 0.

As explained in the Introduction, the analysis is naturally split in four cases, depending on whether $1 < p < \infty$ or $p = 1$ and $\Omega = \mathbb{R}^n$ or $\Omega = \mathrm{B}$ (an arbitrary ball in $\mathbb{R}^n$). We rule out the case $n = 1$, as solving (1.1) in this case essentially reduces to integration.

2.1. The case $1 < p < \infty$.

As outlined in the Introduction, standard Calderón–Zygmund theory [21] implies the following:

Theorem 2.1. *Let $\mathbb{A}$ be elliptic, $1 < p < \infty$. Then*

(a) *If $u \in \mathscr{S}'(\mathbb{R}^n, V)/\ker\mathbb{A}$ and $f \in \mathrm{L}^p(\mathbb{R}^n, W)$ satisfy $\mathbb{A}u = f$, it follows that $u \in \dot{\mathrm{W}}^{k,p}(\mathbb{R}^n, V)$.*

(b) *If $u \in \mathscr{D}'(\mathrm{B}, V)$, $f \in \mathrm{L}^p(\mathrm{B}, W)$ satisfy $\mathbb{A}u = f$, then $u \in \mathrm{W}^{k,p}_{\mathrm{loc}}(\mathrm{B}, V)$.*

To prove the second statement, one employs a modification of the Deny–Lions Lemma to show that $u \in \mathrm{W}^{k-1,p}_{\mathrm{loc}}(\mathrm{B}, V)$ (see, e.g., [56, Lem. 2.2]), so that $\mathbb{A}(\rho u) = \tilde{f} \in \mathrm{L}^p(\mathrm{B}, W)$ for cut–off functions $\rho \in \mathrm{C}_c^\infty(\mathrm{B})$ equal to 1 on increasing subsets of B. From this point of view, the assumption on u in (a) can be replaced with $u \in \mathrm{W}^{k-1,p}_{\mathrm{loc}}(\mathbb{R}^n, V)$ which satisfies a smallness condition at infinity. A standard such condition would be

$$\mathscr{L}^n\left(\{x \in \mathbb{R}^n \colon |u(x)| > \lambda\}\right) < \infty \tag{2.4}$$

for all $\lambda > 0$, see, e.g., [47, Ch. 3.2].

The statement in Theorem 2.1(b) is optimal in the class of elliptic operators. This can be seen by taking $n = 2$, $V := \mathbb{R}^2$, $\mathrm{B} := \mathbb{D}(1,1)$, $p := 2$, and the Wirtinger derivative

$$\mathbb{A}_1 u := \frac{1}{2}\begin{pmatrix} \partial_1 u_1 - \partial_2 u_2 \\ \partial_2 u_1 + \partial_1 u_2 \end{pmatrix}. \tag{2.5}$$

Then one takes, in complex notation, $u(z) := \log z$ defined, say, on $\mathbb{C} \setminus (-\infty, 0]$, so that $\mathbb{A}_1 u = 0$ in B, but $\nabla u(z) = z^{-1}$, which is not in $\mathrm{L}^2(\mathrm{B}, \mathbb{C})$.

It is natural to ask under what additional assumptions on $\mathbb{A}$ one achieves global regularity in Theorem 2.1(b). This was proved by Smith in the, perhaps forgotten, work [61]:

Theorem 2.2. *Let $\mathbb{A}$ be elliptic, $1 < p < \infty$. The following are equivalent:*

(a) *If $u \in \mathrm{L}^p(\mathrm{B}, V)$, $f \in \mathrm{L}^p(\mathrm{B}, W)$ satisfy $\mathbb{A}u = f$, then $u \in \mathrm{W}^{k,p}(\Omega, V)$.*
(b) *$\dim\{u \in \mathscr{D}'(\mathbb{R}^n, V) \colon \mathbb{A}u = 0\} < \infty$, i.e., the null–space of $\mathbb{A}$ is finite dimensional.*

In fact, Smith's original condition is that of $\mathbb{C}$*–ellipticity*, i.e.,

$$\det\left(\mathbb{A}^*[\xi]\mathbb{A}[\xi]\right) \neq 0 \text{ for all } \xi \in \mathbb{C}^n \setminus \{0\}. \tag{2.6}$$

The fact that (2.6) is equivalent to Theorem 2.2(b) is essentially proved in [61]. For a brief proof, see [38, Prop. 3.1]. We will discuss Theorem 2.2 in Section 3.1.

In light of Theorems 2.1, 2.2, it seems natural to ask the following:

Question 2.3. *Is ellipticity sufficient to have that if $u \in \mathscr{D}'(\mathrm{B}, V)/\ker\mathbb{A}$, $f \in \mathrm{L}^p(\mathrm{B}, W)$ such that $\mathbb{A}u = f$, is it then always the case that $\mathrm{D}^k u \in \mathrm{L}^p(\mathrm{B}, V \odot^k \mathbb{R}^n)$?*

Equivalently, does the Korn–type inequality

$$\inf_{\mathbb{A}v = 0 \text{ in } \mathrm{B}} \|\mathrm{D}^k(u - v)\|_{\mathrm{L}^p(\mathrm{B}, V \odot^k \mathbb{R}^n)} \lesssim \|\mathbb{A}u\|_{\mathrm{L}^p(\mathrm{B}, W)} \tag{2.7}$$

hold for elliptic operators $\mathbb{A}$? This statement can be shown to hold for $\mathbb{C}$–elliptic operators by use of Smith's inequality (see [38, Lem. 5.5] for the precise version) coupled with the Poincaré–type inequality [38, Prop. 4.2]. Interestingly, it also holds for the elliptic, but not $\mathbb{C}$–elliptic operator (2.5) [34]. Unfortunately, the proof relies crucially on the particular form of the generalized Cauchy formula.

2.2. **The case $p = 1$.** In the limiting case $p = 1$, the analogue of Theorem 2.1 holds only in trivial cases, i.e., if A as in (2.1) is injective; equivalently, there exists a linear map $B \in \mathrm{Lin}(W, V \odot^k \mathbb{R}^n)$ such that $\mathrm{D}^k u = B(\mathbb{A}u)$ for all $u \in \mathrm{C}^\infty(\mathbb{R}^n, V)$, which implies the pointwise estimate $|\mathrm{D}^k u| \lesssim |\mathbb{A}u|$. This result is termed as *Ornstein's non–inequality* [53, 45].

However, one can ask whether estimates on the lower order derivatives hold, such as the Gagliardo–Nirenberg–Sobolev inequality or the more particular inequality of Strauss [70] for $\mathbb{A}u := \mathcal{E}u := \frac{1}{2}\left(\mathrm{D}u + (\mathrm{D}u)^\mathsf{T}\right)$. The first generalization of this result, for a class first order elliptic operators $\mathbb{A}$, is due to Bourgain and Brezis in [13, Thm. 25], obtained by duality from the far reaching result [13, Thm. 10], as a consequence of their work in [10, 11, 12] and, jointly with Mironescu, in [14, 15, 16].

The generalization to operators of arbitrary order and necessity of the algebraic assumption are due to Van Schaftingen, who showed in [80, Thm. 1.3] that solutions $u \in \mathscr{S}'(\mathbb{R}^n, V)/\ker\mathbb{A}$ of the elliptic system (1.1) with $f \in \mathrm{L}^1(\mathbb{R}^n, W)$ have the Sobolev regularity $\mathrm{D}^{k-1}u \in \mathrm{L}^{n/(n-1)}$ if and only if the operator $\mathbb{A}$ does *not* admit a fundamental solution. It is worth mentioning that, although VAN SCHAFTINGEN's work is closely related to that of BOURGAIN, BREZIS, and MIRONESCU, his approach is independent and self–contained, see [77, 78]. The statement can be refined on the fractional scale [79], giving comprehensive Sobolev regularity in spite of the L^1–non–inequality:

Theorem 2.4 ([80, Thm. 8.1])**.** *Let $\mathbb{A}$ be elliptic, $s \in (0,1]$. The following are equivalent:*

(a) *If $u \in \mathscr{S}'(\mathbb{R}^n, V)/\ker\mathbb{A}$ and $f \in \mathrm{L}^1(\mathbb{R}^n, W)$ satisfy $\mathbb{A}u = f$, it follows that $u \in \dot{\mathrm{W}}^{k-s,n/(n-s)}(\mathbb{R}^n, V)$.*
(b) *If $u \in \mathscr{D}'(\mathrm{B}, V)$, $f \in \mathrm{L}^1(\mathrm{B}, W)$ satisfy $\mathbb{A}u = f$, then $u \in \mathrm{W}^{k-s,n/(n-s)}_{\mathrm{loc}}(\mathrm{B}, V)$.*
(c) *If $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n, V)$, $w \in W$ satisfy $\mathbb{A}u = \delta_0 w$, then $w = 0$.*

Aspects of this Theorem are presented in Section 3.2. Condition (c), termed *cancellation*, was introduced in [80, Def. 1.2] as

$$\bigcap_{\xi \in \mathbb{R}^n \setminus \{0\}} \mathbb{A}[\xi](V) = \{0\}, \tag{2.8}$$

which is obtained by Fourier transforming the equation for fundamental solutions of $\mathbb{A}$.

In analogy to Question 2.3, one can ask whether the Korn–Sobolev–type inequality

$$\inf_{\mathbb{A}v=0 \text{ in } \mathrm{B}} \| \mathrm{D}^{k-1}(u-v)\|_{\mathrm{L}^{n/(n-1)}(\mathrm{B}, V\odot^{k-1}\mathbb{R}^n)} \lesssim \|\mathbb{A}u\|_{\mathrm{L}^1(\mathrm{B},W)}$$

hold for elliptic and canceling operators $\mathbb{A}$. Again, this is true of $\mathbb{C}$–elliptic operators [39, Prop. 2.5]. If $\mathbb{A}$ is assumed elliptic, then it is easy to see that cancellation is necessary for the estimate. We do not know of any examples of elliptic and canceling but not $\mathbb{C}$–elliptic instances of $\mathbb{A}$ for which the inequality holds.

The question of identifying conditions that ensure global regularity in Theorem 2.4(b) was tackled by GMEINEDER and the author in [38]:

Theorem 2.5. *Let $\mathbb{A}$ be elliptic, $s \in (0,1]$. The following are equivalent:*

(a) *If $u \in \mathrm{L}^1(\mathrm{B}, V)$, $f \in \mathrm{L}^1(\mathrm{B}, W)$ satisfy $\mathbb{A}u = f$, then $u \in \mathrm{W}^{k-s,n/(n-s)}(\Omega, V)$.*
(b) $\dim\{u \in \mathscr{D}'(\mathbb{R}^n, V) \colon \mathbb{A}u = 0\} < \infty$.

In view of Theorem 2.2, it does seem plausible that $\mathbb{C}$–ellipticity (2.6) is the right condition, but it is not a priori evident how does it compare with the canceling condition (2.8). In [38, Sec. 3] we show that, indeed, $\mathbb{C}$–ellipticity *strictly* implies the canceling condition. To provide a counterexample, the elliptic, non–$\mathbb{C}$–elliptic operator $\mathbb{A}_1$ defined in (2.5) cannot be used, as it satisfies

$$\mathbb{A}_1\left(\frac{(x_1, -x_2)}{x_1^2 + x_2^2}\right) = \delta_0(\pi, 0),$$

thus failing the canceling condition. In fact, to find an elliptic and canceling, first order operator that is not $\mathbb{C}$–elliptic on $\mathbb{R}^n$, one must require $n > 2$, as we will discuss below. In turn, if $n = 3$, one finds, with $V \simeq \mathbb{R}^2$,

$$\mathbb{A}_2 u := \begin{pmatrix} \partial_1 u_1 - \partial_2 u_2 \\ \partial_2 u_1 + \partial_1 u_2 \\ \partial_3 u_1 \\ \partial_3 u_2 \end{pmatrix}.$$

We will discuss Theorem 2.5 in Section 4.1. Momentarily, we zoom in on the difference between the algebraic conditions and show that $\mathbb{C}$–ellipticity implies a stronger canceling condition. For first order operators, the two conditions are equivalent.

Proposition 2.6. *Let $\mathbb{A}$ be as in (2.1). Suppose that $\mathbb{A}$ is $\mathbb{C}$–elliptic. Then*

$$\bigcap_{\xi\in H\setminus\{0\}} \mathbb{A}[\xi](V)=\{0\} \tag{2.9}$$

for any hyperplane $H\leqslant\mathbb{R}^n$ of dimension at least two.

Conversely, suppose that $\mathbb{A}$ is elliptic of order $k=1$ and that (2.9) holds for all $H\leqslant\mathbb{R}^n$ with $\dim H\geq 2$. Then $\mathbb{A}$ is $\mathbb{C}$–elliptic.

The characterization of $\mathbb{C}$–ellipticity by (2.9) only holds for first order operators. For higher orders $k>1$, one considers the operator $\mathbb{B}:=\nabla^{k-1}\circ\mathbb{A}_1$, where $\mathbb{A}_1$ is given in (2.5). Since $n=2$, (2.9) reduces to (2.8). The fact that $\mathbb{B}$ is elliptic and canceling, but not $\mathbb{C}$–elliptic is proved in [38, Counterexample 3.4]. The converse statement confirms the fact already noticed in [38, Lem. 3.5(b)], that, if $n=2$, the class of first order, elliptic and canceling operators coincides with the class of $\mathbb{C}$–elliptic operators. We prove Proposition 2.6 and discuss possible refinements of condition (2.9) and their plausible relation to properties of the singular parts of measures with differential structure in Section 5.2.

