Maths in the City: Oxford

Short description

This is the route and notes to accompany our Maths in the City tour of Oxford. You can also see the tour at http://www.mathsinthecity.com/tours/maths-city-oxford.

Description

This is the Maths in the City walking tour of Oxford. We'll be looking at symmetry, geometry, GPS and engineering using footprints, string, chalk, woks and marbles!

The tour is suitable for anyone of any age and includes a lot of demonstrations that illustrate the maths behind what you see. If you choose to take your own tour of Oxford and want to make it a bit more interactive, look at the ‘Demonstration’ sections listed in the full tour to see what materials you need to bring with you.

Tour steps

1. Rewley House
2. Sackler Libary – a round peg in a square hole
3. Frieze symmetries at the Ashmolean Museum
4. The Beehive, Oxford
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6. Penrose tiles at Wadham College, Oxford
7. The roof of the Sheldonian Theatre

1. Rewley House

Rewley House is the home of the Department for Continuing Education at the University of Oxford, where Maths in the City is based. It is also the starting point of our walking tour of Oxford!

1 Wellington Square, Oxford, OX1 2JA, UK
2. Sackler Library – a round peg in a square hole

To a mathematical eye a circle seems a strange choice for a building's shape, due to the space wasted between the circular Sackler Library and its straight-sided neighbours. But architectural historian Giles Worsley said the architect, Robert Adam, "choose a rotunda to get the greatest volume on the site with the least impact. The circular form ensures that the building always appears to be receding, minimising its bulk."

St John St, Oxford, Oxfordshire OX1 2, UK

3. Frieze symmetries at the Ashmolean Museum

Artists have used friezes to decorate buildings for thousands of years. The symmetries of these patterns are key to their aesthetic beauty, and also to their mathematical significance.

Ashmolean Museum, Beaumont St, Oxford OX1 2, UK

Viewpoint:

In the courtyard in front of the museum
In search of symmetry

Symmetry is something we all appreciate, whether in beautiful faces, stunning flowers or pleasing patterns. And we have been exploring symmetry for thousands of years, as can be seen in the carved stone balls produced over 3000 years ago in the neolithic period in Scotland. These are carefully decorated, most with evenly spaced carvings or knobs. And most remarkably more than half have 4, 6 or 8 symmetrically placed knobs, showing a striking similarity to the highly symmetric platonic solids that would be discovered a thousand years later by the Greeks.

Three carved stone balls from the British Museum’s collection. Image © Trustees of the British Museum

One way symmetry has been explored over the centuries by artists from all cultures is through decorative patterns and friezes, such as those decorating the Ashmolean museum.

Magic tricks and foot prints

Frieze patterns, such as the simple beaded pattern around the columns of the Ashmolean, are strips decorated with a repeating pattern. As they are repeating patterns they all have translational symmetry – you can slide the pattern along until it matches itself exactly.

Symmetries are just like magic tricks. If you close your eyes and I perform one of these symmetry operations on the frieze pattern, it will appear unchanged when you open your eyes. This beaded pattern holds many other symmetries apart from translation. [What other symmetries can you see in
this beaded pattern?] It would appear the same if I reflected the pattern horizontally or vertically, if I spun it through a half turn, or if I slide the pattern along and then reflected it horizontally (called a glide reflection).

The group of symmetries for this frieze pattern can be remembered by the simple name of "Spinning Jump". You can see the pattern of footprints left behind by spinning between jumps has the same symmetries as the beaded pattern around the column.

If you jump out this pattern and leave behind the cut-out footprints, it makes it easy to demonstrate that it has the same symmetries as the beaded frieze. Pick any of the symmetries, place four coloured cut-out footprints on the pattern, perform the symmetry transformation with these coloured footprints, and it will be clear that the footprint pattern will appear unchanged.

When symmetries fail

The spinning jump pattern has many symmetries but other frieze patterns, for example the "Sidle", are far less symmetrical.

What symmetries does this pattern have? Jump out the pattern, marking it with cut-out footprints. Use coloured cut-out prints to explore which symmetries hold, and which don’t, for this pattern.

