Finite-time singularity formation for Euler vortex sheet

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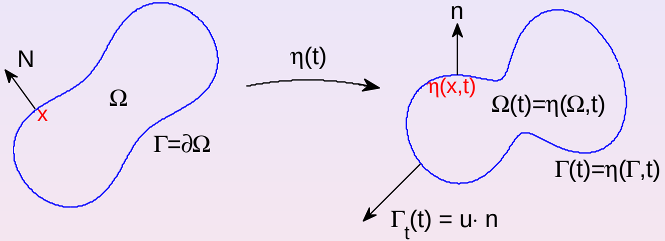
Free surface Euler equations

Picture



- Free-surface $\Gamma(t)$ is transported by the fluid velocity
- Lagrangian variables: Let $\eta(\cdot,t) : \Omega \to \Omega(t)$ denote the flow map of u

$$\eta_t = u \circ \eta, \quad \eta(x,0) = x \quad \text{or} \quad \eta = Id + \int_0^t v$$

with $v(x,t) := u(\eta(x,t),t)$. Also, $\text{div}u = 0 \Rightarrow \det \nabla\eta = 1$.

The Eulerian description

The system of PDE

$$\begin{aligned} u_t + u \cdot \nabla u + \nabla p = 0 & \quad \text{in } \Omega(t) = \eta(\Omega(0), t) \\ \text{div} u = 0 & \quad \text{in } \Omega(t) \\ pn = -\sigma \nabla_\tau \tau & \quad \text{on } \Gamma(t) \ (\sigma \geq 0) \\ u = u_0 & \quad \text{on } \Omega(0) \\ \Omega(0) = \Omega_0 \,. & \end{aligned}$$

Basic Unknowns for the PDE

The solution involves the following quantities:

- Velocity vector $u = (u^1, u^2)$
- Pressure function p
- Moving domain $\Omega(t) = \eta(\Omega_0, t)$

A Brief History of the Free-Boundary Euler equations

- Water waves equation: Assume $u = \nabla\phi$, where ϕ is the velocity potential.
- Then curl $u = 0$ and ϕ is harmonic – problem reduces to the motion of the free-surface (complex analysis in 2-D)
- Local well-posedness for water waves: Wu (1997,1999), Ambrose & Masmoudi (2005,2009), Lannes(2005), Alazard, Burq, Zuily (2014)
- Local well-posedness for Euler: Lindblad (2005), Coutand & Shkoller (2007), Shatah& Zeng (2008, 2011), Zhang& Zhang (2008)
- Water waves *global in time existence* for small data on infinite domain : Wu (2009), Germain, Masmoudi & Shatah (2012), Ionescu & Pusateri (2015), Alazard & Delort (2015), Deng, Ionescu, Pausader & Pusateri (2016), Ifrim & Tataru (2016).
- Small data global in time existence is entirely open for the physical case of bounded domains.
- What about finite time singularity formation ? Curvature blow-up of free-surface, cusp formation, loss of injectivity....

The splash and splat singularities

Definition (Castro, Cordoba, Fefferman, Gancedo, Gomez-Serrano (2013))

Splash Singularity – the smooth free surface $\Gamma(t)$ self-intersects at a point x_0.

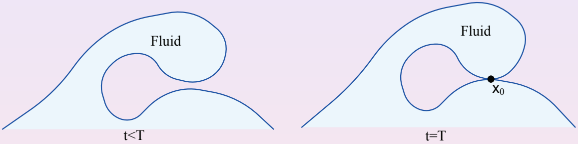


Figure 1: The splash singularity wherein the top of the crest touches the trough at a point x_0 in finite time T.

Definition (Castro, Cordoba, Fefferman, Gancedo, Gomez-Serrano (2013))

Splat Singularity – $\Gamma(t)$ self-intersects on a smooth surface Γ_0.