2.3. **Pointwise estimates.** Apart from Sobolev regularity, one can also consider pointwise regularity for the elliptic system (1.1), i.e., $\mathbb{A}u=f$. With $f\in\mathrm{L}^p$, it is easy to see that unless $\mathbb{A}$ is of high order, there is no hope to obtain any regularity on the C^l–scale. The notion introduced by Calderón and Zygmund in [22] as a replacement is that of k-th order L^p–Taylor expansions. To be precise, we say that $u\in\mathrm{t}^{k,p}(x)$ if and only if there exists a polynomial $P^k_x u$ of degree at most k such that

$$\left(\fint_{\mathrm{B}_r(x)}|u-P^k_x u|^p\,\mathrm{d}y\right)^{1/p}=o(r^k)\text{ as }r\downarrow 0. \tag{2.10}$$

One can use this to define the approximate gradients of u at x as $\nabla^l u(x):=\mathrm{D}^l P^k_x u(x)$ for $0\leqslant l\leqslant k$. Of course, if $u\in\mathrm{W}^{k,p}$, then $u\in\mathrm{t}^{k,p}(x)$ for $\mathscr{L}^n$–a.e. x [22, Thm. 12], and, moreover, $\nabla^l u=\mathrm{D}^l u$ $\mathscr{L}^n$–a.e. for $0\leqslant l\leqslant k$. If $p=\infty$, the left hand side of (2.10) is replaced by the L^∞–norm of $u-P^k_x u$, in which case u has k classical derivatives at x. Details on $\mathrm{t}^{k,p}$–spaces and their relation to Sobolev spaces can be found in Ziemer's monograph [82, Ch. 3].

The main result in [22, Thm. 1] states that if we fix $1<p<\infty$ and $u\in\mathrm{W}^{k,p}$ solves a k–th order elliptic system $\mathbb{A}u=f$ with $f\in\mathrm{t}^{l,p}(x)$, then $u\in\mathrm{t}^{k+l,q}(x)$, with q given by the Sobolev embedding in an obvious way. Should we give a variant of their Theorem in the case $p=1$, by Ornstein's non–inequality, the assumption $u\in\mathrm{W}^{k,1}$ is not reasonable. In fact, as showed by the author in [56, Thm. 1.4] using the results in [41, 3], if $\mathbb{A}u=f\in\mathrm{L}^1$, then $u\in\mathrm{t}^{k,1}(x)$ for $\mathscr{L}^n$–a.e. x, although the measurable map $\nabla^k u$ can fail to be locally integrable. More generally, it is shown that $\mathrm{D}^l u\in\mathrm{t}^{k-l,q}(x)$ a.e. for $0\leqslant l<k$ and q *strictly* less than the corresponding Sobolev exponent. If $k<n$, it is shown in [56, Thm. 1.3(a)] that the Sobolev exponent is attained if and only if the operator $\mathbb{A}$ is, in addition, canceling. Details and precise statements are given in Section 4.2.

In fact, one can say something about C^k–regularity of solutions of $\mathbb{A}u = f$ in L^1, but only up to small sets. Recall the classical Lusin Theorem, stating that if u is a measurable function defined on a bounded Borel subset Ω of $\mathbb{R}^n$, then u equals a continuous function $\tilde{u}$ up to a set of arbitrary small Lebesgue measure. In fact, as noticed, e.g., in [3, Sec. 2.4], if $u \in \mathrm{t}^{k,1}(x)$ for a.e. $x \in \Omega$, then $\tilde{u}$ can be chosen of class C^k. In particular, this is true (locally) of solutions of elliptic systems $\mathbb{A}u = f$ with $f \in \mathrm{L}^1$. It is worth mentioning that the C^2–Lusin property for solutions of $\Delta u = \mu$, for a bounded measure μ [3, Prop. 4.4], gives sufficient conditions on the (rough) coefficients of the continuity equation in two dimensions, so that it has unique bounded solution [2, Thm. 5.2, Sec 2.14(iv)].

2.4. **$\mathrm{W}^{\mathbb{A},p}$–spaces.** The results stated above can be concisely rephrased in the language of $\mathrm{W}^{\mathbb{A},p}$–spaces, defined for $\mathbb{A}$ as in (2.1), $1 \leqslant p \leqslant \infty$, and open $\Omega \subset \mathbb{R}^n$ as

$$\mathrm{W}^{\mathbb{A},p}(\Omega) := \{u \in \mathrm{L}^p(\Omega, V) \colon \mathbb{A}u \in \mathrm{L}^p(\Omega, W)\},$$

which are Banach under the obvious norm. Separability, reflexivity and density of smooth maps (or lack thereof) are obtained analogously to the Sobolev space $\mathrm{W}^{k,p}$. Recall from the Introduction that the homogeneous space $\dot{\mathrm{W}}^{\mathbb{A},p}(\mathbb{R}^n)$ is the closure of $\mathrm{C}_c^\infty(\mathbb{R}^n, V)$ in the (semi–)norm $u \mapsto \|\mathbb{A}u\|_{\mathrm{L}^p}$.

Aside from notational convenience, the $\mathrm{W}^{\mathbb{A},p}$–spaces arise naturally as solution spaces for minimizers of energy functionals of the type

$$\mathscr{E}[u] := \int_\Omega F(x, \mathbb{A}u(x))\,\mathrm{d}x \tag{2.11}$$

over deformations $u \colon \Omega \to V$ satisfying a boundary condition, when the energy density F is assumed to have p–growth. Such problems arise in continuum mechanics, for instance in geometrically–linear elasticity, elasto–plasticity, plasticity [59, 60, 33, 42, 7, 69, 48], to mention a few. The set–up of (2.11) can be compared with the $\mathcal{A}$–free framework introduced by Fonseca and Müller in [32] and later studied, among others, in [18, 30, 8]. In this later set–up, one seeks to minimize integrals

$$\mathscr{G}[w] := \int_\Omega G(x, w(x))\,\mathrm{d}x \tag{2.12}$$

over vector fields w of zero mean satisfying a linear differential constraint $\mathcal{A}w = 0$. For example, if $\mathcal{A} = \mathrm{curl}$, then w is a gradient. In Remark 5.7, we will indicate how one can algebraically reduce (2.12) to (2.11) under the assumption that the symbol map of $\mathcal{A}$ has constant rank. Little is known concerning lower semi–continuity of integrals such as $\mathscr{G}$ in the absence of this condition; in particular, one can argue that $\mathrm{W}^{\mathbb{A},p}$–spaces come up naturally from the mathematical point of view as well.

In the linear growth case, in which (2.11) is posed in $\mathrm{W}^{\mathbb{A},1}(\Omega)$, $\mathbb{A}$–gradients of minimizing sequences can develop concentrations due to the fact that the natural coercivity of the energy is insufficient to guarantee weak compactness in L^1, phenomenon which is well understood in the gradient case, see [5, 6, 31, 37]. Instead, minimizing sequences converge weakly–* to generalized minimizers in the space of maps of bounded $\mathbb{A}$–variation, i.e.,

$$\mathrm{BV}^{\mathbb{A}}(\Omega) := \{u \in \mathrm{L}^1(\Omega, V) \colon \mathcal{M}(\Omega, W)\},$$

where $\mathcal{M}$ denotes the space of bounded measures. A typical example is the space BD of maps of bounded deformation, i.e., $\mathbb{A} = \mathcal{E}$, which is the natural solution space for problems in plasticity theory [7, 66, 67, 68, 69]. Relaxation results for linear growth problems such as (2.12) and (2.11) were proved recently in [8, 19].

The remainder of this section will consist of a quite dry list, comprising of the results mentioned so far, translated in the language of $\mathrm{W}^{\mathbb{A},p}$–spaces. Some generality will be occasionally lost or gained in translation. All results that hold in $\mathrm{W}^{\mathbb{A},1}$ are also true in $\mathrm{BV}^{\mathbb{A}}$ by strict or area–strict approximation [19, Thm. 2.8].

Theorem 2.7 (Theorem 2.1)**.** *Let $\mathbb{A}$ be as in (2.1), $1<p<\infty$. Then $\mathbb{A}$ is elliptic if and only if $\dot{\mathrm{W}}^{\mathbb{A},p}(\mathbb{R}^n)\simeq\dot{\mathrm{W}}^{k,p}(\mathbb{R}^n,V)$, i.e.,*

$$\|\,\mathrm{D}^k u\|_{\mathrm{L}^p(\mathbb{R}^n,V\odot^k\mathbb{R}^n)}\lesssim\|\mathbb{A}u\|_{\mathrm{L}^p(\mathbb{R}^n,W)}$$

for all $u\in\mathrm{C}_c^\infty(\mathbb{R}^n,V)$.

Theorem 2.8 (Theorem 2.2)**.** *Let $\mathbb{A}$ be as in (2.1), $1<p<\infty$. Then $\mathbb{A}$ has finite dimensional null–space if and only if $\mathrm{W}^{\mathbb{A},p}(\mathrm{B})\simeq\mathrm{W}^{k,p}(\mathrm{B},V)$. Equivalently,*

$$\|\,\mathrm{D}^k u\|_{\mathrm{L}^p(\mathrm{B},V\odot^k\mathbb{R}^n)}\lesssim\|\mathbb{A}u\|_{\mathrm{L}^p(\mathrm{B},W)}+\|u\|_{\mathrm{L}^p(\mathrm{B},V)}\tag{2.13}$$

for all $u\in\mathrm{C}^\infty(\bar{\mathrm{B}},V)$.

Recall that $\mathbb{A}$ having finite dimensional null–space, henceforth abbreviated *FDN*, i.e., Theorem 2.2(b), is equivalent to $\mathbb{C}$–ellipticity [38, Prop. 3.1].

Theorem 2.9 (Theorem 2.4)**.** *Let $\mathbb{A}$ be as in (2.1). Then $\mathbb{A}$ is elliptic and canceling if and only if $\dot{\mathrm{W}}^{\mathbb{A},1}(\mathbb{R}^n)\hookrightarrow\dot{\mathrm{W}}^{k-1,n/(n-1)}(\mathbb{R}^n,V)$, i.e.,*

$$\|\,\mathrm{D}^{k-1}u\|_{\mathrm{L}^{n/(n-1)}(\mathbb{R}^n,V\odot^{k-1}\mathbb{R}^n)}\lesssim\|\mathbb{A}u\|_{\mathrm{L}^1(\mathbb{R}^n,W)}$$

for all $u\in\mathrm{C}_c^\infty(\mathbb{R}^n,V)$.

Theorem 2.10 (Theorem 2.5)**.** *Let $\mathbb{A}$ be as in (2.1). Then $\mathbb{A}$ has FDN if and only if $\mathrm{W}^{\mathbb{A},1}(\mathrm{B})\hookrightarrow\mathrm{W}^{k-1,n/(n-1)}(\mathrm{B},V)$. Equivalently,*

$$\|\,\mathrm{D}^{k-1}u\|_{\mathrm{L}^{n/(n-1)}(\mathrm{B},V\odot^{k-1}\mathbb{R}^n)}\lesssim\|\mathbb{A}u\|_{\mathrm{L}^1(\mathrm{B},W)}+\|u\|_{\mathrm{L}^1(\mathrm{B},V)}$$

for all $u\in\mathrm{C}^\infty(\bar{\mathrm{B}},V)$.

Theorem 2.11 ([56, Thm. 1.3(a)])**.** *Let $\mathbb{A}$ be as in (2.1), $k<n$, $1\leqslant j\leqslant k$. The following are equivalent:*

(a) *$\mathbb{A}$ is elliptic and canceling.*
(b) *For all $u\in\mathrm{W}^{\mathbb{A},1}_{\mathrm{loc}}$, we have that $\mathrm{D}^{k-j}u\in\mathrm{t}^{j,n/(n-j)}(x)$ for $\mathscr{L}^n$–a.e. x.*

The necessity of ellipticity for Theorem 2.11 is slightly surprising, as it is not necessary, in general, for the embedding $\dot{\mathrm{W}}^{\mathbb{A},1}\hookrightarrow\dot{\mathrm{W}}^{k-j,n/(n-j)}$, which is used in [56] to show that (a) implies (b) (unless $j=1$ [80, Sec. 5.1]). The discrepancy is due to the fact that the statement in (b) is local.