Unlike the Spinning Jump, which had all possible symmetries, the only symmetries which hold for the Sidle are translation and vertical reflection. Any other transformation will change the pattern. [Can you spot this frieze pattern on the Ashmolean?] This pattern can been seen in the egg and dart frieze running near the top of the building.
What other frieze patterns are there?

Some of the other frieze patterns you can see on the Ashmolean are the plait on the ceiling of the portico:

![Plait on the ceiling](image1)

the plait running along the top of the walls:

![Frieze pattern on the wall](image2)

and the spiralling pattern above the windows:

![Spiralling pattern above the windows](image3)
All three of these patterns have the same symmetries, a group of symmetries known as the "Spinning Hop".

*What symmetries does this pattern have? Use the footprint pattern and coloured cut-outs to explore the symmetries.*

Apart from translation, the only other symmetry that holds for these patterns is rotation: if you spin the pattern by 180 degrees it will appear unchanged.

**The seven frieze groups**

Mathematically we analyse frieze patterns by the combination of symmetries that can coexist within the frieze and how these symmetries interact with each other. You might think there are endless possibilities of different combinations of symmetries, but actually there are only 7 possible ways that these symmetries can occur together in frieze patterns. In addition to the Spinning Jump, Sidle and Spinning Hop groups we've already seen, there are four other frieze groups:

- the "Step"

- the "Spinning sidle"
the "Hop"

and the "Jump".

Artists have been playing with all seven of these patterns for thousands of years. But it was only in the nineteenth century that mathematicians were able to prove that no other frieze groups existed and every possible frieze pattern could be described by one of these seven frieze groups.

**Group theory**

Mathematically symmetry is studied using *group theory*: each of the seven frieze groups above is an example of a mathematical group. One of the founders of group theory was the mathematician Évariste Galois. He made significant contributions to mathematics during his short life before he died in a duel in 1932. The night before the duel he passed the time writing letters and mathematical papers, which included him coining the word "group" for this type of mathematical object.

Group theory is used widely outside of mathematics too, in studying the symmetries of crystal structures and molecules in chemistry and in the Standard Model of particle physics currently being explored at the Large Hadron Collider.

We have only been able to spot three of the frieze groups decorating the Ashmolean. Can you spot anymore? And if you spot other examples of this beautiful piece of mathematics here, or in any city around the world, please let us know!

**Demonstration**

To make it easier for the group to see, hand around printouts of the image of the neolithic carved stone balls, the friezes of the Ashmolean and the footprint patterns for the seven frieze groups (you can download this [footprint bingo](#)).

We've included questions (in italics in the description above) that you can ask the group to get them involved.
To demonstrate the symmetries of these frieze patterns we jumped out the footprint patterns of the symmetry groups and marked the patterns with cut-outs. As indicated in the captions of the footprint patterns above, an easy way to explore the symmetries is to use extra coloured footprints that you can move to see if a transformation leaves the pattern unchanged.

4. The Beehive, Oxford

In St John’s College, Oxford, one of the buildings is hexagonal in shape. Was this hexagonal structure a whim of the architect? Why are most buildings square? What does all of this have to do with bees?
St John’s College, St Giles, Oxford OX1 3JP, UK.

Viewpoint:

North Quad, St John’s College [Note: need to arrange entry with St John’s prior to tour – take email confirmation along!]

Visiting times:
By prior arrangement
Students and busy bees

From children’s pictures to architect’s drawings, most of us imagine houses and buildings as being made up of squares and rectangles; not surprising given the majority of the buildings we see in the city around us are based on right angles. But we do have some fascinating examples of differently shaped buildings here in Oxford. Some are circular, such as the Radcliffe Camera or the Sackler Library, some have curved sides such as the front of the Sheldonian. But only one building in Oxford is hexagonal.

When this student accommodation was built in the North Quad of St John’s college in 1960 the architect made the unusual decision of using hexagonal rooms. This design is rare in the built environment but is found more frequently in nature. [Where do we see hexagons in nature?] The honey comb in beehives, which also serves as accommodation for the next generation, is formed from hundreds of hexagonal wax cells packed together. So it’s not a surprise that this building has come to be known as the Beehive.