Finite time splash and splat singularity for the one phase Euler equations

Theorem (2-D, Castro, Cordoba, Fefferman, Gancedo, Gomez-Serrano (2013))

Finite-time splash and splat singularity for irrotational 2-D water waves equations using complex analysis, analytic functions, and conformal transformations.

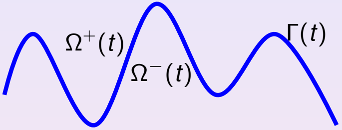
Theorem (3-D, Coutand, Shkoller (2014))

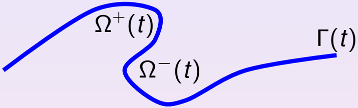
Finite-time splash and splat singularity for free-surface Euler equations with energy methods in Sobolev spaces.

Comment on both results

- The natural norm of the problem stays finite for all time (independently of how close to contact we are).
- Geometric singularity without natural norm blow-up.

Finite time splash and splat singularity ruled out for the two-phase Euler equations (vortex sheet problem)





Theorem (2-D, Fefferman, Ionescu, Lie (2016))

Finite-time splash and splat singularity ruled out for two-phase 2-D water waves equations with surface tension using complex analysis, analytic functions, and conformal transformations.

Theorem (2-D, Coutand, Shkoller (2016))

Finite-time splash and splat singularity ruled out for two-phase 2-D water waves equations with surface tension using elliptic methods.

Finite time splash and splat singularity ruled out for the two-phase Euler equations (vortex sheet problem)

Comment on both results

- The formation of a self-intersection is not ruled out by these results. It is the case when the self-intersecting $\Gamma(t)$ is locally smooth (relative to a parameterisation), as well as the velocity field on it, which is excluded.
- These results show the formation of a self-intersection requires a blow-up of the natural norm of the problem.
- How to identify situations leading to a blow-up of the natural norm of the problem ?

The 2-D Euler vortex sheet with surface tension

The system of PDE

$$
\begin{aligned}
\rho^{\pm}(u_t^{\pm} + u^{\pm} \cdot \nabla u^{\pm}) + \nabla p^{\pm} &= -\rho^{\pm}(0, g) && \text{in } \Omega^{\pm}(t) \\
\operatorname{div} u^{\pm} &= 0 && \text{in } \Omega^{\pm}(t) \\
(p^{+} - p^{-})n &= \sigma \nabla_{\tau} \tau && \text{on } \partial\Omega^{+}(t) \\
u^{-} \cdot n &= u^{+} \cdot n && \text{on } \partial\Omega^{+}(t) \\
u^{-} \cdot n &= 0 && \text{on } \partial\Omega \\
(u^{\pm}(0), \Omega^{\pm}(0)) &= (u_0^{\pm}, \Omega_0^{\pm}) \\
\Omega^{\pm}(t) &= \eta^{\pm}(\Omega_0^{\pm}, t)\,.
\end{aligned}
$$

Basic Unknowns for the PDE

- Velocity vector and pressure in each phase $\Omega^{\pm}(t)$: $u^{\pm} = (u_1^{\pm}, u_2^{\pm})$, $p^{\pm}$.
- Moving bubble domain $\Omega^{+}(t)$

Known properties of Euler vortex sheet

Theorem (Local in time well-posedness)

- *Cheng, Coutand, Shkoller (2008).*
- *Shatah, Zeng (2008, 2011).*

Comment on both results

- These established independently that the problem is locally in time well-posed for a norm like

$$N(t) = \|\eta^-\|_{H^{\frac{9}{2}}(\Omega_0^-)} + \|u^+\|_{H^3(\Omega^+(t))} + \|u^-\|_{H^3(\Omega^-(t))} .$$

- How to identify situations leading to a blow-up of the norm of the problem, given the analysis shows an estimate of the type

$$N(t) \leq t^\alpha P(\sup_{[0,t]} N) + C_0 \, ?$$

Gravity driven singularity formation: initial assumptions

Picture

Initial symmetry assumptions

If

- Ω, $\overline{\Omega_0^+} \subset \Omega$ symmetric with respect to vertical axis $x_1 = 0$,
- $n_2 < -C < 0$ in a neighborhood Γ_1 (of size at least 1) of the lowest part of the vertical projection of Ω_0^+ over $\partial\Omega$ (a graph satisfies this condition).
- $u_1(0)$ odd in x_1, $u_2(0)$ even in x_1,

then this stays the same for all time of existence. The pressure p is even as solution of an elliptic problem.