3. Some detail on the proofs

3.1. **Smith's Theorem.** This section is devoted to the discussion of Theorem 2.8; we regard (2.13) as a *Korn–type inequality on domains.* It is well–known that Korn's Inequality holds on bounded domains, in the sense that, for $1<p<\infty$,

$$\|\,\mathrm{D}u\|_{\mathrm{L}^p(\mathrm{B})}\lesssim\|\mathcal{E}u\|_{\mathrm{L}^p(\mathrm{B})}+\|u\|_{\mathrm{L}^p(\mathrm{B})},\tag{3.1}$$

for all $u \in \mathrm{C}^\infty(\bar{\mathrm{B}}, \mathbb{R}^n)$, where $\mathrm{B} \subset \mathbb{R}^n$ is a generic ball (recall that $\mathcal{E}u = \frac{1}{2}(\mathrm{D}u + (\mathrm{D}u)^{\mathsf{T}})$. The original argument used to prove (3.1), was to write all second derivatives of a field u as first derivatives of components of $\mathcal{E}u$, i.e.,

$$\partial^2_{ij} u_k = \partial_i \left(\frac{\partial_j u_k + \partial_k u_j}{2} \right) - \partial_k \left(\frac{\partial_i u_j + \partial_j u_i}{2} \right) + \partial_j \left(\frac{\partial_k u_i + \partial_i u_k}{2} \right),$$

and then use Nečas's Negative Norm Theorem [42]. In his work [61], Smith characterized operators for which such an approach is possible, i.e., he showed that there exists a homogeneous operator $\mathbb{B}$ and an integer l such that

$$\mathrm{D}^l u = \mathbb{B}(\mathbb{A}u) \tag{3.2}$$

if and only if $\mathbb{A}$ is $\mathbb{C}$–elliptic (2.6). Sufficiency of $\mathbb{C}$–ellipticity for (3.2) can be proved by use of the Hilbert Nullstellensatz. Smith then showed that, by integration by parts of the l–th averaged Taylor polynomial of a map $u \in \mathrm{C}^\infty(\bar{\mathrm{B}}, V)$, one gets that

$$u(x) = \mathcal{P}u(x) + \int_{\mathrm{B}} \mathcal{K}(x,y)\mathbb{A}u(y)\,\mathrm{d}y \tag{3.3}$$

for all $x \in \mathrm{B}$ and a smooth kernel $\mathcal{K}\colon \mathbb{R}^n \times \mathbb{R}^n \setminus \{y = x\} \to \operatorname{Lin}(W, V)$ such

$$|\,\mathrm{D}^j_y \mathcal{K}(x,y)| \lesssim |x-y|^{k-n-j}$$

for $j = 0, 1, \ldots$ and any $x \neq y$. The V–valued polynomials $\mathcal{P}u$ have degree at most l. To show that (2.13) holds, one then shows that the extension operator

$$Eu(x) = \rho(x) \left(\mathcal{P}u(x) + \int_{\mathrm{B}} \mathcal{K}(x,y)\mathbb{A}u(y)\,\mathrm{d}y \right)$$

is bounded $\mathrm{W}^{\mathbb{A},p}(\mathrm{B}) \to \mathrm{W}^{\mathbb{A},p}(\mathbb{R}^n)$ by boundedness of singular integrals and Riesz potentials between L^p spaces. Here ρ is a test function that is equal to 1 in a neighborhood of B. One then has

$$\|\,\mathrm{D}^k u\|_{\mathrm{L}^p(\mathrm{B})} \leqslant \|\,\mathrm{D}^k Eu\|_{\mathrm{L}^p(\mathbb{R}^n)} \lesssim \|\mathbb{A}u\|_{\mathrm{L}^p(\mathrm{B})} + \|u\|_{\mathrm{L}^p(\mathrm{B})},$$

which was desired.

It turns out that $\mathbb{C}$–ellipticity is also necessary for (2.13) to hold. To see this, we first note that (3.3) implies that, if

$$\mathbb{A}u = 0 \text{ in } \mathrm{B}, \tag{3.4}$$

then u is a polynomial of degree at most l. One may regard this as an "extreme boundary regularity" statement, as can be seen by revisiting the example given in the introduction, i.e., $\mathbb{A} := \mathbb{A}_1$ being the anti–holomorphic derivative in (2.5). Then for $u(z) = \log z$ in a disc with boundary containing zero, (3.4) holds, but u is clearly not smooth up to the boundary. We will show that this behavior is in fact generic of elliptic, non–$\mathbb{C}$–elliptic operators. Indeed, consider such an operator $\mathbb{A}$, so there exist $\xi \in \mathbb{C}^n \setminus \mathbb{R}^n$, $v \in (V + \mathrm{i}V) \setminus V$ such that $\mathbb{A}[\xi]v = 0$. Consider a holomorphic map f and let u be formally defined by the plane wave

$$u(x) := f(x \cdot \xi)v. \tag{3.5}$$

Here the dot product is taken in $\mathbb{C}^n \times \mathbb{C}^n$, with a slight abuse of notation, since we still want $x \in \mathbb{R}^n$. Then one can show by careful computation that

$$\mathbb{A}\Re u = 0 = \mathbb{A}\Im u$$

in $\mathscr{D}'(\mathbb{R}^n,V)$ [38, Prop. 3.1]. Moreover, by letting $f(z)=z^{-1}$ and choosing the ball $\mathrm{B}\subset\mathbb{R}^n$ carefully, one can see that solutions of (3.4) need not be smooth up to the boundary.

Another point that is now easy to see, present in SMITH's original work [61], is that if we take $f(z)=\exp(jz)$, we obtain necessity of $\mathbb{C}$–ellipticity for (2.13).

Lastly, we can also extrapolate that $\mathbb{C}$–ellipticity is equivalent to the null–space of $\mathbb{A}$ having finite dimension. This point was used in the recent works [19, 38] to obtain embeddings in the limiting case $p=1$ and motivates the definition:

Definition 3.1. *An operator $\mathbb{A}$ as in* (2.1) *is said to have* finite dimensional null–space *(FDN) if and only if* $\dim\{u\in\mathscr{D}'(\mathbb{R}^n,V)\colon \mathbb{A}u=0\}<\infty$.

We collect the facts described above, to obtain the following extended version of SMITH's Theorem:

Theorem 3.2. *Let $\mathbb{A}$ be as in* (2.1)*. The following are equivalent:*

- (a) *$\mathbb{A}$ is $\mathbb{C}$–elliptic.*
- (b) *There exist an integer l and a homogeneous, linear differential operator $\mathbb{B}$ such that $\mathrm{D}^l=\mathbb{B}\circ\mathbb{A}$ holds.*
- (c) *The representation formula* (3.3) *holds.*
- (d) *There exists an integer l such that distributions satisfying $\mathbb{A}u=0$ in B are polynomials of degree at most l.*
- (e) *Distributions satisfying $\mathbb{A}u=0$ in B are maps in $\mathrm{C}^\infty(\bar{\mathrm{B}},V)$.*
- (f) *For $1<p<\infty$, the* KORN*–type inequality* (2.13) *holds.*
- (g) *$\mathbb{A}$ has FDN.*

In particular, the polynomials in (d) *are independent of* B.

One can also correlate $\mathrm{W}^{k-1/p,p}$–regularity of traces of $\mathrm{W}^{\mathbb{A},p}(\mathrm{B})$–maps to the FDN condition. Somewhat surprisingly, it is known that, for first order operators, this also covers the case $p=1$ [19], although this cannot follow from Theorem 3.2, by Ornstein's non–inequality. As far as the higher order case is concerned, we point out the recent work [49] of MIRONESCU and RUSS, where it is shown that the trace space of $\mathrm{W}^{k,1}(\mathbb{R}^n_+)$ is given by the Besov space $\mathrm{B}^{k-1}_{1,1}(\mathbb{R}^{n-1})$. Interestingly, this space is strictly contained in $\mathrm{W}^{k-1,1}(\mathbb{R}^{n-1})$ in general [20, Rk. A.1]. In view of [38, Thm. 1.3], we conclude this section with the following:

Question 3.3. *Let $\mathbb{A}$ as in* (2.1) *have FDN. Is it the case that there exists a linear, surjective, continuous trace operator* $\mathrm{Tr}\colon\ \mathrm{W}^{\mathbb{A},1}(\mathbb{R}^n_+)\to\mathrm{B}^{k-1}_{1,1}(\mathbb{R}^{n-1})$*?*

3.2. **VAN SCHAFTINGEN's Theorem.** We move on to the discussion of Theorem 2.9, stating that the estimate

$$\|\mathrm{D}^{k-1}u\|_{\mathrm{L}^{n/(n-1)}(\mathbb{R}^n,V\odot^{k-1}\mathbb{R}^n)}\lesssim\|\mathbb{A}u\|_{\mathrm{L}^1(\mathbb{R}^n,W)}\tag{3.6}$$

holds for all $u\in\mathrm{C}^\infty_c(\mathbb{R}^n,V)$ if and only if $\mathbb{A}$ is elliptic and canceling (EC).

The estimate (3.6) contrasts both standard harmonic analysis estimates, which are often of weak–type in L^1, as well as Ornstein's non–inequality. We aim to explain why (3.6) holds precisely for EC operators and build a bridge between the algebraic description (2.8) of the canceling condition for an operator $\mathbb{A}$ and the PDE description in Theorem 2.4(c). To this end, we assume that $\mathbb{A}$ is elliptic, although this is not necessary for the time being, as we will discuss in Section 5.1.

It was shown in [17, Lem. 2.1] that we have a representation

$$\mathrm{D}^{k-1}u=\mathrm{D}^{k-1}G*\mathbb{A}u=:I^{\mathbb{A}}_1(\mathbb{A}u)\tag{3.7}$$

for $u\in \mathrm{C}_c^\infty(\mathbb{R}^n,V)$, where $\mathrm{D}^{k-1}G\in \mathrm{C}^\infty(\mathbb{R}^n\setminus\{0\},\mathrm{Lin}(W,V\odot^{k-1}\mathbb{R}^n))$ is $(1-n)$–homogeneous. Here G is defined by the standard one–sided inverse

$$\widehat{G}(\xi):=(\Delta_{\mathbb{A}}[\xi])^{-1}\mathbb{A}^*[\xi] \tag{3.8}$$

for $\xi\neq 0$, where $\Delta_{\mathbb{A}}:=\mathbb{A}^*\mathbb{A}$. Such an inverse exists by ellipticity of $\mathbb{A}$. One can then show that G extends to a tempered distribution which is $\mathrm{L}^1_{\mathrm{loc}}$–integrable and enjoys the suitable smoothness. The notation $I_1^{\mathbb{A}}$ is justified by the fact that, indeed, by homogeneity and smoothness of G, the *Riesz–type potential* $I_1^{\mathbb{A}}$, which can be defined on L^1, is indeed bounded pointwisely by the Riesz potential I_1, i.e.,

$$|(I_1^{\mathbb{A}}f)(x)|\lesssim (I_1|f|)(x) \tag{3.9}$$

for a.e. $x\in\mathbb{R}^n$ and any $f\in \mathrm{L}^1(\mathbb{R}^n,W)$. We recall that $I_1g:=|\cdot|^{1-n}*g$ for scalar valued $g\in\mathrm{L}^1(\mathbb{R}^n)$. It is well known that I_1 is bounded $\mathrm{L}^1\to\mathrm{L}^{n/(n-1)}_{\mathrm{weak}}$ [63, Ch. V]. In particular, ellipticity of $\mathbb{A}$ is sufficient to give the weak–type estimate

$$\lambda\mathscr{L}^n\left(\{x\colon |(I_1^{\mathbb{A}}f)(x)|>\lambda\}\right)^{(n-1)/n}\lesssim \|f\|_{\mathrm{L}^1} \tag{3.10}$$

for *all* $f\in\mathrm{L}^1(\mathbb{R}^n,W)$. This estimate cannot in general be improved to a strong estimate as f can be chosen to approximate the identity, e.g., if f is a sequence of standard mollifiers.

The reasoning above shows, roughly speaking, that a straightforward harmonic analysis approach to reduce the estimate (3.6) to boundedness of (essentially scalar) Riesz potentials is doomed to fail. In turn, if one takes into account the vectorial structure of the operator $\mathbb{A}$ and views $I_1^{\mathbb{A}}$ as a map defined from $\mathrm{im}\mathbb{A}\cap\mathrm{L}^1\lneq\mathrm{L}^1$ into $\mathrm{L}^{n/(n-1)}_{\mathrm{weak}}$, then perhaps the estimate can be strengthened, as is the case with the Gagliardo–Nirenberg inequality.