Making the most of your wall

So why should bees, or architects, use hexagons? For bees, making the wax for the walls of the honeycomb is a very expensive business. (A single bee produces just 1/12th of a teaspoon of honey in their entire lifetime. Bees in a hive need to consume 6-8 pounds of honey to produce 1 pound of wax, which means they collectively need to fly more than 6 times around the world to produce that amount of wax!) So understandably they would want to use this expensive resource most efficiently, building the largest cells possible for a given amount of wax.

If you know how much wall you have to use, say a fixed number of bricks or a fixed amount of honey, how should you build your room so that it encloses the largest space possible?
You can explore how to increase the area enclosed by a fixed length of wall using a loop of string and asking volunteers to add a corner to the room, one corner at a time.

You can explore this using a loop of string for your fixed length of wall. Starting with (an admittedly very useless) room with just two corners, each time you add in another corner (going from triangular, to square, to pentagonal, etc), you increase the area the string encloses. If you carry on adding corners you shape becomes more and more like a circle, and it is a circle which encloses the most area for fixed perimeter. (If you’ve got a mathematical bent why not try deducing the area of these shapes, and prove that it increases as the number of sides increases. You can check you calculations for the areas with those listed on Wikipedia)

Making the most of your space

A circle is the most efficient shape for a room on its own – it encloses the largest possible space for a given length of wall. Although we have some examples of circular buildings, including those here in Oxford, it isn’t a good choice for the shape of rooms within a building. [Why aren’t circular rooms common?] Circles don’t fit neatly together and wasted gaps of space would be left between circular rooms.

In order to make the most of your space you need a shape that tessellates or tiles the floor space of your building, just like the paving stones that neatly cover the courtyard near the Beehive.

There are only three regular shapes that can tile the plane – triangles, squares and hexagons. So by choosing hexagons, the architects of the beehive, both this building and that of the bee, have chosen the most efficient shape for a room – the hexagonal walls enclose the largest possible areas while wasting no space between them.
Bees have known about the benefits of hexagons for millennia, and we have suspected that hexagons were the most efficient way to divide up a flat plane from at least 300AD when Pappus of Alexandria posed this as a question. However this Honeycomb Conjecture was only proved mathematically just over a decade ago, by Thomas Hales in 1999.

**Making the most of your energy**

Although the architects of the Beehive were aware of all this maths, we can be pretty sure that the bees haven’t learnt any geometry. But nature, like mathematicians and architects, is keen on efficiency and will form shapes and arrangements that require the least amount of energy. The bees actually start by making roughly circular cells as these are the most efficient use of their wax. But as these pack together the walls bend to create a hexagonal arrangement. You can see how this happens by tossing a handful of marbles in a curved wok – the spherical marbles naturally settle into a hexagonal pattern.

Mathematically hexagonal rooms have many benefits, but they have some practical drawbacks too. Any of the students who have lived in these rooms will tell you that it can be quite hard to fit normal, predominantly right-angled, furniture into these hexagonal rooms. Some students apparently even went so far as to take to their beds with a saw to make them fit! Which is why most of us won’t be living in a beehive soon.

**Demonstration**

We’ve included questions (in italics in the description above) that you can ask the group to get them involved.

You can use a large loop of string (one about 6m long works well) to explore how the area of a room (enclosed by a fixed length of wall) changes as you alter the shape. Ask for two volunteers and ask them to pull the loop taut, giving you a room you with just two corners – it’s not a very useful room as it encloses no room at all. The room becomes much more useful if you add a third corner, creating a triangular room which now encloses some space. Continue asking for new volunteers to add a corner to the room and note that the area enclosed continues to increase.

It is also useful to have a wok (or similarly curved dish) and a bag full of marbles. If you empty the marbles into the dish they will settle into a hexagonal pattern, demonstrating that making the honeycomb arrangement uses the least energy.

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**5. You are here – GPS and geometry**

Many of us now rely on a little electronic help in finding our way, making the most of the GPS in our phones and satnavs. GPS shows us the way thanks to some simple geometry and a little help from Einstein.

