From Many-particle Relativistic Field Theory to the Schrodinger Equation

David Stuart

Centre for Mathematical Sciences
University of Cambridge, UK

Introduction

Classical Hamiltonian $H(\pi,\phi) = \int_{\mathbb{R}} \mathcal{H}(\pi,\phi)\,dx$ where

$$\mathcal{H}(\pi,\phi) = \frac{1}{2}\left(\pi^2 + \partial_x\phi^2\right) + \mathcal{U}(\phi)\,.$$

with double-well potential function

$$\mathcal{U}(\phi) = \frac{m^4}{2g^2}\left(1 - \frac{g^2\phi^2}{m^2}\right)^2 = \frac{1}{2}\,g^2\left(\phi^2 - \Phi_0^2\right)^2. \tag{1}$$

Two classical vacua $\pm\Phi_0$, where $\Phi_0 = m/g$. Constant configuration Φ_0 minimizes energy with boundary conditions

$$\lim_{|x|\to\infty} \phi(x) = \Phi_0\,; \tag{2}$$

The classical soliton,

$$\Phi_S(x) = \frac{m}{g}\,\tanh mx\,, \qquad \Pi_S(x) = 0\,, \tag{3}$$

minimizes energy amongst configurations which interpolate between the two vacua as its asymptotic boundary values, i.e.,

$$\Phi_S(x) \to \pm\Phi_0 \quad \text{as} \quad x \to \pm\infty\,. \tag{4}$$

Soliton is not unique due to translation invariance: the set of energy minimizers is $\{(\Phi_S(\cdot - \xi), 0)\}_{\xi\in\mathbb{R}}$. The energy of an energy minimizer equals the minimum value of H on the set of finite energy configurations verifying (4). It is the classical rest mass of the soliton, given by

$$\mathbb{M}_{cl} = \frac{4m^3}{3g^2} = \frac{\mathsf{m}_{cl}}{g^2}, \qquad \mathsf{m}_{cl} = \frac{4m^3}{3}. \tag{5}$$

Quantization . Fields now operator valued distributions which must satisfy Heisenberg commutation relation

$$\boxed{\Big[\,\phi(t,x)\,,\,\dot{\phi}(t,y)\,\Big] = i\delta(x-y)}$$

as a constraint.

Existence: regularize the problem, construct Hamiltonian as operator on Hilbert space and take a limit as regularization disappears. To study soliton must carry out this procedure in a comparable way: i.e. must involve the same subtractions ("Wick ordering") and regularizations of the fields (which are related by scattering theory.)

Main result, stated briefly: construct <u>comparable</u> spatially cut-off quantum Hamiltonians corresponding to vacuum and soliton boundary conditions

$$H_g^{vac} \quad \text{and} \quad H_g^{sol}$$

and prove that as $g \to 0$, with strong operator convergence locally uniformmly in time t

Theorem 1

$$\exp[-itH_g^{vac}] \to \exp[-itH_0]$$

but

$$\exp[-itH_g^{sol} + it\frac{\mathrm{m}_{cl}}{g^2} + it\Delta\mathbb{M}_{scl}] \to \exp[-it\frac{P^2}{2\mathrm{m}_{cl}}] \oplus \exp[-itH$$

$\mathrm{m}_{cl} = 4m^3/3$ *scaled classical soliton mass;*

$\Delta\mathbb{M}_{scl} = -m(\frac{3}{\pi} - \frac{1}{2\sqrt{3})}$ *semi-classical mass correction due to Dopplicher-Haslacher-Neveu;*

$\exp[-it\frac{P^2}{2\mathrm{m}_{cl}}]$ *describes free motion of NR quantum particle*

H_0 *(and* H_0^{sol}*) are free field (and free field in soliton background)*

Now explain these concepts.

I Classical Nonrelativistic particle

Particle: mass M concentrated at a point

$$\mathbf{X}(t) \in \mathbb{R}^3 \quad \text{at time } t$$

No internal structure.

Newton : if no forces act on a particle it moves at uniform velocity

$$\mathrm{m}\frac{d^2\mathbf{X}}{dt^2} = 0$$

Conservation laws:

$$\mathbf{P} = \mathrm{m}\frac{d\mathbf{X}}{dt} \ (momentum)$$

$$E = \frac{\mathbf{P}^2}{2\mathrm{m}} \ (kinetic\, energy)$$

People used to think that ... when a thing moves it is in a state of motion. This is now known to be a mistake. **Bertrand Russel**

II Quantum Nonrelativistic Particle

The energy momentum relation $E = \frac{\mathbf{P}^2}{2\mathrm{m}}$ turns into a dispersion relation

$$\frac{1}{\hbar}\,\omega_k \;=\; \frac{k^2}{2\mathrm{m}}$$

for waves

$$\exp[ikx - i\omega_k t]$$

which are the basic solutions of the Schrödinger equation:

$$i\hbar\frac{\partial\psi}{\partial t} \;=\; -\frac{\hbar^2}{2\mathrm{m}}\,\frac{\partial^2\psi}{\partial x^2}$$

with initial data $\psi(x,0) = \psi_0(x)$.