Gravity driven singularity formation: Statement

Theorem (Finite-time singularity formation, Coutand (2017))

There exists $\epsilon > 0$ *(depending on the dimensions of* Ω*) such that if*

- $\rho^+ > \rho^-$, *(higher density for the bubble* $\Omega^+(t)$*)*
- $\|u^\pm(0)\|_{L^2(\Omega^\pm(0))} + |\partial\Omega_0^+| + \frac{1}{|\Omega_0^+|}\left|\int_{\Omega_0^+} x_2\,dx\right| + \|curl u_0^\pm\|_{L^2(\Omega^\pm)} \leq \epsilon\,,$

then for some finite time $T > 0$*, the vertical projection of* $\Omega^+(t)$ *stays contained in* Γ_1 *and either*

- $N(t) \to \infty$ *as* $t \to T$*,*
- *or there will be finite-time self-intersection of* $\partial\Omega^+(T)$*,*
- *or* $\partial\Omega^+(T)$ *intersects* $\partial\Omega$*.*

Comment

Finite time self-intersection or intersection with $\partial\Omega$ lead to a blow-up of $\|\nabla u^\pm\|_{L^\infty(\Omega^\pm(t))} + |\nabla_\tau \tau\|_{L^\infty(\Gamma(t))}$ as $t \to T$.

Gravity driven singularity formation: Idea of proof

Outline of approach

- Basic energy estimate for controlling L^2 norm of velocity and length of interface independently of time.
- Tracking motion of centre of gravity of moving bubble.
- Obtain an equation for this motion showing some surface energy.
- Perform elliptic estimates away from the bubble to control undesirable terms that may oppose fall of bubble.
- Obtain a nice DI for the surface energy providing finite time blow-up.

Definition (Centre of gravity of $\Omega^+(t)$)

$$x^+(t) = \frac{1}{|\Omega^+|} \int_{\Omega^+(t)} x \, dx = \frac{1}{|\Omega_0^+|} \int_{\Omega_0^+} \eta^+ \, dx \,,$$

since $\det \nabla \eta^+ = 1$ and $\Omega^+(t) = \eta^+(\Omega_0^+, t)$.

Gravity driven singularity formation: Centre of mass

Velocity and momentum of centre of mass

- $$v^+(t) = \frac{dx^+}{dt} = \frac{1}{|\Omega_0^+|}\int_{\Omega_0^+} v^+ \, dx = \frac{1}{|\Omega_0^+|}\int_{\Omega^+(t)} u^+ \, dx\,.$$

  Thus, $|v^+(t)| \le \frac{1}{\sqrt{|\Omega_0^+|}} \underbrace{\|u^+\|_{L^2(\Omega^+(t))}}_{controlled}$.

- $$\underbrace{\rho^+|\Omega_0^+|}_{=m^+} \frac{dv^+}{dt}(t) = \rho^+ \int_{\Omega_0^+} \frac{dv^+}{dt}(x,t)\, dx = \rho^+ \int_{\Omega^+(t)} u_t^+ + u^+\cdot\nabla u^+ \, dx\,.$$

  $$m^+\frac{dv^+}{dt}(t) = -\int_{\Omega^+(t)} \nabla p^+ + \rho^+ g(0,1)\, dx = \int_{\partial\Omega^+(t)} p^+ n^- \, dl(t) - m^+ g(0,1)$$

  $$m^+\frac{dv^+}{dt}(t) + m^+ g(0,1) = \int_{\partial\Omega^+(t)} (p^- n + \sigma \underbrace{\nabla_\tau \tau)}_{zero\ integral} \, dl(t) = \int_{\partial\Omega^+(t)} \underbrace{p^- n}_{sign?} dl(t)\,.$$