To rule out the counterexample to (3.10), we want to rule out concentrations

$$f_r: x\mapsto r^{-n}f(r^{-1}x)\text{ as } r\downarrow 0$$

for $f\in\mathrm{C}_c^\infty(\mathbb{R}^n,W)\cap\mathrm{im}\mathbb{A}$. It is easy to see that $f_r\overset{*}{\rightharpoonup}\delta_0\int f$ in $\mathcal{M}(\mathbb{R}^n,W)$ as $r\downarrow 0$. If we more specifically let $f=\mathbb{A}u$ for some $u\in\mathrm{C}^\infty(\mathbb{R}^n,V)$, with the adjusted rescaling

$$u_r(x):=r^{1-n}u(r^{-1}x),$$

we obtain that

$$\mathbb{A}u_r\overset{*}{\rightharpoonup}\delta_0\int_{\mathbb{R}^n}\mathbb{A}u\,\mathrm{d}x. \tag{3.11}$$

In the absence of ellipticity, it is not clear if one can show that the RHS lies in $\mathrm{im}\mathbb{A}$. This is usually achieved by applying an exact annihilator of $\mathbb{A}$ (i.e. analogous to curl for $\mathbb{A}=\mathrm{D}$) to (3.11). Since we assume that $\mathbb{A}$ is elliptic, we can use the projection operator defined by

$$\mathcal{A}[\xi]=\det(\Delta_{\mathbb{A}}[\xi])\left(\mathrm{Id}-\mathbb{A}[\xi](\Delta_{\mathbb{A}}[\xi])^{-1}\mathbb{A}^*[\xi]\right), \tag{3.12}$$

for $\xi\in\mathbb{R}^n\setminus\{0\}$ [80, Rk. 4.1]. Recall that $\Delta_{\mathbb{A}}:=\mathbb{A}^*\mathbb{A}$, such that, clearly, $\ker\mathcal{A}[\xi]=\mathbb{A}[\xi](V)$ for all non–zero ξ. We apply $\mathcal{A}$ to (3.11) and Fourier transform to get that $\int\mathbb{A}u\in\ker\mathcal{A}[\xi]$ for all $\xi\neq 0$. Collecting, we expect the following:

Lemma 3.4 ([80, Prop. 5.5, 6.1])**.** *Let* $\mathbb{A}$ *be elliptic. Then*

$$\bigcap_{\xi\in\mathbb{R}^n\setminus\{0\}} \mathbb{A}[\xi](V) = \{w\in W\colon \mathbb{A}u=\delta_0 w \text{ for some } u\in\mathscr{S}'(\mathbb{R}^n,V)\}$$
$$=\left\{\int_{\mathbb{R}^n}\mathbb{A}u\,\mathrm{d}x\colon u\in \mathrm{C}^\infty(\mathbb{R}^n,V), \mathbb{A}u\in\mathrm{C}^\infty_c(\mathbb{R}^n,W)\right\}$$

In particular, the following are equivalent

(a) $\mathbb{A}$ *is canceling.*
(b) $\int\mathbb{A}u=0$ *for all smooth* u *such that* $\mathbb{A}u$ *has compact support.*
(c) *If* $u\in\mathscr{S}'(\mathbb{R}^n,V)$*,* $w\in W$ *such that* $\mathbb{A}u=\delta_0w$*, then* $w=0$*.*

Rephrasing, $\mathbb{A}$ *is canceling if and only if* $\mathbb{A}$ *admits no fundamental solutions.*

Proof. We will regard the three vector spaces in the statement simply as *first*, *second*, and *third*. We already showed that the third space is contained in the second. Conversely, one considers a mollification of the equation for fundamental solutions. To see that the second space is contained in the first, one simply applies the Fourier transform to $\mathbb{A}u=\delta_0w$ to get that $w=\mathbb{A}[\xi]\hat{u}(\xi)$ for all $\xi\neq0$. Conversely, let w lie in the first space. Define $u:=Gw$, so that $\hat{u}(\xi)=(\Delta_{\mathbb{A}}[\xi])^{-1}\mathbb{A}^*[\xi]w$ for $\xi\neq0$. By ellipticity, the one–sided inverse of $\mathbb{A}[\xi]$ is an isomorphism $\mathbb{A}[\xi](V)\leftrightarrow V$, so that $\mathbb{A}[\xi]\hat{u}=w$ for $\xi\neq0$, hence $\mathbb{A}u=\delta_0w$. □

We record that ellipticity is not necessary for Lemma 3.4 to hold, as we will discuss in Section 5.1.

It is clear that for an elliptic, non–canceling operator, we have by Lemma 3.4(c) the existence of $w\neq0$ such that $\mathbb{A}v=\delta_0w$ for some $v\in\mathscr{S}'$ which can be chosen such that

$$\mathrm{D}^{k-1}v=\mathrm{D}^{k-1}G*\mathbb{A}v=\mathrm{D}^{k-1}G*\delta_0w=\mathrm{D}^{k-1}Gw. \tag{3.13}$$

Since $\mathrm{D}^{k-1}G$ is $(1-n)$–homogeneous, we get that $\mathrm{D}^{k-1}v$ is not locally $\mathrm{L}^{n/(n-1)}$–integrable. It is then a matter of cutting–off and truncating suitably to contradict the embedding (3.6) (see [80, Prop. 5.5] for full detail).

Summarizing, we sketched a proof of necessity of cancellation for the embedding (3.6), provided that $\mathbb{A}$ is elliptic. Ellipticity is also necessary, as can be seen from [80, Cor. 5.2], where a modification of the plane wave (3.5) (with real ξ,v) is used. It is worth mentioning that, somewhat surprisingly, ellipticity is not necessary for Sobolev–type estimates on the weaker derivatives, as is discussed in [80, Sec. 5.1]. To the best of our knowledge, there is no theory describing estimates such as

$$\|\mathrm{D}^{k-j}u\|_{\mathrm{L}^{n/(n-j)}}\lesssim\|\mathbb{A}u\|_{\mathrm{L}^1}$$

for $u\in\mathrm{C}^\infty_c(\mathbb{R}^n,V)$, for non–elliptic operators $\mathbb{A}$ as in (2.1) of order $k\geq j$; except, of course, if $j=0,1$, in which case there are none.

It is remarkable that EC is, in fact, sufficient for the estimate. This was proved by duality at the level of the annihilator $\mathcal{A}$, i.e.,

$$\int_{\mathbb{R}^n}\langle f,\varphi\rangle\,\mathrm{d}x\lesssim\|f\|_{\mathrm{L}^1}\|\mathrm{D}\varphi\|_{\mathrm{L}^n}, \tag{3.14}$$

for $\mathcal{A}$–free L^1–fields f and $\varphi\in\mathrm{C}^\infty_c(\mathbb{R}^n,W)$. In fact, the estimate (3.14) holds of all co–canceling operators $\mathcal{A}$, introduced in [80, Def. 1.3] as satisfying

$$\bigcap_{\xi\in\mathbb{R}^n\setminus\{0\}}\ker\mathcal{A}[\xi]=\{0\}. \tag{3.15}$$

The first instance of (3.14) was proved by BOURGAIN and BREZIS in [12] for $\mathcal{A} = \operatorname{div}$ (see also [13, Thm. 1′]). An elementary proof was given in [77, Thm. 1.5], by a slicing argument, using the Morrey–Sobolev embedding on hyperplanes, an integration by parts formula, and Hölder's inequality; see also [81, Direct proof of Theorem 1.3]. Both proofs have the estimate on circulation along closed curves [16, Prop. 4] as a starting point (see also [76]). The latter argument was generalized to the k–th order invariant operator $\mathcal{A}f := \sum_{|\alpha|=k} \partial^\alpha f_\alpha$ in [78, Thm. 4] and algebraically extended to all co–canceling operators (3.15) in [80, Prop. 2.3]. For more detail, see the survey [81, Sec. 3.4].

The duality argument can be formulated by noting that the annihilator $\mathcal{A}$, defined in (3.12), of and EC operator $\mathbb{A}$ is co–canceling, so that (3.14) implies

$$\|\mathrm{D}^{k-1}u\|_{\mathrm{L}^{n/(n-1)}} \lesssim \|\mathbb{A}u\|_{\dot{\mathrm{W}}^{-1,n/(n-1)}} \lesssim \|\mathbb{A}u\|_{\mathrm{L}^1}$$

for all $u \in \mathrm{C}_c^\infty(\mathbb{R}^n, V)$, where the first inequality follows by boundedness of singular integrals.

We note that for the estimate (3.6), the competitor maps have zero boundary values. In fact, with appropriate scaling, (3.6) is equivalent to

$$\|\mathrm{D}^{k-1}u\|_{\mathrm{L}^{n/(n-1)}(\mathrm{B},V\odot^{k-1}\mathbb{R}^n)} \lesssim \|\mathbb{A}u\|_{\mathrm{L}^1(\mathrm{B},W)} \tag{3.16}$$

for $u \in \mathrm{C}_c^\infty(\mathrm{B}, V)$ [38, Lem. 5.7]. It is natural to ask the related question: For $u \in \mathrm{C}^\infty(\bar{\mathrm{B}}, V)$, under which conditions on $\mathbb{A}$ does the estimate

$$\|\mathrm{D}^{k-1}u\|_{\mathrm{L}^{n/(n-1)}(\mathrm{B},V\odot^{k-1}\mathbb{R}^n)} \lesssim \|\mathbb{A}u\|_{\mathrm{L}^1(\mathrm{B},W)} + \|u\|_{\mathrm{L}^1(\mathrm{B},V)} \tag{3.17}$$

hold? As mentioned in Section 2.4, this question was answered by GMEINEDER and the author in [38, Thm. 1.3], and generalizes the Gagliardo–Nirenberg inequality on domains and the Korn–Sobolev inequality of STRANG and TEMAM in [69, Thm. 2.1]:

$$\|u\|_{\mathrm{L}^{n/(n-1)}(\mathrm{B},\mathbb{R}^n)} \lesssim \|\mathcal{E}u\|_{\mathrm{L}^1(\mathrm{B},\mathbb{R}^{n\times n}_{\mathrm{sym}})} + \|u\|_{\mathrm{L}^1(\mathrm{B},\mathbb{R}^n)}$$

for all $u \in \mathrm{C}^\infty(\bar{\mathrm{B}}, \mathbb{R}^n)$. We postpone the discussion of (3.17) to Section 4.1, and discuss another paper pertaining to the difference between (3.16) and (3.17).

For the remainder of this section, we restrict our attention to the case $k = 1$, of first order operators. It was shown in [19] that a trace embedding

$$\|u\|_{\mathrm{L}^1(\partial\mathrm{B},V)} \lesssim \|\mathbb{A}u\|_{\mathrm{L}^1(\mathrm{B},W)} + \|u\|_{\mathrm{L}^1(\mathrm{B},V)} \tag{3.18}$$

holds for all $u \in \mathrm{C}^\infty(\bar{\mathrm{B}}, V)$ if and only if $\mathbb{A}$ has FDN. The proof of (3.18) in the case $\mathbb{A} = \mathrm{D}$ is relatively simple, as one can employ the Fundamental Theorem of Calculus near the boundary of B. This tool is obviously unavailable in general and the idea employed in [19, Sec. 4] is to substitute it by replacing u on small balls B_j near the boundary of B by suitable projections onto the null–space of $\mathbb{A}$ in B_j. FTC is replaced by estimates on chains of balls by use of a Poincaré–type inequality, reminiscent of the idea in [44, Sec. 2]. The FDN condition crucially enters the estimation when the $\mathrm{L}^1(\partial\mathrm{B}\cap\mathrm{B}_j)$–norm of the approximation is bounded by Hölder's inequality against the $\mathrm{L}^\infty(\partial\mathrm{B}_j)$–norm, which can be controlled by the $\mathrm{L}^1(\mathrm{B}_j)$–norm since the null–space of $\mathbb{A}$ in B_j is finite dimensional.

Conversely, if FDN, and, hence, complex–ellipticity, fails, one takes u as in (3.5) with complex ξ, v and $f(z) = z^{-1}$ and a ball $\mathrm{B} \subset \mathbb{R}^n$ such that $0 \in \partial\mathrm{B}$. A simple computation shows that $u \in \mathrm{L}^1(\mathrm{B}) \setminus \mathrm{L}^1(\partial\mathrm{B})$. This example has already been discussed in Section 3.1, from which we recall that $\mathbb{A}u = 0$, so the trace embedding fails.

We keep the restriction $k = 1$ for clarity of exposition and conclude that the FDN condition is indeed a condition intimately linked to boundary regularity, since it is equivalent to both

(a) if $\mathbb{A}u = 0$ in B, then $u \in \mathrm{C}^\infty(\partial\,\mathrm{B})$,
(b) if $u, \mathbb{A}u \in \mathrm{L}^1(\mathrm{B})$, then $u \in \mathrm{L}^1(\partial\,\mathrm{B})$.

On the other hand, for elliptic $\mathbb{A}$, cancellation is a condition of interior regularity, as it is equivalent to both

(a) if $\mathbb{A}u \in \mathcal{M}(\mathbb{R}^n, W)$, then $u \in \mathrm{L}^{n/(n-1)}_{\mathrm{loc}}$,
(b) if $\mathbb{A}u = \delta_0 w$ and u satisfies the smallness condition (2.4), then $u = 0$.

In fact, if we assume that $\mathbb{A}$ is elliptic, non–canceling, and, without loss of generality, that $0 \in \mathrm{B}$, and let $u := Gw$, so that $\mathbb{A}u = \delta_0 w$ for $w \neq 0$, then clearly $u \notin \mathrm{C}^\infty(\partial\,\mathrm{B})$, so $\mathbb{A}$ cannot have FDN. Here G is given by (3.8). This basic idea will be extended in Section 4.1 to show that FDN (strictly) implies EC.

4. Recent contributions

4.1. **The Sobolev–type embedding on domains.** We proceed with the overview of the answer to the question of identifying computable conditions for (3.17):

Theorem 4.1 ([38, Thm. 1.3])**.** *Let $\mathbb{A}$ be as in* (2.1)*. The following are equivalent:*

(a) *$\mathbb{A}$ has FDN.*
(b) *$\mathbb{A}$ is EC and there exists an extension operator $\mathbf{E}\colon \mathrm{W}^{\mathbb{A},1}(\mathrm{B}) \to \mathrm{W}^{\mathbb{A},1}(\mathbb{R}^n)$ which is linear and bounded.*
(c) *$\mathrm{W}^{\mathbb{A},1}(\mathrm{B}) \hookrightarrow \mathrm{L}^{n/(n-1)}(\mathrm{B}, V)$ holds, i.e.,*
$$\|\,\mathrm{D}^{k-1}u\|_{\mathrm{L}^{n/(n-1)}(\mathrm{B},V\odot^{k-1}\mathbb{R}^n)} \lesssim \|\mathbb{A}u\|_{\mathrm{L}^1(\mathrm{B},W)} + \|u\|_{\mathrm{L}^1(\mathrm{B},V)}$$
for all $u \in \mathrm{C}^\infty(\bar{\mathrm{B}}, V)$.

We first show that FDN implies EC using Lemma 3.4(c), thereby formalizing the heuristics at the end of the previous Section. We refer the reader to [38, Lem. 3.2] for a proof using Lemma 3.4(b).