Oxford, UK
Viewpoint:

This site can actually be anywhere in any city, you just need to be able to draw some chalk circles on open ground. We often use the walkway between Parks Road and The Lamb & Flag. Pretend you’ve gotten lost and the activity will make more sense.

Military hardware in your pocket

Many of us now use a little electronic help to find our way around. We have satnav in our cars, GPS enabled phones in our pockets, even our cameras can geotag our photos, telling us exactly where we were when we took our holiday snaps.

The Global Positioning System (GPS) was developed by the US military and initially was very expensive and cumbersome, the only other people who used it were specialists such as Arctic explorers. But as it has become more affordable and more compact, many of us now use this remarkable technology every day. And surprisingly, the way GPS works is all down to some simple geometry along with a little help from Einstein.

Listening to satellites

Your phone (or satnav or camera) acts as a GPS receiver listening out for a signal from one of the nearest GPS satellites. There are 31 GPS satellites, their orbits criss-crossing the Earth in such a way that there is always at least 4 satellites in the sky above any spot on Earth at any time.

These satellites constantly bellow out their locations to the universe and your phone receives this message, along with the time at which it was sent. This message is a radio signal and therefore travels at the speed of light. [How can we calculate where we are in relation to the satellite?] With this information, your phone performs a simple calculation to determine the distance it is from the satellite.

\[
\text{Distance from satellite} = (\text{time it took message to reach us}) \times \text{speed of light}
\]

Suppose it took half a millisecond for the message from the satellite to reach your phone. The speed of light is about \(3 \times 10^8 \text{ ms}^{-1}\), so a radio signal travels about 300 km in a microsecond. From this, our phone would calculate it is 150km away from this GPS satellite.

You are here

All your phone now knows is that you are a certain distance, in our example 150km, away from the known position of the satellite. [Where can we be in relation to the satellite?]
In our 3D space if you are 150km away from a satellite, you are somewhere on a sphere with that satellite as its centre and with a radius of 150km. The principle of finding your location with this method is the same in two dimensions as in three, and it’s a little easier to picture, and to draw! In two dimensions you will be somewhere on a circle with a 150km radius that is centred on the satellite’s location.

At any time there will also be at least 3 other satellites overhead. Your phone uses the messages from these other satellites to calculate that you are also on circles of particular radii centred on each of those satellites.

Suppose your phone has calculated that you are 100km away from a second satellite, so you are somewhere on the circle centred on that satellite with a 100km radius. These two circles must intersect at least once (at your location) – they could either just touch or more likely they overlap and intersect in two places.

To nail down exactly on which of these two places you are standing, your phone needs a third satellite. These three circles must intersect at at least one place (your location) and in fact they intersect in only one place. Your phone has calculated that you are here!

**Working in 3D**

In two dimensions, you only need 3 circles to pinpoint where you are; two circles intersect in at most two points, requiring a third circle to identify which of these is your location. The phone does these calculations in three dimensions, solving where spheres centred on the satellites intersect. [How many satellites do you think are needed in three dimensions? How can two, three and four spheres intersect?] In three dimensions at least 4 satellites are needed; two spheres intersect in a circle, this circle intersects the third sphere in two points, and the fourth sphere pinpoints which of these is your location.

Although it is easier for us to picture this as intersecting circles and spheres, your phone does not know where you are by actually drawing circles. Instead it uses the equations for spheres centred on the satellites and solves these equations to find where the spheres intersect.

**Right on time?**

In order for GPS to work, your phone needs to know exactly how long it takes the radio signals to travel from the satellites. It calculates this from the difference between the time the satellite sent the message and the time your phone received it; so it is vital that the clocks on the satellites and the clock in your phone agree. The smallest error in this time would be hugely magnified when it is multiplied by the speed of light in the distance calculation.

The GPS satellites orbit the Earth twice each day, travelling at more than 14,000 km/h relative to your phone here on Earth. Einstein’s theory of special relativity says that clocks tick more slowly when moving at such high speeds. A clock travelling at the speed of a GPS satellite would lose about 7 microseconds a day compared to if it were on the ground.