Equation for the centre of mass

Lemma (Equation of centre of mass)

$\theta : [0,1] \to \partial\Omega$ being a parameterisation of $\partial\Omega$,

$$m^+ \frac{dv_2^+}{dt} = \rho^- \int_{\partial\Omega} \frac{|u^-|^2}{2} \underbrace{n_2}_{<0 \text{ on bottom}} dl - \rho^- \frac{d}{dt} \int_{\Omega^-(t)} u_2^- \, dx - \underbrace{(\rho^+ - \rho^-)}_{>0} |\Omega_0^+| g$$

$$+ \frac{d}{dt} \int_0^1 \int_0^s \rho^- u^- \cdot \tau(\theta(\alpha), t) |\theta'(\alpha)| d\alpha \, n_2(\theta(s)) \, |\theta'(s)| \, ds \, .$$

Proof. With n denoting the outer unit normal to $\Omega^-(t)$,

$$\int_{\Omega^-(t)} \nabla p^- \, dx = \int_{\partial\Omega^+(t)} p^- \, n \, dl(t) + \int_{\partial\Omega} p^- \, n \, dl \, .$$

This provides by substitution in our equation of motion:

$$m^+ \frac{dv^+}{dt} = - \int_{\partial\Omega} p^- \, n \, dl + \int_{\Omega^-(t)} \nabla p^- \, dx - m^+ g(0,1) \, ,$$

Equation for the centre of mass

$$
\begin{aligned}
m^- \frac{dv^s}{dt} &= -\int_{\partial\Omega} p^- \, n \, dl - \rho^- \int_{\Omega^-(t)} u_t^- + u^- \cdot \nabla u^- + g(0,1)\, dx - m^+ g(0,1)\,, \\
&= -\int_{\partial\Omega} p^- \, n \, dl - \rho^- \frac{d}{dt} \int_{\Omega^-(t)} u^- \, dx - (m^+ + \rho^- |\Omega^-|) g(0,1)\,. \qquad (3)
\end{aligned}
$$

Next, we express the Euler equations as

$$
\rho^- u_t^- + \nabla \left(\frac{\rho^- |u^-|^2}{2} + p \right) = -\rho^- g(0,1) + \rho^- \mathrm{curl} u^- \, (-u_2^-, u_1^-)\,.
$$

$$
\rho^f u_t^- \cdot \tau + \nabla_\tau \Big(p + \rho^- \frac{|u^-|^2}{2}\Big) = -\rho^- g \tau_2 + \rho^- \mathrm{curl} u^- \underbrace{u^- \cdot n}_{=0 \text{ on } \partial\Omega} \,.
$$

By integration along $\partial\Omega$,

$$
\begin{aligned}
\Big(p^- + \rho^- \frac{|u^-|^2}{2}\Big)(\theta(s), t) =& \Big(p + \rho^- \frac{|u^-|^2}{2}\Big)(\theta(0), t) \\
& - \int_0^s (\rho^- g \tau_2 + \rho^- u_t^- \cdot \tau)(\theta(\alpha), t) |\theta'(\alpha)| d\alpha\,. \qquad (4)
\end{aligned}
$$

Equation for the centre of mass

Report (4) in (3). This yields a term

$$\int_0^1 \int_0^s \tau_2(\theta(\alpha))|\theta'(\alpha)|d\alpha \; n(\theta(s)) \; |\theta'(s)| \; ds \, .$$

