Cross-diffusion and competitive interaction in Population dynamics

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Outline

Classical models in Population dynamics

$u := u(t) \geq 0$ (and $v := v(t) \geq 0$) : quantities of species at time $t \geq 0$.

Logistic equation ($\sim$1840)

$$d_t u = ru(1-u).$$

Competition for the ressources.

Lotka-Volterra system (1925)

$$\begin{cases} d_t u = u\left(r_1 - r_3 v\right), \\ d_t v = -v\left(r_2 - r_4 u\right). \end{cases}$$

Predator-prey interaction.

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Fisher's equation (1938)

$u := u(t,x) \geq 0$: density of species at time $t \geq 0$, space $x \in \Omega \subset \mathbb{R}^N$.

$$\partial_t u - D\Delta_x u = ru(1-u).$$

Diffusion term $D\Delta_x u$ (with $D > 0$) : natural tendency of the individuals to spread homogeneously.

Reaction-diffusion systems

$u_i := u_i(t,x) \geq 0$: space density of species i (for $i = 1..J$) at time $t \geq 0$.

Classical reaction-diffusion system

$$\partial_t U - \Delta_x [D\, U] = F(U),$$

with $F : \mathbb{R}^I_+ \longrightarrow \mathbb{R}^I$ and $D = \mathrm{diag}(d_1, \ldots, d_J)$ a positive diagonal matrix.

Interactions between individuals of different species affect the growth rate of the populations.

Reaction-cross diffusion systems

$u_i := u_i(t, x) \geq 0$: space density of species i (for $i = 1..J$) at time $t \geq 0$.

General reaction-cross diffusion system

$$\partial_t U - \Delta_x [A(U)] = F(U),$$

with $F : \mathbb{R}^I_+ \longrightarrow \mathbb{R}^I$ and $A : \mathbb{R}^I_+ \longrightarrow (R^*_+)^I$.

Interactions between individuals of different species affect the growth rate ***and the spreading*** *of the populations.*

The SKT system : modeling

$t \geq 0$: time, $x \in \Omega$: space ($\Omega \subset \mathbb{R}^N$: environment),
$u = u(t,x) \geq 0$: space density of first species at time $t \geq 0$,
$v = v(t,x) \geq 0$: space density of second species at time $t \geq 0$.

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Shigesada-Kawasaki-Teramoto system (1979)

$$\begin{cases} \partial_t u - \Delta_x(d_1\, u + d_\alpha\, u^2 + d_\beta\, u\, v) = u\,(r_1 - r_a\, u - r_b\, v), \\ \partial_t v - \Delta_x(d_2\, v + d_\gamma\, v^2 + d_\delta\, u\, v) = v\,(r_2 - r_c\, v - r_d\, u), \\ \nabla_x u \cdot n = \nabla_x v \cdot n = 0 \quad \text{at } \partial\Omega. \end{cases}$$

Interpretation :

- $r_i > 0$ intrinsic growth rate ; $r_a > 0$, $r_c > 0$: intraspecific competition ; $r_b > 0$, $r_d > 0$: interspecific competition ;
- $d_i > 0$: standard diffusivity ;
- $d_\alpha \geq 0$, $d_\gamma \geq 0$: self-diffusion ;
- $d_\beta \geq 0$, $d_\delta \geq 0$: cross-diffusion.

The SKT system : repulsive effect

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In the first equation :

$$-\Delta_x\,[v\,u] = -\underbrace{\nabla_x \cdot [u \nabla_x v]}_{\text{transport}} - \underbrace{\nabla_x \cdot [v \nabla_x u]}_{\text{Fickian diffusion}}\,.$$

$\rightarrow$ **repulsive effect due to the competition.**

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- show strong, nonlinear coupling / no maximum principle,

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- show strong, nonlinear coupling / no maximum principle, existence of global strong solutions : still open,
- can lead to Turing's instability, model the segregation of species.