Lemma 4.2. *Let $\mathbb{A}$ as in* (2.1) *have FDN. Then $\mathbb{A}$ satisfies EC.*

Proof. It is clear from (3.5) that $\mathbb{A}$ is elliptic. Let $w \in W$, $v \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ be such that $\mathbb{A}v = \delta_0 w$. Consider a ball $\mathrm{B} \subset \mathbb{R}^n$ such that $0 \in \partial\,\mathrm{B}$. Then $\mathbb{A}v = 0$ in B, so that v, and hence $\mathrm{D}^{k-1}v$, equal a polynomial in B by Theorem 3.2. By $(1-n)$–homogeneity of $\mathrm{D}^{k-1}v$ and geometry of B, it follows that $\mathrm{D}^{k-1}v = 0$ in $\mathbb{R}^n \setminus T^0_{\mathrm{B}}$, where T^0_{B} denotes the tangent plane of B at 0. By smoothness of G away from 0, we get that $\mathrm{D}^{k-1}v = 0$, so that $0 = \mathbb{A}v = \delta_0 w$. We conclude that $w = 0$, so $\mathbb{A}$ is canceling by Lemma 3.4(c). □

The second step consists of a Jones–type extension [38, Thm. 4.1], which is much in the spirit of [44, Sec. 3]. We would like to single out as a particularly relevant modification the Poincaré–type inequality for FDN operators [38, Prop. 4.2]:
$$\inf_{\mathbb{A}v=0 \text{ in } \mathrm{B}} \|\,\mathrm{D}^j(u-v)\|_{\mathrm{L}^1(\mathrm{B})} \lesssim \mathrm{diam}(\mathrm{B})^{k-j}\|\mathbb{A}u\|_{\mathrm{L}^1(\mathrm{B})},$$
for $u \in \mathrm{C}^\infty(\bar{\mathrm{B}}, V)$, $j = 0 \dots k-1$. This is a generalization of [19, Thm. 3.3].

With the bounded extension $\mathbf{E}\colon \mathrm{W}^{\mathbb{A},1}(\mathrm{B}) \to \mathrm{W}^{\mathbb{A},1}(\mathbb{R}^n)$ in place, we can estimate, by use of Lemma 4.2 and (3.6)
$$\begin{aligned}\|\,\mathrm{D}^{k-1}u\|_{\mathrm{L}^{n/(n-1)}(\mathrm{B})} &\leqslant \|\,\mathrm{D}^{k-1}\mathbf{E}u\|_{\mathrm{L}^{n/(n-1)}(\mathbb{R}^n)} \lesssim \|\mathbb{A}\mathbf{E}u\|_{\mathrm{L}^1(\mathbb{R}^n)}\\ &\lesssim \|\mathbb{A}u\|_{\mathrm{L}^1(\mathrm{B})} + \|u\|_{\mathrm{L}^1(\mathrm{B})},\end{aligned}$$

which proves (c).

Conversely, assuming that $\mathbb{A}$ does not have FDN, the aforementioned example of SMITH ($u_j(x) := \exp(jx\cdot\xi)v$ for non–zero, complex ξ, v such that $\mathbb{A}[\xi]v = 0$) can be used to disprove (3.17) provided $k > 1$. To cover the case $k = 1$, we showed that the embedding $\mathrm{W}^{\mathbb{A},1}(\mathrm{B}) \hookrightarrow \mathrm{L}^p(\mathrm{B}, V)$ is compact for $1 \leqslant p < n/(n-1)$ [38, Thm. 4.6]. This fact is itself a generalization of the Theorem of SUQUET in the case $\mathbb{A} = \mathcal{E}$ [71]. The proof consists of a careful application of the Riesz–Kolmogorov compactness criterion. It is then easy to conclude that $\mathbb{A}$ has FDN by using the Peetre–Tartar Equivalence Lemma [74, Lem. 11.1].

It remains to confirm that the implication of EC by FDN is strict, otherwise Theorem 4.1(b) would contain redundancy. We restrict the discussion here to the case $k = 1$ and refer the reader to [38, Sec. 3] for a more comprehensive debate. By Theorem 3.2, we know that it is not difficult to construct an elliptic, non–FDN operator, since such an operator should have (2.5) as a building block. Interestingly, if $n = 2$, it is not possible to build such a canceling operator, as is shown in [38, Lem. 3.5(a)]. However, as soon as we allow for $n = 3$, we see that the operator

$$\mathbb{A}_2 u := \begin{pmatrix} \partial_1 u_1 - \partial_2 u_2 \\ \partial_2 u_1 + \partial_1 u_2 \\ \partial_3 u_1 \\ \partial_3 u_2 \end{pmatrix}, \tag{4.1}$$

defined for $u\colon \mathbb{R}^3 \to \mathbb{R}^2$, and mentioned in Section 2.2 satisfies EC, but not FDN.

This fact has two somewhat surprising consequences. Let $\mathbb{A}$ be an elliptic, first order operator and recall that (3.6) is equivalent to $\mathrm{W}_0^{\mathbb{A},1}(\mathrm{B}) \hookrightarrow \mathrm{L}^{n/(n-1)}(\mathrm{B}, V)$ by a scaling argument [38, Lem. 5.7]. Here $\mathrm{W}_0^{\mathbb{A},1}(\mathrm{B})$ denotes the closure of $\mathrm{C}_c^\infty(\mathrm{B}, V)$ in the (semi–)norm $u \mapsto \|\mathbb{A}u\|_{\mathrm{L}^1}$. Then:

(a) If $\mathbb{A}$ is in addition canceling, but not FDN, there is *no* bounded, linear extension operator $\mathbf{E}\colon \mathrm{W}^{\mathbb{A},1}(\mathrm{B}) \to \mathrm{W}^{\mathbb{A},1}(\mathbb{R}^n)$. This phenomenon cannot be observed by looking at $\mathrm{W}^{1,1}$ and indicates that the analysis of vectorial operators can hide non–obvious phenomena.

(b) If $\mathbb{A}$ is not canceling, we still have the embedding $\mathrm{W}_0^{\mathbb{A},1}(\mathrm{B}) \hookrightarrow \mathrm{L}^p(\mathrm{B}, V)$ for all $1 \leqslant p < n/(n-1)$, e.g., by (3.10). If $\mathbb{A}$ is canceling, non–FDN, we have that $\mathrm{W}_0^{\mathbb{A},1}(\mathrm{B}) \hookrightarrow \mathrm{L}^{n/(n-1)}(\mathrm{B}, V)$ but there exists $u \in \mathrm{W}^{\mathbb{A},1}(\mathrm{B})$ such that $\mathbb{A}u = 0$ but u has *no* higher integrability [38, Lem. 3.6].

Of course, the map u constructed in (b) lies in $\mathrm{L}^{n/(n-1)}_{\mathrm{loc}}(\mathrm{B}, V)$ by (3.6), which highlights the connection between the Sobolev–type embedding on domains (3.17) and the trace embedding (3.18). Although the relation between the two embeddings was established indirectly, both being equivalent to $\mathbb{A}$ having FDN, we do not have any direct proof of one implying the other.

4.2. **L^p–differentiability of $\mathrm{BV}^{\mathbb{A}}$–maps.** Apart from embedding theorems, one can, of course, inquire about pointwise properties of $\mathbb{A}$–weakly differentiable functions. Such properties would be, for instance, the existence a.e. of a *k–th order L^p–Taylor expansion*, as introduced by CALDERÓN and ZYGMUND in [22] as a generalization on the L^p–scale of classical differentiability. On their scale, classical differentiability at a point corresponds to the endpoint $p = \infty$ and weak derivatives equal L^p–derivatives almost everywhere, showing consistency of their notion. With the risk of redundancy, we recall for the reader's convenience that $u \in \mathrm{t}^{k,p}(x)$

if and only if there exists a polynomial $P_x^k u$ of degree at most k such that

$$\left(\fint_{\mathrm{B}_r(x)} |u - P_x^k u|^p \,\mathrm{d}y \right)^{1/p} = o(r^k) \text{ as } r \downarrow 0.$$

If $k = 1$, we abbreviate and say that u is L^p*–differentiable at* x. If $p = \infty$, we replace the quantity on the right hand side by the L^∞–norm of $u - P_x^k u$, in which case u is k times differentiable in the classical sense at x.

If $1 \leqslant p < \infty$, L^p–Taylor expansions of $\mathrm{W}^{k,p}$–maps are completely described in the original paper [22, Thm. 12]. Since Theorem 2.1(b) holds under the mild assumption of ellipticity, in the case of $\mathrm{W}^{\mathbb{A},p}$, we are only interested in $p = 1$. In fact, ellipticity is always necessary for L^q–differentiability of $\mathrm{W}^{\mathbb{A},p}$–maps [56, Lem. 4.2]. Interestingly, even though by Ornstein's Non–inequality, for $u \in \mathrm{BV}^{\mathbb{A}}_{\mathrm{loc}}$, $\mathrm{D}^k u$ need not be a Radon measure, one can still show that, for elliptic $\mathbb{A}$, the k–th approximate differential of u at x (defined as $\nabla^k u(x) := \mathrm{D}^k P_x^k u(x)$) exists for a.e. $x \in \mathbb{R}^n$. Of course, the measurable map $x \mapsto \nabla^k u(x)$ fails to be locally integrable in general.

The question of L^p–differentiability of $\mathrm{BV}^{\mathbb{A}}$–maps is treated in the author's work [56]. In analogy with boundedness of Riesz–type potentials, as discussed in Section 3.2, it was shown that ellipticity of $\mathbb{A}$ is equivalent to existence of L^p–Taylor expansions with sub–critical exponents:

Theorem 4.3 ([56, Thm. 1.4])**.** *Let $\mathbb{A}$ be as in (2.1). Then $\mathbb{A}$ is elliptic if and only if either of the following holds:*

(a) *If $1 \leqslant j \leqslant \min\{k, n-1\}$, $1 < p < n/(n-j)$, we have that*

$$\mathrm{D}^{k-j} u \in \mathrm{t}^{j,p}(x) \textit{ for } \mathscr{L}^n\textit{–a.e. } x \in \mathbb{R}^n.$$

for all $u \in \mathrm{BV}^{\mathbb{A}}_{\mathrm{loc}}$.

(b) *If $j = n \leqslant k$, $1 < p < \infty$, we have that*

$$\mathrm{D}^{k-j} u \in \mathrm{t}^{j,p}(x) \textit{ for } \mathscr{L}^n\textit{–a.e. } x \in \mathbb{R}^n.$$

for all $u \in \mathrm{BV}^{\mathbb{A}}_{\mathrm{loc}}$.

(c) *If $n < k$, we have that*

$$\mathrm{D}^{k-n-1} u \in \mathrm{t}^{n+1,\infty}(x) \textit{ for } \mathscr{L}^n\textit{–a.e. } x \in \mathbb{R}^n.$$

for all $u \in \mathrm{BV}^{\mathbb{A}}_{\mathrm{loc}}$.

In particular, for any ball $\mathrm{B} \subset \mathbb{R}^n$ and any $\varepsilon > 0$, there exists $E \subset B$ and $\tilde{u} \in \mathrm{C}^k(\mathbb{R}^n, V)$ such that $\mathscr{L}^n(\mathrm{B} \setminus E) < \varepsilon$ and $u = \tilde{u}$ in E. Also, in (c), u is k times differentiable a.e. (in the classical sense).

This Theorem is a simple consequence of the main result [3, Thm. 3.4] (built on the work in [41]), coupled with the iteration of [22, Thm. 11]. See [56, Sec. 2, Lem. 4.2] for the complete proof. The C^k–Lusin property follows directly from the fact that $u \in \mathrm{t}^{k,1}(x)$ for a.e. x, as explained in Section 2.3. It is quite interesting that this property survives, although for a general $\mathrm{BV}^{\mathbb{A}}$–map u it may well be that $\mathrm{D}^k u \notin \mathcal{M}_{\mathrm{loc}}$ and even $\nabla^k u \notin \mathrm{L}^1_{\mathrm{loc}}$.

The physically relevant case BD, covered by (a), was singled out by Alberti, Bianchini, and Crippa in [3, Prop. 4.2], where it is shown that maps of bounded deformation are L^p–differentiable a.e. for all $1 \leqslant p < n/(n-1)$. Previous work on this topic has been done in [4, Sec. 7], where Theorem 4.3(a) with $p = 1$ has been shown to hold in BD.

It is natural to ask under which additional conditions on $\mathbb{A}$ can the critical exponents in Theorem 4.3 be reached. To the author's knowledge, this was previously known only in the case $\mathbb{A}=\mathrm{D}^k$, from the works [23, 28]. A first step in this direction was made recently by GMEINEDER and the author in [39, Thm. 1.1], where we showed that if $k=1$ and $\mathbb{A}$ has FDN, then, indeed, any $u\in \mathrm{BV}^{\mathbb{A}}_{\mathrm{loc}}$ is such that $u\in \mathrm{t}^{1,n/(n-1)}(x)$ for a.e. $x\in\mathbb{R}^n$.

