Minimal hypersurfaces with bounded index and area

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- The work of Almgren '68 and Pitts '81 (also Schoen-Simon '81) guarantees the existence of at least one closed, smooth, minimal and embedded minimal hypersurface $M^n \hookrightarrow N^{n+1}$ in any closed Riemannain manifold N when $n \leq 6$. It is a conjecture of Yau that there should be infinitely many distinct $M \hookrightarrow N$.
- Due to the recent work of Marques-Neves '14, Yau's conjecture has been proved when $Ric_N \geq \alpha > 0$. Thus it makes sense to try to understand the space (where M is closed)

$$\mathfrak{M}(N) = \{M^n \hookrightarrow N^{n+1} | M \text{ is smooth, minimal and embedded.}\}$$

- One way to get a handle on this space is to understand the relationship between the Betti numbers (or topology) of M, $b_i(M)$ and the geometric-analytic properties; $index(M)$, $\mathcal{H}^n(M)$ and the total curvature $\int_M |A|^n$.

Throughout this talk N^{n+1} will be a closed Riemannian manifold with $2 \leq n \leq 6$ and $M^n \hookrightarrow N$ a smooth embedded minimal hypersurface which is also closed.

We call M minimal if for all $\phi_t : (-\varepsilon, \varepsilon) \times N \to N$, one parameter families of diffeomorphisms with $\phi_0 = Id$, $M_t = \phi_t(M)$ we have

$$\frac{\partial}{\partial t} Vol(M_t)|_{t=0} = -\int_M H \cdot v \, \mathrm{d}V_M = 0.$$

Where for $x \in M$, $v(x) = \left(\frac{\partial \phi_t(x)}{\partial t}|_{t=0} \right)^{\perp_M} \in \Gamma(\mathrm{N}M)$ and $H = \mathrm{tr}_M A$ is the mean curvature of M.

The index of M, $index(M)$ is the Morse index - the number of negative eigenvalues associated with the Jacobi (second variation) operator for minimal hypersurfaces $M \subset N$:

$$\frac{\partial^2}{\partial t^2} Vol(M_t)|_{t=0} = \int_M |\nabla^\perp v|^2 - |A|^2|v|^2 - Ric_N(v,v)\, \mathrm{d}V_M.$$

- Call M stable if $Q(v,v) := \int_M |\nabla^\perp v|^2 - |A|^2|v|^2 - Ric_N(v,v)\, \mathrm{d}V_M \geq 0$ for all v,
- and locally minimising if $Q(v,v) > 0$ for all $v \neq 0$.
- If M is two-sided, $v = f\nu$ some $f \in C^\infty(M)$ and

  $$Q(f\nu, f\nu) = \int_M |\nabla f|^2 - (|A|^2 + Ric_N(\nu,\nu))f^2\, \mathrm{d}V_M,$$

  thus if $Ric_N > 0$ there are no stable, two sided, minimal hypersurfaces in N.

Given a sequence $\{M_k^n\} \hookrightarrow N^{n+1}$ if we know that

$$\mathcal{H}^n(M_k) + \sup_{x \in M_k} |A_k(x)| \leq \Lambda < \infty$$

then there exists some M (minimal, smooth, closed, embedded) such that (up to subsequence) $M_k \to M$ smoothly and graphically. When $Ric_N > 0$ and $n = 2$ we can do better:

Theorem (Choi-Schoen '85)

Let N be a compact 3-dimensional manifold with positive Ricci curvature. Then the space of compact embedded minimal surfaces of fixed topological type in N is compact in the C^k topology for any $k \geq 2$ - i.e. smooth graphical convergence with multiplicity one.

Remarks

1. For $n \geq 3$ control on topology can never give such a strong compactness theorem. Letting $S^4 \hookrightarrow \mathbb{R}^5$ be the round sphere, W. Y. Hsiang '83 proved there exists a sequence $M_k^3 \hookrightarrow S^4$ such that $M_k^3 \cong S^3$, and $\mathcal{H}^3(M_k^3) \leq \Lambda$ but $M_k^3 \to M$ where $M = \frac{T^2 \times [0,1]}{\sim}$ is singular at antipodal points.
2. When $n = 2$ and $Ric_N > 0$, by results of Ejiri-Micallef '08 and Choi-Wang '83 we get that

$$index(M) + \mathcal{H}^2(M) \leq C(N) \frac{16\pi}{\alpha} \left(\frac{1}{|\pi_1(N)|} - \chi(M) \right).$$

Question: are index and volume the correct quantities to control for higher dimensions?