Known results of existence

Amann : existence of local (in time) strong solutions + extension criteria.
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Global solutions : triangular system ($d_\delta = 0$)

• Matano, Mimura ; Shim ; Yagi ; Lou, Ni, Wu : in dimension $N = 1$ or 2,
• Choi, Lui, Yamada ; Phan *et al.* : in any dimension with a small cross-diffusion coefficient ; or in presence of self-diffusion in the first equation ; or in presence of self-diffusion in the second equation with $N \leq 10$.

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• Deuring ; Yagi : in any dimension with a small cross-diffusion coefficient,
• Li, Zhao : when $d_1 = d_2$,
• Chen, Jüngel : weak solutions thanks to the Lyapunov functional $\int_\Omega u \log u - u + l \int_\Omega v \log v - v$.

A *generalized* SKT system

We now consider on $\mathbb{R}_+ \times \Omega$

$$\begin{cases} \partial_t u - \Delta_x \left[\left(d_1 + d_\alpha\, u^\alpha + d_\beta\, v^\beta\right) u\right] = u\,(r_1 - r_a\, u^a - r_b\, v^b), \\ \partial_t v - \Delta_x \left[\left(d_2 + d_\gamma\, v^\gamma + d_\delta\, u^\delta\right)\; v\right] = v\,(r_2 - r_c\, v^c - r_d\, u^d), \end{cases}$$

where α, β, γ, δ, a, b, c, $d > 0$, together with the homogeneous boundary conditions

$$\nabla_x u(t,x) \cdot n(x) = \nabla_x v(t,x) \cdot n(x) = 0 \quad \text{for } (t,x) \in \mathbb{R}_+ \times \partial\Omega,$$

and some initial data $u_{in} \geq 0$, $v_{in} \geq 0$.

The triangular case : *a priori* estimates

$$\begin{cases} \partial_t u - \Delta_x \left[(d_1 + d_\alpha\, u^\alpha + d_\beta\, v^\beta)\, u\right] = u\,(r_1 - r_a\, u^a - r_b\, v^b), \\ \partial_t v - \Delta_x \left[(d_2 + d_\gamma\, v^\gamma)\ v\right] = v\,(r_2 - r_c\, v^c - r_d\, u^d). \end{cases}$$

Available estimates :

- Maximum principle : $0 \le v(t,x) \le C$,
- Entropy estimate : $\|\nabla_x v^\beta\|_{L^2([0,T]\times\Omega)} \le C(1 + \|u^d\|_{L^1([0,T]\times\Omega)})$,
- Entropy estimate : $\|\nabla_x \log u\|_{L^2([0,T]\times\Omega)} \le C(1 + \|u^d\|_{L^1([0,T]\times\Omega)})$,

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  $\longrightarrow$ to close the estimates, we need some control on the integrability of u.

A crucial tool : duality lemma

Lemma (Pierre Schmitt, 2000)

Let M be a smooth function on $[0, T] \times \overline{\Omega}$ with positive value. Then any classical solution $u \geq 0$ of

$$\begin{cases} \partial_t u - \Delta_x(Mu) \leq K \text{ in } [0, T] \times \Omega, \\ \nabla_x(Mu)(t, x) \cdot n(x) = 0 \text{ on } [0, T] \times \partial\Omega, \end{cases}$$

satisfies

$$\left\| Mu^2 \right\|_{L^1([0,T]\times\Omega)} \leq C(\Omega, T, u(0, \cdot), K)[1 + \|M\|_{L^1([0,T]\times\Omega)}].$$

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$\longrightarrow$ Back to the SKT system : estimate on $\left\| u^{2+\alpha} \right\|_{L^1([0,T]\times\Omega)}$.

The triangular case : main result

Theorem (T.)

Let Ω be a smooth bounded domain of $\mathbb{R}^N$ ($N \in \mathbb{N}^*$). Suppose $d_\beta > 0$, $d_\delta = 0$ and

$$(d_\alpha > 0,\ d < 2 + \alpha,\ a < 1 + \alpha) \text{ or } (d_\alpha = 0,\ d \leq 2,\ a \leq 1).$$

Let $(u_{in} \geq 0,\ v_{in} \geq 0)$ be in $L^2(\Omega) \times L^\infty(\Omega)$.