- Quantum particle has no internal structure;

- lives in a state characterized e.g. by Fourier transform

  $$f(k) = \widehat{\psi}_0(k) \in L^2$$

  as

  $$\psi(x,t) \;=\; \frac{1}{\sqrt{2\pi}}\int f(k)\,\exp[ikx - i\omega_k t]\,dk$$

III Relativistic Particle The relativistic energy momentum relation

$$E^2 = \mathbf{P}^2 + m^2$$

turns into the dispersion relation

$$\omega_k^2 = k^2 + m^2 \qquad (\hbar = 1)$$

and thence the relativistic wave equation

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + m^2 \phi = 0 \, .$$

Problem of negative energies $E = \pm\sqrt{\mathbf{P}^2 + m^2}$ resolved by saying

- ϕ is not a wave function;
- it is a quantum field operator describing creation and annihilation of particles;
- interpretation as multi-particle theory essential.

- ϕ is a distribution taking values in space of unbounded operators on a Hilbert space, constrained by Heisenberg relation

$$[\phi(t,x)\,,\,\dot{\phi}(t,y)] \,=\, i\delta(x-y)\,,$$

(*The Reason for Anti-particles* by Richard Feynman.) Leads to three sources of trouble: ultra-violet, infra-red and particle number.

The free field describes multi-particle theory: sequence of n-particle wave functions $\psi_n(x_1,\ldots,x_n;t)$ evolving according to

$$i\partial_t\,\psi_n \,=\, \sum_{j=1}^{n}\,\sqrt{-\partial_j^2+m^2}\,\psi_n$$

where $\sqrt{-\partial_j^2+m^2}$ is pseudo-differential operator acting in j^{th} argument of ψ_n.

Non-quandratic terms in Hamiltonian couple the ψ_n.

IV Fock space is the (complete) Hilbert direct sum of the symmetric n-fold tensor powers of $L^2(\mathbb{R})$, i.e.

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathrm{Sym}^n(L^2(\mathbb{R}))\,.$$

A typical element, $\Psi \in \mathcal{H}$, is a sequence of functions $\{\Psi_n\}_{n=0}^{\infty}$, where $\Psi_n \in L^2(\mathbb{R}^n)$ is symmetric with respect to interchange of any pair of coordinates.

$$\|\Psi\|^2 = \sum \|\Psi_n\|^2_{L^2(\mathbb{R}^n)}\,.$$

The vacuum has $\Psi_0 = 1$ and $\Psi_n = 0$ for $n \geq 1$. Call it Ω or $|\,0\,\rangle$.

Annihilation and creation operators are given, respectively, by

$$(a_k\Psi)_{n-1}(k_1,\dots,k_{n-1}) = \sqrt{n}\Psi_n(k,k_1,\dots,k_{n-1})\,,$$

$$(a_k^\dagger\Psi)_{n+1}(k_1,\dots,k_{n+1}) = \sum_{j=1}^{n+1} \frac{\delta(k-k_j)}{\sqrt{n+1}}\,\Psi_n(k_1,\dots,\widehat{k_j},\dots,k_{n+1})\,.$$

(Really define operator valued distributions or quadratic forms.)

V The Free Field

Given dispersion relation $\omega_k = \sqrt{k^2 + 4m^2}$, we define the fields

$$\begin{aligned} \varphi(x) &= \frac{1}{\sqrt{2\pi}} \int \frac{1}{\sqrt{2\omega_k}} \left(a_k e^{ikx} + a_k^\dagger e^{-ikx} \right) dk \,, \\ \pi(x) &= \frac{1}{\sqrt{2\pi}} \int -i \sqrt{\frac{\omega_k}{2}} \\ &\qquad \times \left(a_k e^{ikx} - a_k^\dagger e^{-ikx} \right) dk \,. \end{aligned}$$

Really operator valued distributions

$$\varphi(f) = \int \frac{1}{\sqrt{2\omega_k}} \left(a_k \, \widehat{f}(-k) + a_k^\dagger \, \widehat{f}(k) \right) dk \,,$$

where $\widehat{f}(k) = (2\pi)^{-1/2} \int e^{-ikx} f(x) \, dx \in \mathcal{S}(\mathbb{R})$ is the Fourier transform.