This is partly counteracted by the fact that the satellite’s high orbit means the gravity due to the Earth’s mass is far weaker than on the Earth’s surface. Einstein’s theory of general
relativity says that clocks tick more quickly the further they are from a massive object; a clock at the distance of a GPS satellite would gain 45 microseconds a day compared to if it were on the ground.

But taking these two effects into consideration the clock on a GPS satellite would still tick faster than clocks on the ground, gaining 38 microseconds a day compared to the time it would show if it were on the ground. This would make the satellite useless for navigation – your distance calculations would be out by more than 10km within one day of the satellite's launch!

To account for these relativistic effects engineers set the GPS clocks to tick slower on the ground than our normal clocks. This means that when they are in orbit the GPS clocks will synchronise with the ticking of the clocks on the ground, including the one in your phone.

It is very surprising to find an application of Einstein’s theory of relativity, something that seems so esoteric, in a device that many of us use everyday. So the next time you find your way using your phone, you’ll know that the geometry of circles, and a little relativity, are showing you the way.

**Demonstration**

**Props:**

- Three loops of strings (it’s nice if they are different lengths) and some chalk.

It’s a good idea to try the demonstration in situ so you can get a feel for where to lay out your string. It takes up more room than you expect. And obviously make sure it is ok to draw on the ground with chalk before you run the tour!

When you arrive ask the group to stand in a rough circle. Start by asking them if anyone used a satnav to get to Oxford today, or a phone to navigate their way to the starting point of the tour.

Before you explain the second section “Listening to satellites” ask if anyone has a GPS-enabled phone or camera with them, and if you can borrow it. You can then run the demo referring to this phone as it uses GPS to calculate it’s location.

**Demo:**

Questions are suggested in italics in the text above to help engage the group.

If people have trouble understanding how to convert time into distance in the “Listening to satellites” section, you could write the equation out with chalk on the ground.

In the section “You are here” use the loops of string to represent the distance to the satellite. For the first satellite, stretch the loop of string out on the ground explaining that it represents our distance from the satellite, marking the position of one end of the loop as the location of the satellite. Ask for a volunteer to act as the satellite, standing on the mark with their foot in the loop. *Can anyone suggest where we can be in relation to the satellite if we know they are a fixed distance away?* Then ask for another volunteer to act as the phone and...
mark out where we could be in relation to the satellite, pulling the loop taut and using chalk in the other end to mark out a circle using the satellite/foot as a pivot.

For the second satellite, stretch the loop out so that one end lies within the first circle, and mark the position of the other end (allowing a little give) as the second satellite. Again ask for a volunteer to act as the satellite, standing on the mark with one foot in the loop, and ask the phone volunteer to again mark out the circle.

For the third satellite, stretch the loop out so that one end sits at one of points where the first two circles intersect. Mark the position of the other end (again allowing a little give) as the third satellite. Then ask for a volunteer to act as the satellite and your phone volunteer to inscribe the final circle. The three circles should intersect at one point (you might have to help a little with the positioning) and you can ask the phone volunteer to stand on the final spot to indicate they have found their location!

6. Penrose tiles at Wadham College, Oxford

[No longer have access here – may need to reorder tour and use Maths Institute?]

No matter where you stand, the pattern in the pavement outside the student bar at Wadham College never repeats. This is because it is a Penrose tiling, named after the mathematician Roger Penrose who invented it in the 1970s. Penrose tilings not only have many interesting mathematical properties, they also explain the structure of some unusual metallic crystals, called quasicrystals, that were discovered in the 1980s and won Dan Shechtman the Nobel Prize for Chemistry in 2011.
7. The roof of the Sheldonian Theatre

The fascinating and inspired mathematics behind the construction of the Sheldonian Theatre allowed it to have the largest unsupported roof the world of the 17th century had ever seen. Broad St, Oxford, Oxfordshire OX1 3, UK

Viewpoint:

On the paving in front of the steps in front of the Clarendon Building, on the corner of Broad and Parks streets.

Visiting times:
Any time. If you want to go inside the Sheldonian Theatre, visit their website for opening times and entry fees.