Then, there exist $u = u(t,x) \geq 0$, $v = v(t,x) \geq 0$ with $(u,v) \in L^{2+\alpha}_{loc}(\mathbb{R}_+ \times \overline{\Omega}) \times L^\infty_{loc}(\mathbb{R}_+ \times \overline{\Omega})$ such that (u,v) is a weak solution of the triangular generalized SKT system with homogeneous Neumann boundary conditions and with initial data u_{in}, v_{in}.

The non-triangular case : estimates

Let us go back to the full generalized SKT system :

$$\begin{cases} \partial_t u - \Delta_x \left[\left(d_1 + d_\alpha\, u^\alpha + d_\beta\, v^\beta \right) u \right] = u \left(r_1 - r_a\, u^a - r_b\, v^b \right), \\ \partial_t v - \Delta_x \left[\left(d_2 + d_\gamma\, v^\gamma + d_\delta u^\delta \right)\ v \right] = v \left(r_2 - r_c\, v^c - r_d\, u^d \right), \end{cases}$$

Available estimates :

- Hidden Lyapunov functional : when $\beta = \delta = 1$ [Chen Jüngel] ; when $\beta < 1$, $\delta < 1$ [Desvillettes Lepoutre Moussa] ; when $0 < \beta\delta < 1$ [Desvillettes Lepoutre Moussa T.],

The non-triangular case : entropy structure

We write the system as ($R = 0$ to simplify)

$$\partial_t U - \Delta_x [A(U)] = R(U) = 0.$$

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Let $\phi : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$. Mutiply (scalar product) by $D\phi$ and integrate :

$$\begin{aligned}
\mathrm{d}_t \int_\Omega \phi(U) &= \int_\Omega D\phi \cdot \Delta_x[A(U)] \\
&= -\int_\Omega (\nabla_x U) \cdot D^2\phi(U) DA(U) \nabla_x U \\
&=: -\int_\Omega Q_U(\nabla_x U, \nabla_x U).
\end{aligned}$$

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If $|\nabla_x U|^2 \lesssim_U Q_U(\nabla_x U, \nabla_x U)$, we obtain estimates on $\nabla_x U$.

The non-triangular case : entropy structure

Back to the generalized SKT system (with $R = 0$, no self-diffusion) : we take

$$\phi(u, v) = \frac{\beta}{1-\beta}[v - v^{\beta} - 1 + 1/\beta] + \frac{\gamma}{1-\gamma}[u - u^{\gamma} - 1 + 1/\gamma].$$

The obtained matrix $Q_{(u,v)}$ is symmetric, with positive trace, and

$$\det Q_{(u,v)} \sim_{(u,v)} \beta\gamma(1 - \beta\gamma).$$

The non-triangular case : estimates

Let us go back to the full generalized SKT system :

$$\begin{cases} \partial_t u - \Delta_x \left[\left(d_1 + d_\alpha \, u^\alpha + d_\beta \, v^\beta \right) u \right] = u \left(r_1 - r_a \, u^a - r_b \, v^b \right), \\ \partial_t v - \Delta_x \left[\left(d_2 + d_\gamma \, v^\gamma + d_\delta u^\delta \right) \, v \right] = v \left(r_2 - r_c \, v^c - r_d \, u^d \right), \end{cases}$$

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- Estimates closed thanks to the duality lemma.

→ Existence of (very) weak solutions with supplementary assumptions on a, b, c, d.

The non-triangular case : main result

Theorem (Desvillettes, Lepoutre, Moussa, T.)

Let Ω be a smooth bounded domain of $\mathbb{R}^N$ ($N \in \mathbb{N}^*$). Suppose $0 < \delta < 1/\beta < 1$, $d_\alpha = d_\gamma = 0$ *and*

$$a < 1,\ b < \beta + c/2,\ d < 2.$$

Let $(u_{in} \geq 0,\ v_{in} \geq 0)$ *be in* $L^2(\Omega) \times L^{\max(\beta,2)}(\Omega)$.