Notice vacuum expectation infinite:

$$\langle 0 | \varphi(x)^2 | 0 \rangle = \| \varphi(x) \Omega \|^2 = \frac{1}{4\pi} \int \frac{dk}{\omega_k} = +\infty \,.$$

Wick ordering - move annihilation operators to right - gives

$$\langle 0 : | \varphi(x)^2 : | 0 \rangle = 0 \,.$$

Physically : removes self interaction of particles on themselves.

VI Regularized fields Let $\delta_1 \in C_0^\infty(\mathbb{R})$ be a non-negative, even function with $\delta_1(x) = 0$ for $|x| \geq 1$, and satisfying $\int \delta_1(x)\,dx = 1$. For $\kappa > 0$ define $\delta_\kappa(x) = \kappa\delta_1(\kappa x)$, so that the operator $\delta_\kappa *$ is an approximation to the identity. Regularized fields:

$$\varphi_\kappa(x) = \int \frac{\chi_\kappa(k)}{\sqrt{2\omega_k}} \left(a_k e^{ikx} + a_k^\dagger e^{-ikx}\right) dk\,,$$

$$\pi_\kappa(x) = \int -i\chi_\kappa(k) \sqrt{\frac{\omega_k}{2}} \left(a_k e^{ikx} - a_k^\dagger e^{-ikx}\right) dk\,,$$

where $\chi_\kappa(k) = \chi(k/\kappa)$ with $\chi(k) = \widehat{\delta}_1(k)$.

Regularization amounts to a smooth momentum cut-off at scales large compared to κ since $\chi_\kappa(k) = \widehat{\delta}_1(k/\kappa)$:

$$\gamma_\kappa = \langle\, 0\,|\varphi_\kappa(x)^2|\,0\,\rangle = \int \frac{|\chi_\kappa(k)|^2\,dk}{2\omega_k} < +\infty\,.$$

But Wick ordering interferes with boundedness below:

$$\begin{aligned} :\varphi_\kappa(x)^4: &= \varphi_\kappa(x)^4 - 6\gamma_\kappa\varphi_\kappa^2 + 3\gamma_\kappa^2 \\ &= (\varphi_\kappa - 3\gamma_\kappa)^2 - 6\gamma_\kappa^2 \\ &\geq -6\gamma_\kappa^2 \end{aligned}$$

so the pointwise lower bound diverges as cut-off removed.

Wick Operators Given a function or distribution $w \in \mathcal{S}(\mathbb{R}^{m+n})$, written$w = w(\underline{k}, \underline{k}')$ for $\underline{k} = (k_1, \ldots k_m)$ and $\underline{k}' = (k'_1, \ldots k'_n)$, the Wick operator on Fock space is given by

$$\mathbb{W}_w = \int_{\mathbb{R}^{n+m}} a^\dagger(k_m) \ldots a^\dagger(k_1) w(\underline{k}, \underline{k}') \\ \times a(k'_1) \ldots a(k'_n) d\underline{k} d\underline{k}' .$$

Here $d\underline{k}' = \prod_{j=1}^n dk'_j$ and $d\underline{k} = \prod_{j=1}^m dk_j$.

Writing $\mathbb{N} = \int a^\dagger(k) a(k)\, dk$ for the number operator as usual, we have the following bounds in the case that the kernel is square integrable:

$$\|(\mathbb{1} + \mathbb{N})^{-m/2} \mathbb{W}_w (\mathbb{1} + \mathbb{N})^{-n/2}\| \leq \|w\|$$

and, more generally for $a + b \geq m + n$,

$$\|(\mathbb{1} + \mathbb{N})^{-a/2} \mathbb{W}_w (\mathbb{1} + \mathbb{N})^{-b/2}\| \\ \leq (1 + |m - n|^{|m-n|/2}) \|w\|$$

where on the left hand side $\| \cdot \|$ means Fock space operator norm, while on the right hand side $\|w\|$ means the norm of the kernel w as an operator $\mathrm{Sym}^n(L^2(\mathbb{R})) \to \mathrm{Sym}^m(L^2(\mathbb{R}))$.