Rain dancing

When Sir Christopher Wren was asked to design the Sheldonian Theatre in the 1660s, he began with visions of the great amphitheatres of Ancient Rome. Oxford having rather more rain than Rome, his design of a modern amphitheatre was in need of a roof.
How would you use timber beams to build a simple roof?

The most obvious solution span the walls with the timber beams. But the dimensions of the Sheldonian are 70 foot by 80 foot and this simple roof design would need beams far longer than the timber beams available at the time. To overcome this Wren planned to use internal columns to support the roof but this was vetoed by the university officials – they didn’t want columns impeding any dancing at the venue! Today the Sheldonian is used for graduation ceremonies but back in the 17th century they had much wilder parties in mind!

Solving the puzzle of an unsupported roof

Wren couldn’t use internal columns to support the roof and he couldn’t span the space with individual beams. Therefore to complete the Sheldonian, Wren would need to build the largest unsupported roof the 17th Century had ever seen.

Luckily Wren had studied mathematics here at Oxford, taught by John Wallis. Wallis was the Savilian Professor of Geometry, a chair that still exists at Oxford today, currently held by Nigel Hitchens. It was Wallis’ ingenious design that provided Wren with the answer he needed.

Wren’s problem was that the walls could only support the roof at one end of the timbers. At first sight it might seem impossible to build a stable roof with beams that are only supported in one place – but if you have ever balanced on a seesaw then you’re already part of the way to the solution. Just like a seesaw balancing on its single pivot, Wallis’ ingenious idea hinged on statics: balancing all the forces involved in so they cancel out making the seesaw, or roof truss, stable.
If the timber beams are only supported at one end by the walls, how can you arrange them so that together they provide stable, strong roof? The answer is to interlock the beams. You can see a great video demonstrating the construction on Amy Mason's Sheldonian site.

Wallis’ solution

Wallis’ devised an ingenious pattern of interlocking beams, so that every beam was supported at both ends – either by the walls or by other beams – while every beam also supported the ends of two other beams. So for every beam, the downward forces from those resting on it are balanced by the upward forces from the beams, or wall, supporting it. In an impressive feat of calculation, Wallis demonstrated that his geometrical flat floor could carry loads when supported by the walls alone by solving a set of 25x25 simultaneous equations using just pen and paper!

So Wren’s roof, inspired by Wallis’ design, not only keeps dancers (and graduates and their families) dry, it can also support significant loads. The Oxford University Press stored books on the first floor for many years, proving that you can build a strong stable roof supported by mathematics instead of columns.
Demonstration

Props needed:
- 3 flat sticks, about a metre long (the beams)
- 3 maths books
- Laminated picture of Wallis’ design

If you have a large tour group (over 15) you’ll need to divide them into two groups, with a set of 3 beams and 3 books for each group. Identify 3 volunteers in each group and place them in the centre of the group in a triangle roughly a beam length apart. These three volunteers will act as the walls supporting the roof, where the whole group will work together to design the roof.

Your groups can explore the principles at this site by trying to answer the questions in italics under the pictures in the explanation above:

*How would you use timber beams to build a simple roof?*

With the three volunteers standing about a beam’s length apart, ask them to build a simple roof. Mostly likely they will build one using single beams to span the walls. You can demonstrate how this design fails if the room is bigger than the length of the beams, by asking them to take a small step backwards so they are now more than a beam length apart.

*If the timber beams are only supported at one end by the walls, how can you arrange them so that together they provide stable, strong roof?*

The most successful approach with tour groups has been to just stand back and let the group get on with experimenting and solving this puzzle. If they are struggling after several minutes you could suggest they need to have the beams supporting each other, interweaving them in some way. The answer will be something like the following construction:
- End of first beam rests on one wall (you temporarily support other end)
- End of second beam rests on next wall and other end on middle of first beam
- End of third beam rests on last wall, other end on middle of second beam, and the centre of third beam supporting end of first beam.

To demonstrate that the structure is not only stable but can support significant weights, place several books in the centre of the arrangement (if the beams interlock close to the centre) or book at each of the three points where the beams cross. Encourage other people in the group to try holding the ends of the beams to feel the solidity of the arrangement.