Then, there exist $u = u(t,x) \geq 0$, $v = v(t,x) \geq 0$ *with* $(u,v) \in L^2_{loc}(\mathbb{R}_+ \times \overline{\Omega}) \times L^{\beta+c}_{loc}(\mathbb{R}_+ \times \overline{\Omega})$ *such that* (u,v) *is a very weak solution of the generalized SKT system with homogeneous Neumann boundary conditions and with initial data* u_{in}, v_{in}.

Remark

This result extends a previous result of Desvillettes, Lepoutre, Moussa when $\beta < 1$, $\delta < 1$.

Relaxation model

Remark on the modeling

The decomposition "cross-diffusion term = transport + diffusion" is not a *justification* of the SKT model.

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Relaxation model proposed by Iida, Mimura, Ninomiya

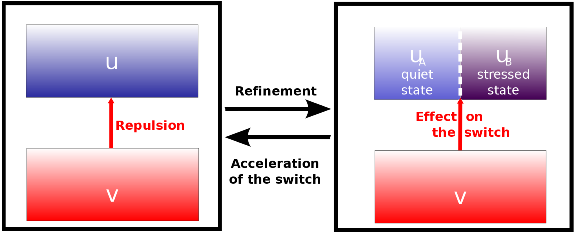

Figure: Triangular SKT system IMN model

Relaxation model

$u_A^\varepsilon = u_A^\varepsilon(t,x) \geq 0$: density of population of first species in quiet state,
$u_B^\varepsilon = u_B^\varepsilon(t,x) \geq 0$: density of population of first species in stressed state,
$v^\varepsilon = v^\varepsilon(t,x) \geq 0$: density of second species.

Iida-Mimura-Ninomiya system (2006)

$$\begin{cases} \partial_t u_A^\varepsilon - d_A \, \Delta_x u_A^\varepsilon = [1 - (u_A^\varepsilon + u_B^\varepsilon) - v^\varepsilon] \, u_A^\varepsilon + \dfrac{1}{\varepsilon} [k(v^\varepsilon) \, u_B^\varepsilon - h(v^\varepsilon) \, u_A^\varepsilon], \\ \partial_t u_B^\varepsilon - d_B \, \Delta_x u_B^\varepsilon = [1 - (u_A^\varepsilon + u_B^\varepsilon) - v^\varepsilon] \, u_B^\varepsilon - \dfrac{1}{\varepsilon} [k(v^\varepsilon) \, u_B^\varepsilon - h(v^\varepsilon) \, u_A^\varepsilon], \\ \partial_t v^\varepsilon - \Delta_x v^\varepsilon = [1 - v^\varepsilon - (u_A^\varepsilon + u_B^\varepsilon)] \, v^\varepsilon, \end{cases}$$

- the species 1 exists in a quiet state A and a stressed state B ($d_B > d_A$),
- the stress is induced by the presence of the species 2,
- the rate of switch is of order $1/\varepsilon \gg 1$.

Relaxation model : acceleration of the switch

Equations for the densities of species

$$\begin{cases} \partial_t(u_A^\varepsilon + u_B^\varepsilon) - \Delta_x \left[(d_A \dfrac{u_A^\varepsilon}{u_A^\varepsilon + u_B^\varepsilon} + d_B \dfrac{u_B^\varepsilon}{u_A^\varepsilon + u_B^\varepsilon})(u_A^\varepsilon + u_B^\varepsilon) \right] \\ \qquad\qquad = \left[1 - (u_A^\varepsilon + u_B^\varepsilon) - v^\varepsilon\right](u_A^\varepsilon + u_B^\varepsilon), \\ \partial_t v^\varepsilon - \Delta_x v^\varepsilon = \left[1 - v^\varepsilon - (u_A^\varepsilon + u_B^\varepsilon)\right] v^\varepsilon. \end{cases}$$

Computation of the formal limit

If $(u_A^\varepsilon, u_B^\varepsilon, v^\varepsilon) \to (u_A, u_B, v)$ (in a strong sense) when $\varepsilon \to 0$ then $h(v)u_A = k(v)u_B$, i. e. $\frac{u_A}{u_A + u_B} = \frac{k(v)}{h(v)+k(v)}$ and $\frac{u_B}{u_A+u_B} = \frac{h(v)}{h(v)+k(v)}$.