The Wick polynomial $\int : \varphi_\kappa(x)^4 : b(x)\, dx$ determined by a regularized field and a spatial cut-off $b \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ determines a Wick operator (with obvious conventions for $j = 0, 4$):

$$\sum_{j=0}^{4} \binom{4}{j} \int_{\mathbb{R}^4} a^\dagger(k_1) \dots a^\dagger(k_j) w(\underline{k}, \underline{k}') \\ \times a(-k_{j+1}) \dots a(-k_4) dk_1 \dots dk_4$$

where

$$v(k) = \hat{b}(-\sum_{j=1}^{4} k_j) \prod_{j=1}^{4} \frac{\chi_\kappa(k_j)}{2\omega_{k_j}} \in \mathcal{S}'(\mathbb{R}^4)\,.$$

The preceding Wick operator bounds applied to this give for any $\epsilon > 0$ a number C_ϵ such that

$$\left\| \frac{\int : \varphi_\kappa(x)^4 : b(x)\, dx - \int : \varphi_{\kappa'}(x)^4 : b(x)\, dx}{(\mathbb{1} + \mathbb{N})^4} \right\| \\ \leq \frac{C_\epsilon}{(\min\{\kappa, \kappa'\})^{\frac{1}{2}-\epsilon}}$$

VII Schrödinger equation:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2\mathrm{m}}\frac{\partial^2\psi}{\partial x^2} + V\psi$$

with initial data $\psi(x,0) = \psi_0(x)$.

Feynman (PhD thesis, 1942) reformulated quantum mechanics :

$$\psi(x,t) = \int \exp\Big\{\frac{i}{\hbar}\int_0^t \Big(\frac{1}{2}\mathrm{m}\dot{X}^2 - V(X(s))\Big)ds\Big\} \times \psi_0(X(t)) \boxed{\prod_{0\le s\le t} dX(s)}$$

in terms of "complex probability amplitudes" by summing over paths with $X(0) = x$. Put $\hbar = 1$ from now on.

Mathematical analysis of semi-group via the Feynman-Kac formula: after Wick rotation

$$\underline{t \to -it}$$

to Euclidean time $\exp[-tH]\psi_0(x)$ is given by

$$\mathbb{E}_x\Big[\exp\{-\int_0^t V(X(s))ds\Big\}\,\psi_0(X(t))\Big]$$
$$= \int \exp\Big\{-\frac{1}{\hbar}\int_0^t V(X(s))\,ds\Big\}\,\psi_0(X(t))\,d\mathbb{W}_x(X)$$

(expectation w.r.t. Wiener measure $d\mathbb{W}_x$ on paths starting at $x = X(0)$.)

VIII Rewrite Feynman-Kac formula as

$$\left(F\,,\,e^{-tH}\,G\right)_{L^2} = \int \mathbb{E}_x\left(F(x)\,e^{-\int_0^t J_sV\,ds}J_tG\right)\,dx$$

where $J_sV : C(\mathbb{R}) \to \mathbb{R}$ is the function on path space $X \mapsto V(X(s))$ etc.

X: Gaussian process with covariance $\frac{1}{2\mathrm{m}}\min\{s,t\}$, i.e. evolution is obtained by averaging over all Brownian paths with diffusion $\frac{1}{2\mathrm{m}}$.

For an oscillator

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + \frac{1}{2}\omega^2x^2\psi + V\psi$$

the formula generalizes via introduction of the oscillator process defined as the Gaussian process indexed by $t \in \mathbb{R}$ with covariance

$$\mathbb{E}\left(q(t)q(s)\right) = \frac{e^{-\omega|t-s|}}{2\omega}$$

Averaging over oscillator process we can write $\psi = e^{-tH}\psi_0$ where

$$(F\,,\,\exp[-tH]G) = \mathbb{E}\left(J_0F\,e^{-\int_0^t J_sV\,ds}J_tG\right)$$

where again $J_sV : C(\mathbb{R}) \to \mathbb{R}$ is the function on path space with value $V(q(s))$.

IX Classical Harmonic Oscillator: Hamiltonian

$$H_{osc} = \frac{1}{2}\left(p^2 + \omega^2 x^2\right)$$

Classical free field

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + 4m^2\phi = 0$$

we have, with $\pi = \partial_t\phi = \dot{\phi}$, the Hamiltonian

$$H = \frac{1}{2}\int\left(\pi^2 + \left(\frac{\partial\phi}{\partial x}\right)^2 + 4m^2\phi^2\right)dx$$

or, with, $\widehat{\phi}(k) = (2\pi)^{-\frac{1}{2}}\int e^{-ikx}\phi(x)dx$ etc

$$H = \frac{1}{2}\int\left(|\widehat{\pi}(k)|^2 + \left(k^2 + 4m^2\right)|\widehat{\phi}(k)|^2\right)dk$$

Free field: an infinite collection of oscillators of frequency $\omega_k = \sqrt{k^2 + 4m^2}$.