It does not, however, seem plausible that an assumption describing boundary regularity should describe a statement resembling interior critical higher integrability. Indeed, the author recently showed in [56, Thm. 1.3], that, at least for (a), critical exponents are equivalent to EC:

Theorem 4.4 ([56, Thm. 1.3])**.** *Let $\mathbb{A}$ be as in (2.1). Then*

(a) *If $k<n$, $j=1\dots k$, $\mathbb{A}$ is EC if and only if for all $u\in\mathrm{BV}^{\mathbb{A}}_{\mathrm{loc}}$ we have that $\mathrm{D}^{k-j}u\in\mathrm{t}^{j,n/(n-j)}(x)$ for $\mathscr{L}^n$–a.e. $x\in\mathbb{R}^n$.*

(b) *If $k\geq n$ and $\mathbb{A}$ is canceling, then $\mathbb{A}$ is elliptic if and only if for all $u\in\mathrm{BV}^{\mathbb{A}}_{\mathrm{loc}}$, we have that $\mathrm{D}^{k-n}u\in\mathrm{t}^{n,\infty}(x)$ for $\mathscr{L}^n$–a.e. $x\in\mathbb{R}^n$.*

In particular, in the former case, u has a k–th order $\mathrm{L}^{n/(n-k)}$–Taylor expansion a.e.; in the latter case u is k times differentiable a.e. (in the classical sense).

The proof of sufficiency of EC follows from applying [17, Thm. 1.1, 1.3] to $\rho_r(u-P^k_x u)$, where $\chi_{\mathrm{B}_r(x)}\leqslant\rho_r\leqslant\chi_{\mathrm{B}_{2r}(x)}$ is a smooth cut–off function. The problem is then reduced to Theorem 4.3, which also guarantees existence of $P^k_x u$ for a.e. x.

Necessity of ellipticity is proved using a "rough" plane wave (3.5) that is locally integrable, but nowhere L^p–integrable for some $p>1$. We thus obtain necessity of ellipticity, although, as mentioned in Section 3.2, ellipticity is not necessary for [17, Thm. 1.1], which is crucially used to prove the sufficiency part of Theorem 4.4. The reason for the discrepancy is, loosely speaking, that when we consider pointwise statements, we do not capture the behavior of the plane wave at infinity.

Necessity of cancellation for Theorem 4.4(a) is proved as follows: it is easy to see that $\mathrm{D}^{k-j}v\notin\mathrm{t}^{j,n/(n-j)}(0)$ for $j=1,\dots k$ and $v\in\mathrm{BV}^{\mathbb{A}}_{\mathrm{loc}}$ as defined by (3.13). This is due to the fact that $\mathrm{D}^{k-j}v$ is not $\mathrm{L}^{n/(n-j)}$–integrable in any neighborhood of 0. A Baire Category argument shows existence of $\tilde v\in\mathrm{BV}^{\mathbb{A}}_{\mathrm{loc}}$ such that $\mathrm{D}^{k-j}\tilde v$ is not $\mathrm{L}^{n/(n-j)}$–integrable in any neighborhood of any point in a dense subset of $\mathbb{R}^n$.

It would be interesting to obtain a sharp statement in Theorem 4.4(b) as well. We have already seen that, for $k<n$, a Sobolev–type embedding is equivalent to the pointwise a.e. property of existence of critical L^p–Taylor expansions. It seems that the right problem to tackle would be to specify additional conditions for elliptic $\mathbb{A}$ as in (2.1) with $k\geq n$ such that the inequality

$$\|\mathrm{D}^{k-n}u\|_{\mathrm{L}^\infty}\lesssim\|\mathbb{A}u\|_{\mathrm{L}^1} \tag{4.2}$$

for $u\in\mathrm{C}^\infty_c(\mathbb{R}^n,V)$. This would generalize the embedding $\mathrm{W}^{n,1}(\mathbb{R}^n)\hookrightarrow\mathrm{C}^0_0(\mathbb{R}^n)$ and would complement the analysis of BOUSQUET and VAN SCHAFTINGEN in [17, Thm. 1.3], where it is shown that EC is sufficient for (4.2). Conversely, there are elliptic, non–canceling operators for which the embedding holds ($\mathbb{A}u=u'$ for $n=1$), but there are also vectorial examples:

Let $\mathbb{B}:=(\operatorname{div},\operatorname{curl})$ on $\mathbb{R}^3$ from $\mathbb{R}^3$ to $\mathbb{R}^3$, and define $\mathbb{A}:=\mathbb{B}\circ\Delta$, which is elliptic (as composition of elliptic operators) and non–canceling, which can be easily checked by computation. One can then also check that $\mathbb{B}^*\circ\mathbb{A}=\Delta^2$, so that for $u \in \mathrm{C}_c^\infty(\mathbb{R}^3, \mathbb{R}^3)$ we have

$$u = \mathbb{B}^* \Delta^{-2} \mathbb{A} u.$$

Since in three dimensions the bilaplacian operator has fundamental solution proportional to $x \mapsto |x|$, it follows that its derivative and, hence, the Green's function for $\mathbb{A}$ is bounded. This is to say that $u = G * \mathbb{A}u$, for bounded (0–homogeneous) G. The inequality (4.2) follows by Young's convolution inequality. The conclusion extends immediately to $\mathbb{A}_d := \mathbb{B} \circ \Delta^d$ on $\mathbb{R}^{2d+1}$ from $\mathbb{R}^{2d+1}$ to $\mathbb{R}^{2d+1}$, showing that in all odd dimensions, there are elliptic, non–canceling operators for which (4.2) holds.

It is also not the case that (4.2) holds for all elliptic operators, as can be seen by looking at the fundamental solution of $\mathbb{A} = \Delta$ if $n = 2$.

This and Lemma 3.4 suggest that, in order to determine for which elliptic operators (4.2) holds, one should obtain better understanding of understand fundamental solutions for operators of order $k \geq n$. Even in the scalar case $\dim V = 1 = \dim W$, this is significantly more difficult than if $k < n$, as can be seen from [43, Thm. 7.1.20]. In view of this, we conclude the section with the following, perhaps optimistic, conjecture:

Conjecture 4.5. *Let $\mathbb{A}$ be elliptic of order $k \geq n$. The embedding*

$$\| \mathrm{D}^{k-n} u\|_{\mathrm{L}^\infty} \lesssim \|\mathbb{A}u\|_{\mathrm{L}^1}$$

holds for $u \in \mathrm{C}_c^\infty(\mathbb{R}^n, V)$ if and only if $\mathbb{A}$ is canceling or n is odd.

5. Some new ideas

5.1. Constant–rank operators.

In this section we sketch a proof of the following new estimate, the details of which will make the object of future work:

Theorem 5.1. *Let $\mathbb{A}$ as in (2.1) be a canceling operator of constant rank. Then the estimate*

$$\inf_{\mathbb{A}v=0 \text{ in } \mathbb{R}^n} \| \mathrm{D}^{k-1}(u-v)\|_{\mathrm{L}^{n/(n-1)}} \lesssim \|\mathbb{A}u\|_{\mathrm{L}^1} \tag{5.1}$$

holds for all $u \in \mathrm{L}^1_{\mathrm{loc}} \cap \mathscr{S}'(\mathbb{R}^n, V)$ such that $\mathbb{A}u \in \mathrm{L}^1(\mathbb{R}^n, W)$.

We say that an operator $\mathbb{A}$ is of *constant rank* if there exists an integer r such that $\operatorname{rank} \mathbb{A}[\xi] = r$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$. For example, elliptic operators are of constant rank $r = \dim V$.

A few remarks are in order. Firstly, the constant rank condition appears in the study of compensated compactness [72, 51, 75] and in the study of $\mathcal{A}$–quasiconvex integrals [32, 30, 18, 8]. In fact, except for examples [50, 73, 72], there is no theory of lower semi–continuity for integrals depending on vector fields satisfying differential constraints that do not satisfy the constant rank condition.

To prove Theorem 5.1, we construct an exact annihilator $\mathcal{A}$ of $\mathbb{A}$ which generalizes (3.12) and enables us to use (3.14). As a by–product, we obtain the algebraic fact that variational problems under differential constraints with constant rank symbol have an associated potential.

It will also become apparent from the proof that, if $\mathbb{A}$ is elliptic, the inequality (5.1) reduces to

$$\| \mathrm{D}^{k-1} u\|_{\mathrm{L}^{n/(n-1)}} \lesssim \|\mathbb{A}u\|_{\mathrm{L}^1} \tag{5.2}$$

for $u \in \mathrm{C}_c^\infty(\mathbb{R}^n, V)$, which was discussed in Section 3.2. On the other hand, an inequality such as (5.2) cannot hold for constant rank operators in general. To see this, let $\mathbb{A} := \operatorname{curl}$ act on vector fields $u\colon \mathbb{R}^3 \to \mathbb{R}^3$. By letting $u = \nabla\phi$ for any non–zero scalar field $\phi \in \mathrm{C}_c^\infty(\mathbb{R}^3)$, it is clear that (5.2) fails. This behaviour can be shown to be generic for non–elliptic operators of constant rank, see Remark 5.7. This indicates that the class of non–elliptic operators for which the weaker estimates

$$\begin{aligned} \|\mathrm{D}^{k-j}u\|_{\mathrm{L}^{n/(n-j)}} &\lesssim \|\mathbb{A}u\|_{\mathrm{L}^1} \quad \text{if } 1 < j \leqslant k < n \\ \|\mathrm{D}^{k-n}u\|_{\mathrm{L}^\infty} &\lesssim \|\mathbb{A}u\|_{\mathrm{L}^1} \quad \text{if } k \geq n \end{aligned}$$

hold is contained in the class of operators that do not have constant rank. Indeed, for the estimates [54] and [80, Prop. 5.4], the constant rank condition fails.

We record separately the result for $\mathbb{A} = \operatorname{curl}$, as we could not find the explicit statement in the literature:

$$\inf_{\phi\in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}^3)} \|u - \nabla\phi\|_{\mathrm{L}^{3/2}} \lesssim \|\operatorname{curl} u\|_{\mathrm{L}^1}$$

for all $u \in \mathrm{L}^1_{\mathrm{loc}} \cap \mathscr{S}'(\mathbb{R}^3, \mathbb{R}^3)$ such that $\operatorname{curl} u \in \mathrm{L}^1(\mathbb{R}^3, \mathbb{R}^3)$. This estimate can be used to prove the inequality

$$\|u\|_{\mathrm{L}^{3/2}} \lesssim \|\operatorname{curl} u\|_{\mathrm{L}^1} \tag{5.3}$$

for vector fields $u \in \mathrm{C}_c^\infty(\mathbb{R}^3, \mathbb{R}^3)$ such that $\operatorname{div} u = 0$, which was proved by Bourgain and Brezis in [12, Thm. 2]. Of course, this only shows consistency of our work, since we implicitly use [77, Cor. 1.4], which can be used to prove (5.3) directly [77, Sec. 1]. For contrast, the divergence operator is of constant rank, but not canceling (cp. [13, Rk. 5]).

The main technical tool that we will use is the Moore–Penrose generalized inverse of a matrix, to which we will refer simply as the *pseudo–inverse*, although this terminology is not standard. This will enable us to define suitable convolution operators and exact annihilators of constant rank operators. For simplicity of exposition, we will make no distinction between linear transformations and their matrices.

Following [24], we write $M^\dagger \in \mathbb{R}^{N\times m}$ for the the pseudo–inverse of a matrix $M \in \mathbb{R}^{m\times N}$. Note that the restriction $M\colon (\ker M)^\perp \to \operatorname{im} M$ is an isomorphism. Then $M^\dagger$ is defined as the inverse of this isomorphism on $\operatorname{im} M$ and by zero on $(\operatorname{im} M)^\perp$. As a consequence, the compositions $MM^\dagger$ and $M^\dagger M$ are orthogonal projections onto $\operatorname{im} M$ and $(\ker M)^\perp$, respectively. This geometric property, in fact, characterizes $M^\dagger$. For consistency with (3.12), we note that if M has full rank $N \leqslant m$, then $M^\dagger = (M^*M)^{-1}M^*$.

The upshot of using pseudo–inverses to invert matrix–valued fields, such as the symbol map $\mathbb{A}[\cdot]$, is that they locally preserve smoothness, provided that the matrix fields have constant rank.

With the remarks above in mind, the first fact that we establish is that Lemma 3.4 holds for constant rank operators.

Lemma 5.2. *Let $\mathbb{A}$ as in (2.1) have constant rank. Then*

$$\begin{aligned} \bigcap_{\xi\in\mathbb{R}^n\setminus\{0\}} \mathbb{A}[\xi](V) &= \{w \in W \colon \mathbb{A}u = \delta_0 w \text{ for some } u \in \mathscr{S}'(\mathbb{R}^n, V)\} \\ &= \left\{ \int_{\mathbb{R}^n} \mathbb{A}u \,\mathrm{d}x \colon u \in \mathrm{C}^\infty(\mathbb{R}^n, V), \mathbb{A}u \in \mathrm{C}_c^\infty(\mathbb{R}^n, W) \right\} \end{aligned}$$

The arguments used to prove Lemma 3.4 extend, provided that:

(a) we can Fourier invert the definition $\hat{u}(\xi) := \mathbb{A}^\dagger[\xi]w$ for $\xi \neq 0$ to obtain a fundamental solution $\mathbb{A}u = \delta_0 w$ (here w lies in the intersection of $\mathbb{A}[\xi](V)$).

(b) $\mathbb{A}$ admits an exact annihilator.