Define $f(x) = x_2$, so that $\nabla f = (0,1)$ and $\nabla_\tau f = \tau_2$. Therefore,

$$f(\theta(s)) = f(\theta(0)) + \int_0^s \underbrace{\tau_2(\theta(\alpha))}_{\nabla_\tau f(\theta(\alpha))} \underbrace{|\theta'(\alpha)| \; d\alpha}_{dl} \, . \tag{5}$$

Next, since

$$\int_\Omega (0,1) \; dx = \int_\Omega \nabla f \; dx = \int_{\partial\Omega} f \; n \; dl \, , \tag{6}$$

substituting (5) in (6) provides:

$$|\Omega|(0,1) = \int_0^1 \int_0^s \tau_2(\theta(\alpha))|\theta'(\alpha)|d\alpha \; n(\theta(s)) \; |\theta'(s)| \; ds \, . \tag{7}$$

Locating the moving bubble

Lemma (Conservation of energy)

- *The total energy is independent of time:*

$$E(t) = \frac{\rho^+}{2} \int_{\Omega^+(t)} |u^+(x,t)|^2 \, dx + \frac{\rho^-}{2} \int_{\Omega^-(t)} |u^-(x,t)|^2 \, dx$$
$$+ \underbrace{(\rho^+ - \rho^-)}_{\geq 0} g \underbrace{x_2^+(t)}_{\geq 0} |\Omega^s| + \sigma |\partial\Omega^+(t)| \,,$$

Given our initial assumptions, length of curve stays small and centre of gravity stays away from top:

Lemma (Most of Ω never gets touched by the moving bubble)

- $\Omega^+(t)$ *stays away from the top and lateral sides of* $\partial\Omega$ *by a positive distance independently of time.*
- *Its vertical projection on the bottom of* $\partial\Omega$ *only intersects the part of* $\partial\Omega$ *where* $n_2 < 0$.

Elliptic estimate on $\partial\Omega$ away from bubble

Lemma

We have the existence of $C > 0$ (depending on Ω) such that

$$\int_{\Gamma_1^c} |u^f|^2 \, dl \leq C x_2^+(0)|\Omega_0^+| + C\|\omega^-(0)\|^2_{L^2(\Omega_0^-)} \,,$$

where $\omega = curl \, u^-$ and Γ_1^c is the part of $\partial\Omega$ which is away from the bubble for all time by a positive distance $D > 0$. C is small if Ω has large dimensions.

Proof.

$$u^- = \nabla^\perp \phi \,, \text{in } \Omega^-(t) \,,$$

$$\phi = 0 \,, \text{on } \partial\Omega \,,$$

$$\Delta\phi = \omega = \omega_0(\eta^{-1}(x,t)) \,, \text{in } \Omega^-(t) \,,$$

with $\|\nabla\phi\|^2_{L^2(\Omega^-(t))} = \|u^f\|^2_{L^2(\Omega^-(t))} \leq 2\frac{E(0)}{\rho^-}$.

Elliptic estimate on $\partial\Omega$ away from bubble

With $\tilde{n}$ being an harmonic extension of the normal vector on $\partial\Omega$ to Ω, we have

$$|\tilde{n}(x)|^2\omega = |\tilde{n}(x)|^2\Delta\phi(x,t) = \tilde{\tau}_i(x)\tilde{\tau}_j(x)\frac{\partial^2\phi}{\partial x_i\partial x_j}(x,t) + \tilde{n}_i(x)\tilde{n}_j(x)\frac{\partial^2\phi}{\partial x_i\partial x_j}(x,t)\,. \tag{8}$$

Using

$$\xi\tilde{n}_k\frac{\partial\phi}{\partial x_k} = \xi\nabla_{\tilde{n}}\phi$$

as a test function in (8) for ξ cut-off function such that $\xi = 1$ in Γ_1^c, we have after integration by parts a relation of the type:

$$\frac{1}{2}\int_{\Gamma_1^c}|\nabla_n\phi|^2 - \underbrace{|\nabla_\tau\phi|^2}_{=0 \text{ on } \partial\Omega} dl = \int_\Omega \xi \underbrace{\omega}_{L^2(\Omega^-(t))} |\tilde{n}|^2\nabla_n\phi + \int_\Omega B(\nabla\phi,\nabla\phi)F(\nabla\xi,\xi,\nabla\tilde{n},n)\,dx$$