Nelson: used this to generalize Feynman-Kac to quantum fields, to describe semi-group e^{-tH} acting on the Fock space.

X Feynman-Kac-Nelson Formula We need two facts

- $\exists$ a Gaussian measure γ on $\mathcal{S}'(\mathbb{R})$ giving a model of Fock space (Schrödinger representation) such that

$$\mathcal{F} = \overline{\bigoplus}_n Sym^n\, L^2(\mathbb{R}) = L^2\left(\mathcal{S}'(\mathbb{R}), d\gamma\right)$$

$$\begin{aligned}\mathbb{E}\left(\phi(f)\phi(g)\right) &= \int \phi(f)\phi(g)\,\gamma(d\phi)\\ &= \int \frac{\overline{f(k)}g(k)}{2\omega_k}\,dk\,.\end{aligned}$$

- $\exists$ a Gaussian measure μ on $\mathcal{S}'(\mathbb{R}^2)$ such that

$$\begin{aligned}\mathbb{E}\left(\Phi(f)\Phi(g)\right) &= \int \Phi(f)\Phi(g)\,\mu(d\Phi)\\ &= \iiint \frac{\overline{f(s,k)}e^{-|t-s|\omega_k}g(t,k)}{2\omega_k}\,dkdsdt\,.\end{aligned}$$

$$\left(F\,,\,e^{-tH}\,G\right)_{L^2(\gamma)} = \mathbb{E}\left(J_0F\,e^{-\int_0^t J_sV\,ds}J_tG\right)$$

Here ϕ is the spatial Fourier transform of Euclidean field

$$\phi(t,k) = (2\pi)^{-\frac{1}{2}} \int e^{-ikx} \phi_E(t,x)dx$$

i.e. arguments are time t and spatial Fourier variable k. The Euclidean field is Gaussian process on $\mathcal{S}'(\mathbb{R}^2)$ with covariance

$$\mathbb{E}\Big(\Phi_E(f)\Phi_E(g)\Big) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{\overline{\widehat{f}(k)}\widehat{g}(k)}{k^2 + 4m^2} \, dk \, .$$

At each time t there exists an isometry J_t : $L^2(d\gamma) \to L^2(d\mu)$ given on Wick monomials by

$$J_t : \phi(f)^n : \to : \phi(t,f)^n :$$

Wick monomials obtained by orthogonalization process with respect to the corresponding Gaussian measure. They generate polynomials which are dense in the corresponding L^2 space.

XI Glimm-Jaffe PSC Expansion

Introduce an overall large upper momentum cut-off κ, and sequence

$$\kappa_1 < \kappa_2 < \kappa_3 < \cdots < \kappa_{n-1} < \kappa \leq \kappa_n \qquad \kappa_n = e^{\sqrt{\nu}}$$

and corresponding cut-off Hamiltonians $h_\nu = H^{\kappa_\nu}$ for $1 \leq \nu \leq n-1$, and then and $h_n = H^\kappa$ if $\nu \geq n$. Want bounds independent of κ or equivalently n.

Iterated Duhamel:

$$\begin{aligned} e^{-tH^\kappa} = \; & e^{-th_1} - \int_0^t e^{-(t-s_1)h_2}(H^\kappa - h_1)e^{-s_1 h_1}\, ds_1 \\ & - \int_0^t \int_{s_1}^t e^{-(t-s_2)h_3}(H^\kappa - h_2)e^{-(s_2-s_1)h_2} \\ & \qquad \times (H^\kappa - h_1)e^{-s_1 h_1}\, ds_2 ds_1 \\ & \cdots \\ & - (-1)^n \int_0^t \cdots \int_{s_{n-2}}^t e^{-(t-s_{n-1})H^\kappa}(H^\kappa - h_{n-1}) \\ & \qquad \times \prod_{\nu=2}^{n-1} \left(e^{-(s_\nu - s_{\nu-1})h_\nu}(H^\kappa - h_{\nu-1}) \right) \\ & \qquad \times e^{-s_1 h_1} \prod_{j=1}^{n-1} ds_j \, . \end{aligned}$$

The aim is to prove an *operator* lower bound $H^\kappa \geq -c_0 > -\infty$ which is uniform in κ, in spite of the fact that *pointwise* H_I^κ is not uniformly bounded below. Indeed normal ordering gives

$$
\begin{aligned}
:\varphi_\kappa(x)^4: &= \varphi_\kappa(x)^4 - 6\varphi_\kappa(x)^2\gamma_\kappa + 3\gamma_\kappa^2 \\
&\geq -6\gamma_\kappa^2
\end{aligned}
$$