The fact (b) is proved in Lemma 5.6 below; the fact (a) follows almost immediately, from the following extension of [17, Lem. 2.1]:

Lemma 5.3. *Let $\mathbb{A}$ as in (2.1) have constant rank. Then there exists a kernel $G \in \mathrm{C}^\infty(\mathbb{R}^n \setminus \{0\}, \mathrm{Lin}(W,V))$ that is locally integrable in $\mathbb{R}^n$ such that*

$$u = \pi u + G * \mathbb{A}u, \tag{5.4}$$

where $u, \pi u \in \mathscr{S}'(\mathbb{R}^n, V)$ and $\mathbb{A}(\pi u) = 0$. In particular, $\mathrm{D}^{k-j}G$ is $(j-n)$–homogeneous for $j = 0, \ldots \min\{k, n-1\}$.

We stress that the projection on the kernel of $\mathbb{A}$ cannot be dropped even if the representation formula (5.4) is considered over test functions only. As in the example $\mathbb{A} = \mathrm{curl}$ above, if u is the gradient of a non–zero scalar test function, then $\pi u = u \neq 0$.

The proof of Lemma 5.3 follows the same lines as the proof of [17, Lem. 2.1], with the modification that $\mathbb{A}^\dagger[\cdot]\colon \mathbb{R}^n \setminus \{0\} \to \mathrm{Lin}(W,V)$ extends to a tempered distribution on $\mathbb{R}^n$. This follows since $\mathbb{A}[\cdot]$ is $(-k)$–homogeneous, by definition, and smooth away from zero, by [29, Cor. 3.2]. We stress that the latter follows from the constant rank condition. One then checks that $\hat{G} := -\mathrm{i}^k\, \mathbb{A}^\dagger[\cdot]$ satisfies the required properties. In particular, if $w \in \mathbb{A}[\xi](V)$, for all $\xi \neq 0$, we get that $\mathbb{A}(Gw) = \delta_0 w$, so the fact (a) follows.

The use of the pseudo–inverse in connection with the constant rank condition for first order operators appeared in [40, Thm. 3.5], where a Korn–type inequality is proved. The proof in [40] can be adjusted to cover operators of arbitrary order; alternatively, one can use Lemma 5.3 and boundedness of singular integrals to get:

Proposition 5.4. *Let $\mathbb{A}$ as in (2.1) have constant rank, $1 < p < \infty$. Then*

$$\inf_{\mathbb{A}v = 0 \text{ in } \mathbb{R}^n} \|\mathrm{D}^k(u-v)\|_{\mathrm{L}^p(\mathbb{R}^n, V \odot^k \mathbb{R}^n)} \lesssim \|\mathbb{A}u\|_{\mathrm{L}^p(\mathbb{R}^n, W)}$$

holds for all $u \in \mathrm{L}^1_{\mathrm{loc}} \cap \mathscr{S}'(\mathbb{R}^n, V)$ such that $\mathbb{A}u \in \mathrm{L}^p(\mathbb{R}^n, W)$.

Note that Proposition 5.4 is the full space variant of (2.7). However, the inequality on bounded domains requires a different idea to construct a projection operator, which should cancel out the singularities at the boundary. Proposition 5.4 suggests that the ellipticity assumption may needlessly complicate the understanding of (2.7). We ask the following:

Question 5.5. *Let $\mathbb{A}$ as in (2.1) have constant rank, $1 < p < \infty$. Is it then the case that*

$$\inf_{\mathbb{A}v = 0 \text{ in } \mathrm{B}} \|\mathrm{D}^k(u-v)\|_{\mathrm{L}^p(\mathrm{B}, V \odot^k \mathbb{R}^n)} \lesssim \|\mathbb{A}u\|_{\mathrm{L}^p(\mathrm{B}, W)}$$

holds for all $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathrm{B}, V)$ such that $\mathbb{A}u \in \mathrm{L}^p(\mathrm{B}, W)$?

We remark that the answer is well–known to be positive in the case of the divergence operator [36, Ch. III.1], as shown by Bogovskiĭ in [9].

We turn to statement in (b):

Lemma 5.6. *Let $\mathbb{A}$ as in (2.1) have constant rank. Then there exists a homogeneous, constant rank, linear differential operator $\mathcal{A}$ such that*

$$\ker \mathcal{A}[\xi] = \mathrm{im} \mathbb{A}[\xi]$$

for all $\xi \in \mathbb{R}^n \setminus \{0\}$.

Proof. The idea is to show, in analogy with (3.12), that

$$\mathcal{A}[\xi] := P(\xi)(\mathrm{Id} - \mathbb{A}[\xi]\mathbb{A}^\dagger[\xi]) \tag{5.5}$$

defines a differential operator for some (scalar–valued) polynomial P ; this is to say that $\mathcal{A}[\cdot]$ is itself a W–valued polynomial. This is the true in the elliptic case (3.12) since we obtain $\mathcal{A}[\xi] = \det(\Delta_{\mathbb{A}}[\xi])\,\mathrm{Id} - \mathbb{A}[\xi]\mathrm{adj}(\Delta_{\mathbb{A}}[\xi])\mathbb{A}^*[\xi]$, where the entries of the adjugate matrix are minors of $\Delta_{\mathbb{A}}[\xi]$, hence polynomials. The exactness of the annihilator (5.5) follows from the fact that $\mathbb{A}[\xi]\mathbb{A}^\dagger[\xi]$ is orthogonal projection onto $\mathrm{im}\mathbb{A}[\xi]$.

We will give a computable formula for $P(\xi)$. It was shown in [25] that, for a matrix $M \in \mathbb{R}^{m\times N}$ for which the characteristic polynomial of MM^* is given by

$$Q(\lambda) := \lambda^m + a_1\lambda^{m-1} + \ldots + a_{m-1}\lambda + a_m,$$

then the pseudo–inverse of M is given by

$$M^\dagger = -a_r^{-1}M^*\left[(MM^*)^{r-1} + a_1(MM^*)^{r-2} + \ldots + a_{r-2}MM^* + a_{r-1}\,\mathrm{Id}\right],$$

where r is the largest integer such that $a_r \neq 0$. By considering a singular value decomposition of M, it is easy to show, in addition, that $r = \operatorname{rank} M$. This simple general observation will guarantee sufficiency of the constant rank condition for existence of $P(\xi)$.

Assume now that $\mathbb{A}$ has constant rank r, write $N := \dim V$, $m := \dim W$, and let the characteristic polynomial of $\mathbb{A}[\xi]\mathbb{A}^*[\xi]$ be given by

$$Q_\xi(\lambda) := \lambda^r\left(\lambda^{N-r} + a_1(\xi)\lambda^{N-r-1} + \ldots + a_{r-1}(\xi)\lambda + a_r(\xi)\right),$$

for $\xi \neq 0$. We then have that

$$\begin{aligned}\mathbb{A}[\xi]^\dagger = &-a_r(\xi)^{-1}\mathbb{A}[\xi]^*\left[(\mathbb{A}[\xi]\mathbb{A}^*[\xi])^{r-1} + a_1(\xi)(\mathbb{A}[\xi]\mathbb{A}^*[\xi])^{r-2} + \ldots\right.\\ &\left.\ldots + a_{r-2}(\xi)\mathbb{A}[\xi]\mathbb{A}^*[\xi] + a_{r-1}(\xi)\,\mathrm{Id}\right],\end{aligned}$$

whenever $\xi \neq 0$. It is crucial to note that a_j are polynomials in ξ, as they are linear combinations of minors of $\mathbb{A}[\xi]\mathbb{A}^*[\xi]$, and that r is independent of ξ by the constant rank condition. One can also check carefully that a_j are suitably homogeneous. It remains to choose $P(\xi) := a_r(\xi)$ in (5.5) to conclude. □

Remark 5.7. *The construction can easily be reversed, to construct potentials for constant rank differential constraints. Let* $\mathcal{A}$ *be a homogeneous, linear differential operator of constant rank* r*. One can consider*

$$\mathbb{A}[\xi] := b_r(\xi)(\mathrm{Id} - \mathcal{A}^\dagger[\xi]\mathcal{A}[\xi]),$$

where b_r *arises from* $\mathcal{A}$ *just as* a_r *arises from* $\mathbb{A}$ *in the proof of Lemma 5.6. The 0–homogeneous part* $\mathrm{Id} - \mathcal{A}^\dagger \circ \mathcal{A}[\cdot]$ *was essentially defined in* [32, Sec. 2]*, while its smoothness away from zero was proved using the theory of pseudo–inverses in* [55]*. Of course, if* $\mathcal{A}$ *is elliptic, the potential thus constructed is identically zero.*

Despite the fact that Lemma 5.6 gives a computable method to obtain annihilators, it may be very inefficient. This was already observed in [80, Sec. 4] in the elliptic case in connection with annihilators defined by (3.12). Of course, in general, the situation is the same as can be seen by computing the annihilator of $A := \mathrm{curl}$ on $\mathbb{R}^3$ from $\mathbb{R}^3$ to $\mathbb{R}^3$, for which one can easily check that div is an exact annihilator. With the notation as in the proof of Lemma 5.6, we can compute to get that $r = 2$ and $a_2(\xi) = |\xi|^4$, so that $\mathcal{A}[\xi] = (\mathbb{A}[\xi]\mathbb{A}^*[\xi] - |\xi|^2\,\mathrm{Id})^2 = (\xi \otimes \xi)^2$. It then follows that $\mathcal{A} = \Delta\,\mathrm{Ddiv}$, which is clearly over–complicated. We will seek to refine the construction in Lemma 5.6.

We turn to the proof of the main result of this section:

Proof of Theorem 5.1. Since $\mathbb{A}$ has an exact annihilator $\mathcal{A}$ by Lemma 5.6, it follows that $\mathcal{A}$ is co–canceling, so that, we have by VAN SCHAFTINGEN's inequality (3.14) and Lemma 5.3 that

$$\|\,\mathrm{D}^{k-1}(u-\pi u)\|_{\mathrm{L}^{n/(n-1)}} \lesssim \|\mathbb{A}u\|_{\dot{\mathrm{W}}^{-1,n/(n-1)}} \lesssim \|\mathbb{A}u\|_{\mathrm{L}^1},$$

where the first inequality follows from boundedness of singular integrals, just as in [80, Prop. 4.1]. □

Again, we do not know if a similar estimate can hold on domains, although intuition suggests that this should be the case:

Question 5.8. *Let $\mathbb{A}$ as in (2.1) be of constant rank and canceling. Is it then the case that*

$$\inf_{\mathbb{A}v=0 \text{ in } \mathrm{B}} \|\,\mathrm{D}^{k-1}(u-v)\|_{\mathrm{L}^{n/(n-1)}(\mathrm{B},V\odot^{k-1}\mathbb{R}^n)} \lesssim \|\mathbb{A}u\|_{\mathrm{L}^1(\mathrm{B},W)}$$

holds for all $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n,V)$ such that $\mathbb{A}u \in \mathrm{L}^1(\mathrm{B},W)$?

We conclude this section with the question of necessity of assumptions on $\mathbb{A}$ for Theorem 5.1. We seem to lack the handles to deal with necessity of the constant rank condition, as its failure is, at least at the algebraic level, significantly more difficult to exploit than failure of ellipticity. What seems in reach and plausible is necessity of cancellation, given the constant rank condition.

Question 5.9. *Let $\mathbb{A}$ be a constant rank operator such that (5.1) holds. Is $\mathbb{A}$ necessarily canceling?*

The passage to the next Section may seem abrupt, and, to some extent, it is. The common theme is that VAN SCHAFTINGEN's estimate (3.14) implies that $\mathcal{A}$–free measures μ have $\dot{\mathrm{W}}^{-1,n/(n-1)}$–regularity if $\mathcal{A}$ is co–canceling. This can empirically be viewed as a quantitative restriction on the singular part of μ, qualitatively studied by DE PHILIPPIS and RINDLER in [26, 27]. In the final Section of this work, we will, very tentatively, suggest a correlation between the two results.

5.2. **Properties of measures satisfying differential constraints.** In Section 2.4, we began the discussion of the variational problems (2.12) (the $\mathcal{A}$–*framework*) and (2.11) (the $\mathbb{A}$–*framework*) in the linear growth case. In the classical case $\mathbb{A} = \mathrm{D}$, the study of lower semi–continuity relied on a qualitative property of gradient–measures nowadays referred to as *Alberti's rank–one theorem* [1], which states that the (matrix–valued) polar of the singular part of a gradient measure $\mathrm{D}u$ has rank at most one except on a $\mathrm{D}u^s$–negligible set. More precisely, we have that for all $u \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)$,

$$\frac{\mathrm{d}\mathrm{D}u^s}{\mathrm{d}|\,\mathrm{D}u^s|}(x) \in \{a\otimes b\colon a\in\mathbb{R}^N, b\in\mathbb{R}^n\} \qquad \text{for } |\,\mathrm{D}u^s|\text{–a.e. } x\in\mathbb{R}^n,$$

where $\mathrm{D}u^s$ is the singular part of $\mathrm{D}u$ with respect to $\mathscr{L}^n$–measure, i.e., $\mathrm{D}u = \nabla u\mathscr{L}^n + \mathrm{D}u^s$ for $\nabla u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^{N\times n})$ and $\mathrm{D}u^s \perp \mathscr{L}^n$.