Recall that Q is diagonalised by an L^2-orthonormal basis of eigenfunctions $\{v_k\} \subset \Gamma(NM)$ with eigenvalues

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \cdots \to \infty$$

where

$$-\Delta_M^{\perp} v_k - |A|^2 v_k - Ric_N(v_k, \cdot) = L_M v_k = \lambda_k v_k.$$

Therefore $index(M) = \max\{k | \lambda_k < 0\}$.
We will order the space $\mathfrak{M}(N)$ in the following way, given $p \in \mathbb{N}$ and $0 \leq \Lambda, \mu \in \mathbb{R}$

$$\mathcal{M}_p(\Lambda, \mu) = \{M \in \mathfrak{M}(N) | \lambda_p \geq -\mu, \mathcal{H}^n(M) \leq \Lambda\}.$$

If

$$\mathcal{M}(\Lambda, I) = \{M \in \mathfrak{M}(N) | index(M) \leq I, \mathcal{H}^n(M) \leq \Lambda\}$$

then we see easily that $\mathcal{M}(\Lambda, I) = \mathcal{M}_{I+1}(\Lambda, 0)$ but in general there is no obvious relationship between the two for $\mu > 0$.

Theorem (S '14 (in the case of $\mathcal{M}(\Lambda, I)$), Ambrozio - Carlotto - S '15 (in general))

Let $\{M_k\} \subset \mathcal{M}_p(\Lambda, \mu)$, then there exists $M \in \mathcal{M}_p(\Lambda, \mu)$, $m \in \mathbb{N}$ and a finite set mathcalY $\subset M$ such that (up to subsequence) $M_k \to mM$ smoothly and graphically with multiplicity m on $M \backslash \mathcal{Y}$.

Assuming that $M_k \neq M$ eventually, we have $|\mathcal{Y}| \leq p - 1$ and nullity$(M) \geq 1$[1] or nullity$(\tilde{M}) \geq 1$[2]. Furthermore the following dichotomy holds:

1. *if the number of leaves in the convergence is one then $\mathcal{Y} = \emptyset$*
2. *if the number of sheets is ≥ 2*
   - *M is two-sided implies that M is stable (index$(M) = 0$)*
   - *M is one-sided implies that $\tilde{M}$ is stable.*

[1] i.e. there exists a non-trivial solution to $L_M v = 0$

[2] when M is one-sided $\tilde{M}$ is the two-sided immersion related to M

Corollary (A-C-S '15)

Let N^{n+1} be a closed Riemannian manifold with $Ric_N > 0$ and $2 \leq n \leq 6$. Then given any $0 \leq \mu, \Lambda < \infty$ and $p \in \mathbb{N}$ the class $\mathcal{M}_p(\Lambda, \mu)$ is compact in the C^l topology for all $l \geq 2$.

Notice that by an easy argument we have the existence of $C = C(\Lambda, p, \mu, N)$ such that for any $M \in \mathcal{M}_p(\Lambda, \mu)$

$$\sup_M |A| + \sum_i b_i(M) \leq C,$$

moreover there are only finitely many such M up to diffeomorphism.

A result of Chodosh-Ketover-Maximo '15 implies that in fact the volume bound can be dropped here when $n = 2$.

Open Problem: Can the volume bound be dropped when $n \geq 3$?

Theorem (Buzano - S '16)

Let $\{M_k\} \subset \mathcal{M}_p(\Lambda, \mu)$ for some fixed constants $0 \leq \Lambda, \mu$, $p \in \mathbb{N}$ independent of k. Then with M as in the previous theorem, there exists a finite number (possibly zero) $\{\Sigma_l^n\}_{l=1}^J \hookrightarrow \mathbb{R}^{n+1}$ of properly embedded minimal hypersurfaces with finite total curvature for which

$$\lim_{k\to\infty} \int_{M_k} |A_k|^n = m \int_M |A|^n + \sum_{l=1}^{J} \int_{\Sigma_l} |A_l|^n$$

where m is the multiplicity of the local graphical convergence away from $\mathcal{Y}$ and $J \leq p - 1$. Furthermore when k is sufficiently large, the M_k's are all diffeomorphic to one another.

Corollary

There exists some $C = C(N, p, \Lambda, \mu)$ such that $\mathcal{M}_p(\Lambda, \mu)$ contains at most C elements up to diffeomorphism.

The above has been proved by Chodosh-Ketover-Maximo using different methods when considering the more restrictive class $\mathcal{M}(\Lambda, I)$.

Corollary

There exists some $C = C(N, p, \Lambda, \mu)$ such that for any $M \in \mathcal{M}_p(\Lambda, \mu)$ we have

$$\int_M |A|^n \leq C.$$

By results of Cheng-Tysk ($n \geq 3$) and Ejiri-Micallef ($n = 2$) we know that for any M

$$index(M) \leq C \left(\int_M |A|^n + \mathcal{H}^n(M) \right).$$

Thus we have

Corollary

There exists some $I = I(N, \Lambda, p, \mu)$ such that $\mathcal{M}(p, \mu, \Lambda) \subset \mathcal{M}(\Lambda, I)$.

In other words, if we consider the class of minimal hypersurfaces M so that $\mathcal{H}^n(M) \leq \Lambda$ then a control on index is *equivalent* to controlling just one of the eigenvalues λ_p away from $-\infty$ and either of these conditions is itself equivalent to controlling the total curvature of M.

Thank you for your attention!