□

Return to the ODE and finite-time singularity

$$m^+ \frac{dv_2^+}{dt} = \rho^- \int_{\partial\Omega} \frac{|u^-|^2}{2} \underbrace{n_2}_{<0 \text{ on bottom}} dl - \rho^- \frac{d}{dt} \int_{\Omega^-(t)} u_2^- \, dx - \underbrace{(\rho^+ - \rho^-)}_{>0} |\Omega_0^+| g$$

$$+ \frac{d}{dt} \int_0^1 \int_0^s \rho^- u^- \cdot \tau(\theta(\alpha), t) |\theta'(\alpha)| d\alpha \; n_2(\theta(s)) \; |\theta'(s)| \; ds \, .$$

Integrating this relation from 0 to t, and picking our initial data so that

$$\rho^- (C x_2^+(0) |\Omega_0^+| + C \|\omega^-(0)\|^2_{L^2(\Omega_0^-)}) \leq \frac{\rho^+ - \rho^-}{4} g |\Omega_0^+| \, ,$$

we have an inequality of the type ($C_\Omega^i > 0$):

$$C_\Omega^1 \rho^- \int_{\partial\Omega} |u^-| dl \geq C_\Omega^2 \rho^- \int_0^t \int_{\partial\Omega} |u^-|^2 dl dt + \frac{\rho^+ - \rho^-}{2} |\Omega_0^+| g t - C_0 \, ,$$

Return to the ODE and finite-time singularity

and since

$$\int_0^t \int_{\partial\Omega} |u^-|^2 dl dt \geq \frac{\left(\int_0^t \int_{\partial\Omega} |u^-| dl dt\right)^2}{t|\partial\Omega|},$$

we obtain a nice DI for $\int_0^t \int_{\partial\Omega} |u^-| dl dt$:

$$C_\Omega^1 \rho^- \int_{\partial\Omega} |u^-| dl \geq C_\Omega^2 \rho^- \frac{\left(\int_0^t \int_{\partial\Omega} |u^-| dl dt\right)^2}{t|\partial\Omega|} + \frac{\rho^+ - \rho^-}{2} |\Omega_0^+| gt - C_0 \,.$$

Since $\int_1^\infty \frac{1}{t}\, dt = \infty$, this ensures blow up in finite time T_{max} for $\int_0^t \int_{\partial\Omega} |u^-| dl dt$. Thus, if $N(t)$ was still finite until T_{max} and no self-intersection or contact with $\partial\Omega$ occurred, we still have blow-up of this quantity at T_{max} to ensure breakdown in finite time of smooth solution. □

Comments

- Very general method, works for any included phase (with Ω^- Euler phase) for which we have a priori control of the diameter (3-D non linear elasticity should be suitable, if no vorticity in fluid).
- The principle of derivation of the DI is very different from Sideris (86) and Xin (98) for compressible Euler (wave equation).
- The case when Ω has free surface as well (away from inclusion) presents significant additional challenges (as surface energy DI set on $\partial\Omega$ needs change).
- The simplest case of inclusion is the rigid body (no change of shape), for which this method works as well (without assumptions on the initial height and length this time, but still a small vorticity). We can have more precise statements about the fluid velocity fields and pressure, as well as acceleration of the rigid solid, at the time of contact.