In the absence of Alberti's rank–one theorem, lower semi–continuity results are substantially more difficult to achieve; we refer the reader to the introduction of [58] for detail. By Ornstein's non–inequality, it is only more difficult to tackle the semi–continuity problem for (2.11) or (2.12) in $\mathrm{BV}^{\mathbb{A}}$, where results have been obtained for the first time in [57], in the case $\mathbb{A} = \mathcal{E}$. The lower semi–continuity result in [57, Thm. 1.1] was improved to an integral representation of the lower semi–continuous envelope in [8, Cor. 1.9], facilitated by the outstanding generalization of Alberti's rank–one theorem to the $\mathcal{A}$–framework by De Philippis and Rindler in [26]. To introduce their result, we first note that if we consider the prototypical jump functions $\chi_H w$, where H is a hyperplane of co–dimension one with normal ξ, then $\mathcal{A}(\chi_H w) = 0$ if and only if $\mathcal{A}[\xi]w = 0$. This indicates that the density along a jump across a smooth $(n-1)$–dimensional surface should be valued in the *wave cone*

$$\Lambda_{\mathcal{A}} := \bigcup_{\xi \in \mathbb{R}^n \setminus \{0\}} \ker \mathcal{A}[\xi],$$

introduced in [74, 51] in the context of compensated compactness. The main result [26, Thm. 1.1] states that this is true of the entire singular part of of an $\mathcal{A}$–free measure. More precisely, for any linear, homogeneous, differential operator $\mathcal{A}$ with constant coefficients[1] and any bounded measure μ such that $\mathcal{A}\mu = 0$, having the Radon–Nikodým decomposition $\mu = f\mathscr{L}^n + \mu^s$ with $\mu^s \perp \mathscr{L}^n$, then

$$\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}(x) \in \Lambda_{\mathcal{A}} \quad \text{for } |\mu|^s\text{–a.e. } x \in \mathbb{R}^n.$$

Apart from this remarkable result, little is known about the singular parts of measures satisfying general differential constraints. For the remainder of this section, we will launch some, hopefully not too wild, speculations that may link this topic to the canceling condition (2.8). We begin with a sharpened proof of Lemma 4.2.

Proof of Proposition 2.6. Suppose that $\mathbb{A}$ as in (2.1) is $\mathbb{C}$–elliptic, $H \leqslant \mathbb{R}^n$ has $\dim H = d \geq 2$ and $w \in \bigcap_{\xi \in H \setminus \{0\}} \mathbb{A}[\xi](V)$. We consider coordinates t, y in $H, H^\perp$ respectively. We define

$$\hat{u} := (\mathscr{H}^d \llcorner H)\mathbb{A}^\dagger[\cdot]w \tag{5.6}$$

where $\mathbb{A}^\dagger[\cdot] = (\mathbb{A}^* \circ \mathbb{A}[\cdot])^{-1}\mathbb{A}^*[\cdot]$ as $\mathbb{A}$ is elliptic. Moreover, $\mathbb{A}^\dagger[\cdot]$ extends to a tempered distribution in H by the same reasoning as in [17, Lem. 2.1]. By a simple modification of [43, Thm. 7.1.25] (see also the remark thereafter), we obtain that $\hat{u}$ is well–defined, i.e., a tempered distribution on $\mathbb{R}^n$ given by

$$u(t, y) = \mathscr{F}_H^{-1}(\mathbb{A}^\dagger[\cdot])(t)w \quad \text{for } t \in H, y \in H^\perp,$$

where $\mathscr{F}_H$ denotes the Fourier transform in H. Note also that $\mathbb{A}u$ is classically differentiable, except when the H–variable $t = 0$, and is independent of y, so $\widehat{\mathbb{A}u}$ will concentrate on H. By bijectivity of $\mathbb{A}^\dagger[\xi]\colon V \leftrightarrow \mathbb{A}[\xi](V)$ for $\xi \in \mathbb{R}^n \setminus \{0\}$ and by definition of w, it follows that $\widehat{\mathbb{A}u} = (\mathscr{H}^d \llcorner H)w$. Inverting the Fourier transform and using [43, Thm. 7.1.25] again gives

$$\mathbb{A}u = (\mathscr{H}^{n-d} \llcorner H^\perp)w. \tag{5.7}$$

In particular, $\mathbb{A}u = 0$ in $\mathbb{R}^n \setminus H$, which is connected since $\dim H \geq 2$. By the assumption of $\mathbb{C}$–ellipticity and Theorem 3.2(d), we get that u equals a polynomial in $\mathbb{R}^n \setminus H$. Hence $\mathbb{A}u = 0$ in $\mathbb{R}^n$, so $w = 0$, and (2.9) follows.

Conversely, let $\mathbb{A}$ be elliptic of order $k = 1$ and suppose that $\mathbb{A}$ is not $\mathbb{C}$–elliptic. This implies existence of a complex plane waves (3.5) given by $u(x) = f(x \cdot \eta)v$ for

[1] The result in [26] holds for inhomogeneous operators with variable coefficients as well; we do not include this here to keep consistency with the general notation of the present work.

all $x \in \mathbb{R}^n$ and some complex η, v such that $\mathbb{A}u = 0$ if f is holomorphic. Moreover, by ellipticity of $\mathbb{A}$, it is easy to show that if $\eta = \eta_1 + \mathrm{i}\,\eta_2$ for real η_j, then η_j are linearly independent over $\mathbb{R}$. The analogous statement is true for $v = v_1 + \mathrm{i}\,v_2$. One defines $H := \operatorname{span}\{\eta_1, \eta_2\}$ and lets $f(z) = z^{-1}$. Since $\mathbb{A}_1 f = \pi\delta_0$, where $\mathbb{A}_1$ is the Wirtinger derivative (2.5), it follows that $\mathbb{A}u = \pi\delta_0\mathbb{A}[\eta_1]v_1$ in $\mathscr{D}'(H, V)$. If we write t for the H–coordinate and denote by y the $H^\perp$–coordinate, we have that $u(t, y) = u(t)$, so that

$$\mathbb{A}u = (\mathscr{H}^{n-2} \llcorner H^\perp)\pi\mathbb{A}[\eta_1]v_1.$$

Fourier transforming this equality contradicts (2.9) since $\mathbb{A}[\eta_1]v_1 \in \mathbb{A}[\xi](V)$ for all $\xi \in H \setminus \{0\}$ (of course, $\mathbb{A}[\eta_1]v_1 \neq 0$ by ellipticity of $\mathbb{A}$). □

The idea of the proof of the direct implication is that in (5.6) we essentially construct a fundamental solution for $\mathbb{A}u = \delta_0 w$ for $u\colon H \to V$; this is, of course, exactly the construction in [17, Lem. 2.1] and is made possible by the definition of w (note that H intimately depends on $\mathbb{A}$, otherwise it would be impossible to fruitfully apply $\mathbb{A}$ to maps restricted to H, e.g., if $\mathbb{A} = \mathrm{D}$ and H contains no coordinate axes). We then extend u trivially in the $H^\perp$ directions, to obtain a map defined on $\mathbb{R}^n$ that satisfies $\mathbb{A}u = 0$ except when the H–coordinate $t = 0$. This turns $\mathbb{A}(u \llcorner H) = \delta_0 w$ into $\mathbb{A}u = (\mathscr{H}^{n-d} \llcorner H^\perp)w$. We also remark that the connectedness argument is crucial. If H would be 1–dimensional, then (5.7) would just imply existence of a constant jump across $H^\perp$. As for the converse, we refer the reader to [38, Sec. 3], where some related computations are performed in more detail. To concretely illustrate (5.7) in an example, we will explicitly construct an operator in (5.8).

We record the construction performed in the proof of Proposition 2.6 separately, as the following variant of Lemma 5.2:

Lemma 5.10. *Let $\mathbb{A}$ as in (2.1) have constant rank, $H \leqslant \mathbb{R}^n$, $d = \dim H$. Then*

$$\bigcap_{\xi \in H^\perp \setminus \{0\}} \mathbb{A}[\xi](V) = \{w \in W \colon \exists u \in \mathscr{S}'(\mathbb{R}^n, V) \text{ s.t. } \mathbb{A}u = (\mathscr{H}^d \llcorner H)w\}.$$

The result in Lemma 5.10 can easily be used to give simple estimates on the Hausdorff dimensions of negligible sets of $\mathbb{A}$–measures. To see this, we define $C(\mathbb{A}) \in \{0, 1, \ldots n-1\}$ as the least number d such that there exists a subspace $H \leqslant \mathbb{R}^n$ such that $\dim H = d$ and

$$\bigcap_{\xi \in H^\perp \setminus \{0\}} \mathbb{A}[\xi](V) \neq \{0\}.$$

The integer $C(\mathbb{A})$ can be thought as a measure of how canceling an operator is. Non–canceling operators have $C(\mathbb{A}) = 0$, whereas for FDN operators we have that $C(\mathbb{A}) = n - 1$ by Proposition 2.6 and Theorem 3.2.

To connect this with fine properties of $\mathbb{A}$–measures, define

$$M(\mathbb{A}) := \inf\{\dim \mathbb{A}u \colon u \in \mathrm{BV}^{\mathbb{A}}_{\mathrm{loc}}(\mathbb{R}^n)\},$$

where the dimension $\dim \mu$ of a measure μ is the largest lower bound on the Hausdorff dimension of negligible sets of μ. Lemma 5.10 then implies that

$$C(\mathbb{A}) \geq M(\mathbb{A})$$

for all constant rank operators $\mathbb{A}$.

To the author's best knowledge, the opposite inequality is known only in the cases $\mathbb{A} = \mathrm{D}$ [6, Lem. 3.76], $\mathbb{A} = \mathcal{E}$ [4, Eq. (3.10)] (attributable to Kohn [46]), and when $\mathbb{A}u = (\partial_j^m u)_{j=1}^n$ is a collection of all pure derivatives [65] (to be precise, the result covers the anisotropic case $\mathbb{A}u = (\partial_j^{m_j} u)_{j=1}^n$, $m_j > 0$ for all $j = 1 \dots n$). All the operators listed have $M(\mathbb{A}) = n - 1$ and they have FDN, so, indeed, $C(\mathbb{A}) = M(\mathbb{A})$. Of course, the equality is also true of non–canceling operators of constant rank by Lemma 5.2.

So far, we have mostly seen examples of operators for which $C(\mathbb{A}) = 0$, e.g., $\Delta, (\mathrm{div}, \mathrm{curl})$, or with $C(\mathbb{A}) = n-1$, e.g., $\mathrm{D}^k, \mathcal{E}$. To ensure that we do not work towards a vacuous statement, we modify the example in (4.1) to construct first order, elliptic operators $\mathbb{A}$ such that $C(\mathbb{A}) = d$ for each $0 \leqslant d \leqslant n-1$. To this end, we write $\mathbb{R}^n = \mathbb{R}^{n-d} \oplus \mathbb{R}^d$ with (t, y)–coordinates and write $\mathbb{A}_t$ for the $(\mathrm{div}, \mathrm{curl})$ operator on $\mathbb{R}^{n-d}$; in particular, $\mathbb{A}_t(t|t|^{d-n}) = \delta_0(t)e_1$ for $t \in \mathbb{R}^{n-d}$. By a minor abuse of notation, we say that $e_1 = (1, 0, \dots 0)$, irrespectively of the space where e_1 lies. We define

$$\mathbb{A}u := (\mathbb{A}_t u, \partial_j u_i)_{i>n-d \text{ or } j>n-d} \qquad \text{for } u\colon \mathbb{R}^{n-d} \oplus \mathbb{R}^d \to \mathbb{R}^{n-d} \oplus \mathbb{R}^l, \tag{5.8}$$

so that

$$u(t,y) := \frac{(t, 0_{\mathbb{R}^l})}{|t|^{n-d}} \qquad \text{implies} \qquad \mathbb{A}u = (\mathscr{H}^d \llcorner \{0_{\mathbb{R}^{n-d}}\} \oplus \mathbb{R}^d)e_1,$$

which is also an explicit instance of (5.7). In fact, it is not difficult to see that $M(\mathbb{A}) = d$ as well, since the system $\mathbb{A}u = \mu$ can be decoupled in three simple sub–systems.

Motivated by the considerations above, the duality between the $\mathbb{A}$–framework and the $\mathcal{A}$–framework established for constant rank operators in Lemma 5.6 and Remark 5.7, and by the recent achievement in [26, Thm. 1.1], we conclude the present work with the following:

Question 5.11. *Let $\mathcal{A}$ be a homogeneous, linear, differential operator on $\mathbb{R}^n$ with constant coefficients that satisfies the constant rank condition and let $0 \leqslant d \leqslant n-1$. Does the following statement hold?*

Suppose that d is the least integer l such that

$$\bigcap_{\xi \in H^\perp \setminus \{0\}} \ker \mathcal{A}[\xi] \neq \{0\} \quad \textit{for some} \quad H \leqslant \mathbb{R}^n \quad \textit{such that} \quad \dim H = l.$$

Then for all bounded measures μ such that $\mathcal{A}\mu = 0$ we have that

$$\mathscr{H}^d(S) = 0 \quad \textit{implies} \quad \mu(S) = 0,$$

where $S \subset \mathbb{R}^n$ denotes a Borel set.

Note that having $d = 0$ is equivalent with failure of the co–canceling condition (3.15) and that, in this case, the statement of Question 5.11 is vacuous. On the other hand, this is no contradiction since, if $d = 0$, then $\mathcal{A}(\delta_0 w) = 0$ for some $w \neq 0$, as is shown in [80, Prop. 2.1].

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Andrew Wiles Building, University of Oxford, Woodstock Rd, Oxford OX2 6GG.
E-mail address, B. Raita: raita@maths.ox.ac.uk