Relaxation model : acceleration of the switch

Equations for the densities of species at $\varepsilon = 0$

$$\begin{cases} \partial_t(u_A + u_B) - \Delta_x \left[(d_A \dfrac{k(v)}{h(v)+k(v)} + d_B \dfrac{h(v)}{h(v)+k(v)})(u_A + u_B) \right] \\ \qquad = [1 - (u_A + u_B) - v]\,(u_A + u_B), \\ \partial_t v - \Delta_x v = [1 - v - (u_A + u_B)]\, v. \end{cases}$$

With accurate choices of the functions h and k, the densities $(u_A + u_B, v)$ satisfy the triangular Shigesada-Kawasaki-Teramoto system.

Rigorous asymptotics

Main interests

- qualitative property / modeling justification,
- provides a scheme of approximation $\longrightarrow$ existence theorem.

Previous results on the asymptotics

- for *a priori* uniformly bounded solutions [Iida Mimura Ninomiya 06],
- for stationnary solutions [Izuhara Mimura, 08],
- in dimension 1 [Conforto Desvillettes, 09],
- when the reaction term is Lipschitz continuous [Murakawa 12].

New result on the asymptotics [DT]

- rigorous asymptotics (a.e., up to a subsequence) for a generalized IMN system,
- proof based on entropy and duality methods.

The triangular generalized cross-diffusion system

$$\left.\begin{array}{rr} \partial_t u - \Delta_x \left[Du + uv^{\beta} \right] = u[1 - u^a - v^b] & \text{in } \mathbb{R}_+ \times \Omega, \\ \partial_t v - \Delta_x v = v[1 - v^c - u^d] & \text{in } \mathbb{R}_+ \times \Omega, \\ \nabla_x u(t,x) \cdot n(x) = \nabla_x v^{\varepsilon}(t,x) \cdot n(x) = 0 & \forall t \geq 0, x \in \partial\Omega, \\ u(0,x) = u_{in}(x) \geq 0, \qquad v(0,x) = v_{in}(x) \geq 0 & \forall x \in \Omega. \end{array}\right\} \quad (1)$$

Relaxation model

$$
\left.\begin{aligned}
\partial_t u_A^\varepsilon - d_A\,\Delta_x u_A^\varepsilon &= [1-(u_A^\varepsilon+u_B^\varepsilon)^a-(v^\varepsilon)^b]\,u_A^\varepsilon + \frac{1}{\varepsilon}\,[k(v^\varepsilon)\,u_B^\varepsilon - h(v^\varepsilon)\,u_A^\varepsilon],\\
\partial_t u_B^\varepsilon - d_B\,\Delta_x u_B^\varepsilon &= [1-(u_A^\varepsilon+u_B^\varepsilon)^a-(v^\varepsilon)^b]\,u_B^\varepsilon - \frac{1}{\varepsilon}\,[k(v^\varepsilon)\,u_B^\varepsilon - h(v^\varepsilon)\,u_A^\varepsilon],\\
\partial_t v^\varepsilon - \Delta_x v^\varepsilon &= [1-(v^\varepsilon)^c-(u_A^\varepsilon+u_B^\varepsilon)^d]\,v^\varepsilon,\\
\nabla_x u_A(t,x)\cdot n(x) = \nabla_x u_B^\varepsilon(t,x)\cdot n(x) &= 0 \qquad \forall t\geq 0, x\in\partial\Omega,\\
\nabla_x v^\varepsilon(t,x)\cdot n(x) &= 0 \qquad \forall t\geq 0, x\in\partial\Omega,\\
u_A(0,x) = u_{A,in}(x), \quad u_B(0,x) = u_{B,in}(x) \quad v(0,x) &= v_{in}(x) \qquad \forall x\in\Omega.
\end{aligned}\right\}
\tag{2}
$$

Main theorem : assumptions

Assumption A

- Ω is a smooth bounded domain of $\mathbb{R}^N$,
- $d_B > d_A > 0$, $a, b, c, d > 0$,
- h, k lie in $C^1(\mathbb{R}_+, \mathbb{R}_+)$ and are lower bounded by a positive constant,
- $u_{A,in}, u_{B,in}, v_{in} \geq 0$ such that $u_{A,in}, u_{B,in} \in L^{p_0}(\Omega)$,

$v_{in} \in L^\infty(\Omega) \cap W^{2,1+p_0/d}(\Omega)$ for some $p_0 > 1$, and $\nabla_x v_{in} \cdot n(x) = 0$,
- $a > d$ or $(a \leq 1$ and $d \leq 2)$.