Motion of a rigid body in an Euler fluid in 2-D

Setting

- This time the bubble $\Omega^s(t)$ has a fixed shape. Assuming initial symmetry assumptions at time 0 we have that it falls vertically

$$\Omega^s(t) = \Omega^s(0) + (0, x_2^s(t) - x_2^s(0))\,.$$

- The motion of the rigid solid is governed by $v_2^s(t)$.
- On $\partial\Omega^s(t)$, $u^f \cdot n = v_2^s n_2$.
- The fluid $\Omega^-(t)$ is governed by the Euler equations while the center of gravity of the rigid body is governed by

$$m_s \frac{dv^s}{dt} = \int_{\partial\Omega^s(t)} pn\, dl - m_s(0, g)\,,$$

n exterior to the fluid phase.

- We still assume $\rho^s > \rho^f$.

Motion of a rigid body in an Euler fluid in 2-D: History

- Global in time well-posedness so long as no contact with $\partial\Omega$ proved by Glass & Sueur (2015) for general configurations.
- Finite time contact established by Munnier & Ramdani (2015) with symmetry assumptions ensuring vertical fall and no vorticity, and flat bottom. Some cases with bottom with very special geometries are allowed. They establish that the velocity of contact is either (finite) non zero (like disk) or zero (flatter).
- The method of Munnier & Ramdani (2015) is purely elliptic, (there is no DI for a surface energy) and view this problem as a sequence of Neumann problems for ϕ such that $u^f = \nabla\phi$, in domains with the bottom of $\partial\Omega$ at a distance ϵ from the bottom of the rigid body. They then cast this problem in a strip of given height and length converging to ∞ as $\epsilon \to 0$. Vorticity is not allowed in this setting.

Statement of new results for rigid body case

Our methodology can be applied to this problem. It gives:

Theorem (Coutand(2017))

- *With the symmetry assumptions ensuring vertical motion, if* $v_2^s(0) < 0$ *and* $\|\omega_0\|_{L^2(\Omega^f)} \le \epsilon_0 |v_2^s(0)|$ *(*$\epsilon_0 > 0$ *depending on* Ω*) then there is finite time contact (without assumptions on shape other than symmetric).*
- *For the case of the flat bottom, and strictly convex geometry around lowest point of* $\partial\Omega^s(t)$*, we have*
  $$\lim_{t\to T_{max}^-} \|u^f\|_{L^2(\partial\Omega)} = \lim_{t\to T_{max}^-} \|u^f\|_{L^2(\partial\Omega^s(t))} = \lim_{t\to T_{max}^-} \|u^f\|_{H^1(\Omega^f(t))} = \infty,$$
  while $\displaystyle\int_0^{T_{max}} \|u^f\|^2_{H^1(\Omega^f(t))}\, dt < \infty.$
- *For the same case, if* $\omega_0 = 0$, $\displaystyle\lim_{t\to T_{max}^-} \|p\|_{L^1(\partial\Omega^s(t))} = \lim_{t\to T_{max}^-} \frac{dv_2^s}{dt} = \infty.$

Contrast between free fall in void (constant acceleration) and free fall in fluid (infinite upward acceleration as if trying to avoid collision).

Monotone motion for the centre of gravity of rigid solid

We assume $v_2^s(0) < 0$. Let us assume that there exists a first $t_0 > 0$ such that

$$v_2^s(t_0) = 0\,.$$

Then, from $u^f = \nabla\phi$ which provides for ϕ:

$$\begin{aligned}\Delta\phi(\cdot,t_0) &= 0\,,\ \text{in } \Omega^f(t_0)\,,\\ \nabla\phi(\cdot,t_0)\cdot n &= 0\,,\ \text{on } \partial\Omega\,,\\ \nabla\phi(\cdot,t_0)\cdot n &= v^s(t_0)\cdot n = 0\,,\ \text{on } \partial\Omega^s(t_0)\,,\end{aligned}$$

from which we immediately have $\nabla\phi(\cdot,t_0) = 0$ and thus $u^f(t_0) = 0$. Thus

$$x_2^s(t_0) < x_2^s(0) \Rightarrow E(t_0) = (\rho^s - \rho^f) g x_2^s(t_0)|\Omega^s| < E(0)\,.$$

$\Rightarrow$ If no finite time collision, $\int_0^\infty |v_2^s(t)|\, dt < \infty \Rightarrow$ finite time blow-up in DI.

Non zero contact velocity

Lemma

Assume that the lowest point of the strictly convex $\partial\Omega^s(t)$ is on $x_1 = 0$. Then, independently of time,

$$\int_{\partial\Omega^s(t)} x_1^2 (u^f \cdot \tau)^2 \, dl \leq C\,.$$

Proof Introducing

$$f(t) = \int_{\partial\Omega^s(t)} x_1 n_2 u_1^f u_2^f \, dl\,,$$

and writing it in two different manners: One using $u^f = (u^f \cdot \tau)\, \tau + \underbrace{(u^f \cdot n)}_{= v_2^s n_2}\, n$,

the second one using the divergence theorem to have an integral set on $\Omega^f(t)$,

and comparing the two expressions.

Non zero contact velocity

$u^f = \nabla^\perp \phi$ which provides for ϕ:

$$\begin{aligned} \Delta\phi(\cdot, t_0) =& \omega(\cdot, t)\,, \text{ in } \Omega^f(t)\,, \\ \phi(\cdot, t) =& 0\,, \text{ on } \partial\Omega\,, \\ \phi(\cdot, t) =& v_2^s(t) x_1\,, \text{ on } \partial\Omega^s(t)\,. \end{aligned}$$

as $\nabla \tau x_1 = \tau_1 = n_2$. Thus,

$$\begin{aligned} \int_{\Omega^f(t)} |u^f|^2 \, dx &= -\int_{\Omega^f(t)} \omega\phi \, dx + v_2^s(t) \int_{\partial\Omega^s(t)} \nabla_n \phi \; x_1 \; dl \\ &= -\int_{\Omega^f(t)} \omega\phi \, dx - v_2^s(t) \int_{\partial\Omega^s(t)} u^f \cdot \tau \; x_1 \; dl\,, \end{aligned}$$

Therefore,

$$\|u^f\|^2_{L^2(\Omega^f(t))} \le \|\phi\|_{L^2(\Omega^f(t))} \|\omega_0\|_{L^2(\Omega^f)} + |v_2^s(t)| \left| \int_{\partial\Omega^s(t)} u^f \cdot \tau \; x_1 \; dl \right| .$$

Therefore, with our Poincaré inequality (independent of how close to contact we are), we have if $v_2^s(t)$ converges to zero as contact nears:

$$\|u^f\|_{L^2(\Omega^f(t_0))} \leq C\|\omega_0\|_{L^2(\Omega^f)} \,.$$

This leads to a too small kinetic energy for total energy conservation. □

Equivalence of norms (independently of how close to contact)

Lemma

We have the existence of $C_i > 0$ such that

$$C_1 \int_{\partial\Omega^s(t)} |u^f|^2 \, dl - C_2 \leq \int_{\Omega^f(t)} |\nabla u^f|^2 \, dx \leq C_3 \int_{\partial\Omega^s(t)} |u^f|^2 \, dl + C_4 \, .$$

New formula for acceleration of rigid body

Lemma

For the case $\omega_0 = 0$, we have

$$\frac{2E(0) + 2(\rho^f - \rho^s) g x_2^s |\Omega^s|}{(v_2^s)^2} \frac{dv_2^s}{dt} = \rho^f \underbrace{\int_{\partial\Omega} \frac{|u^f|^2}{2} \, dl}_{\text{blows up}} - \rho^f \underbrace{\int_{\partial\Omega \cap \Gamma_1^c} \frac{|u^f|^2}{2} (n_2 + 1) \, dl}_{\text{bounded}}$$
$$+ (m_s - \rho^f |\Omega_s|) g + 2(\rho^f - \rho^s) g |\Omega^s|$$

- Classical mechanics (in void): constant acceleration, $-g$ until contact.
- Fall in Euler fluid: infinite upward acceleration.

Thank you