Theorem

Theorem (Desvillettes, T.)

Under Assumption A, When $\varepsilon \to 0$, $(u_A^\varepsilon, u_B^\varepsilon, v^\varepsilon)$ *converges (up to a subsequence) for almost every* $(t,x) \in \mathbb{R}_+ \times \Omega$ *to a limit* (u_A, u_B, v) *lying in* $L^{q_0}([0,T]\times\Omega) \times L^{q_0}([0,T]\times\Omega) \times L^\infty([0,T]\times\Omega)$ *for all* $T > 0$*.*
Furthermore,

$$h(v)\, u_A = k(v)\, u_B$$

and $(u := u_A + u_B, v)$ *is a weak solution of system* (1) *with*

$$D + v^\beta = \frac{d_A k(v) + d_B h(v)}{h(v) + k(v)}$$

and initial data $u(0,\cdot) = u_{A,in} + u_{B,in}$, $v(0,\cdot) = v_{in}$*.*

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Proof : entropy and duality methods.

Rigorous asymptotics : sketch of the proof

We fix $T > 0$ and consider a smooth nonnegative solution $(u_A^\varepsilon, u_B^\varepsilon, v^\varepsilon)$.
• Estimates uniformly in ε,
• Convergence of the densities up to a subsequence (compactness : Aubin's lemma),
• Vanishing of $h(v)u_A - k(v)u_B$.

Uniform estimates 1

Tool 1 : solve the equation of v^ε first

$$\partial_t v^\varepsilon - \Delta_x v^\varepsilon = [1 - (v^\varepsilon)^c - (u_A^\varepsilon + u_B^\varepsilon)^d]\, v^\varepsilon.$$

- Maximum principle : $0 \leq v^\varepsilon \leq C_T$.
- Properties of the heat kernel : for all $p > 1$,

$\|\partial_t v^\varepsilon\|_{L^p} + \|\nabla_x^2 v^\varepsilon\|_{L^p} \leq C_T(1 + \|(u_A^\varepsilon + u_B^\varepsilon)^d\|_{L^p})$.

Tool 2 : Duality lemma (case $a \leq 1$, $d \leq 2$)

The total density of species 1 satisfies uniformly in ε :

$$\|u_A^\varepsilon + u_B^\varepsilon\|_{L^2} \leq C_T.$$

Uniform estimates 2

Tool 3 : Entropy

For any $p > 1$, let

$$\mathscr{E}^\varepsilon(t) := \int_\Omega h(v^\varepsilon)^{p-1}\frac{(u_A^\varepsilon)^p}{p}(t) + \int_\Omega k(v^\varepsilon)^{p-1}\frac{(u_B^\varepsilon)^p}{p}(t).$$

Uniform estimates 2

Tool 3 : Entropy

For any $p > 1$, let

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• This functional does *not* increase *too much*,
• the terms in $O(\frac{1}{\varepsilon})$ have a (good) sign,
• consequences : estimates for $u_A^\varepsilon, u_B^\varepsilon$ in Sobolev spaces (uniformly in ε)
+ estimates for $k(v^\varepsilon)\, u_B^\varepsilon - h(v^\varepsilon)\, u_A^\varepsilon$.

Conclusion

Main results

- Existence of (very) weak global solutions,
- Elucidation of the entropic structure,
- Rigorous asymptotics of the relaxation model.

Perspectives and open questions

- Regularity and uniqueness - or blow-up ?
- Phenomena of segregation,
- Different types of interaction (predator-prey, chemotaxis)...

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Thank you.