A Comparison of Classical and Modern Methods for Ordinary Differential Equations

Stuart Hastings

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Based on :

Classical Methods in Ordinary Differential Equations,
With Applications to Boundary Value problems

(Joint with J. B. McLeod, to whose memory these lectures are dedicated)

American Mathematical Society, 2012

Draft, May 15, 2015

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1. Three ways to prove the existence of a solution for a fundamental boundary value problem for boundary layers in fluid mechanics

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5. Some unsolved problems and some recent references

1. Falkner-Skan Equation – functional analysis, dynamical systems, or standard ode methods?

Prandtl boundary layer equations for incompressible laminar flow, where u and v are the horizontal and vertical velocity components, v is kinematic viscosity, ρ is density and p is pressure.

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

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To a first approximation pressure is constant in the boundary layer. Make this assumption and look for the symmetrical boundary layer flow over a wedge of included angle $\beta\pi$. An ode can be obtained by assuming the solution is a function of a single variable combining x and y in a particular way. The result is called a "similarity solution" of the pde.

In the case of the Prandtl equations, if

$$\eta = \sqrt{\frac{u_\infty}{\nu\left(2-\beta\right)}}x^{\frac{\beta-1}{2-\beta}}y$$

we obtain the ode:

$$f''' + ff'' + \beta\left(1 - f'^2\right) = 0,$$

with boundary conditions

$$f\left(0\right) = f'\left(0\right) = 0,\ f'\left(\infty\right) = 1.$$

Here $f' = \frac{df}{d\eta}$ is the (scaled) velocity of the incompressible fluid flowing past a surface and f is the velocity potential, or stream function.

Studied by (among others)

Weyl (1942), Iglisch (1953-55), Coppel (1960), Hartman (1964, book),

Hastings: 1970–72, Serrin (1970), Craven and Peletier, 1972,

H. and Troy (1987-88 – showed the existence of periodic solutions for $\beta < -1$ and $\beta > 1$ and a complicated bifurcation from infinity as β increases beyond 2.)

Swinnerton-Dyer and Sparrow (1995, 2002 – showed that there are many such bifurcations, and described the branches globally; 114 pages of classical techniques).

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".. the present proof relies on detailed calculations of a very specific nature, tied to the exact form of (the Falkner-Skan equation)."

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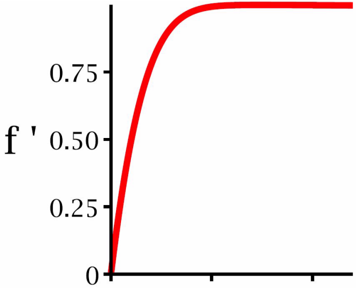

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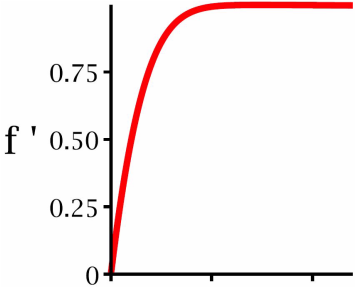

Existence for $\beta \geq 0$ proved by H. Weyl (1942)

Weyl's Existence proof. (Annals of Mathematics, vol 43, 1942, 381-407)

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Proposition: *It is sufficient to find a solution h such that* $h(0) = h'(0) = 0, h''(0) = 1$, *and* $h'(\infty)$ *exists.*

Proof: Given such a solution h, let $f(\eta) = \kappa h(\kappa\eta)$ where $\kappa^2 h'(\infty) = 1$.

$$h'''' + hh''' + (1-2\beta)\, h'h'' = 0$$
$$h(0) = h'(0) = 0,\ \ h''(0) = 1,\ h''(\infty) = 0.$$

Consider this as an ode for $\phi = h''$.

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The goal is to apply a fixed point theorem to the ϕ problem. To set this up, suppose that $g \in C([0,\infty))$ is known, with $|g(\eta)| \le 1$ for all $\eta \ge 0$. Let

$$H(\eta) = \int_0^{\eta} (\eta - s)\, g(s)\, ds.$$

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Lemma: *If* $g \in C([0,\infty))$ *and* $0 \le g(\eta) \le 1$ *for all* $\eta \ge 0$*, and* $H(\eta) = \int_0^\eta (\eta - s)\, g(s)\, ds$*, then this problem has a solution* ϕ which *satisfies the additional conditions* $\phi' + H\phi \le 0$ *and* $\phi \ge 0$ *on* $[0,\infty)$*. Further, there is only one solution satisfying these conditions.*

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Next step: Let $\phi = Qg$ and show that Q has a fixed point. If $Qg = g$ then $\phi = H''$, and reversing the previous scaling gives a solution f to the original problem. (Uniqueness can also be proved. See ode text by P. Hartman.)

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Theorem: *Let X be a locally convex linear topological space, and E a closed convex subset of X. Suppose that F is a compact subset of E and T is a continuous mapping of E into F. Then there is a point $\phi \in F$ such that $T\phi = \phi$.*

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With some work, Weyl verifies the hypotheses of this theorem for the mapping Q to complete the proof. The set E in this case is $\{g \in X \mid 0 \le g(\eta) \le 1 \text{ on } [0, \infty)\}$

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Example: A linear system:

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has a set of solutions determined by the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. For example, if $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ then the phase curves are hyperbolas, while if $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ then the phase curves are circles.

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Then

$$\frac{d}{dt}\left(\frac{1}{2}y^2 - \frac{x^2}{2} + \frac{x^4}{4}\right) = yy' + \left(x^3 - x\right)x' = y\left(x - x^3\right) + \left(x^3 - x\right)y = 0$$

so the phase curves are the graphs of the equations

$$\frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^3 = c$$

for different values of c.

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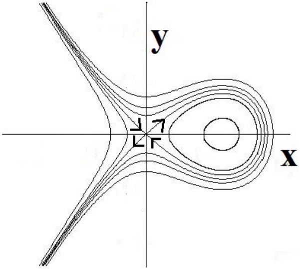

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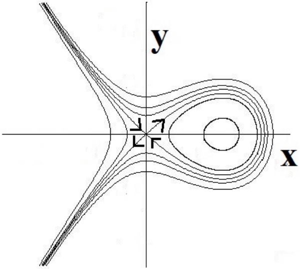

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Near $(0, 0)$ the phase curves are close to hyperbolas, while near $(1, 0)$ they are close to ellipses.

Write the general system as

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}). \tag{1}$$

The smoothness assumption on $\mathbf{f}$ implies that for each $\mathbf{x}_0 \in R^n$ there is a unique solution $\mathbf{x}(\cdot, \mathbf{x}_0)$ of (1) such that $\mathbf{x}(0) = \mathbf{x}_0$. This solution exists on some maximal interval $(\alpha, \omega) \subset R$.

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Any invariant set is the union of trajectories of the system.

An example in R^3 :

$$x' = y$$
$$y' = -x$$
$$z' = 1$$

The phase curves are helices. Any cylinder $x^2 + y^2 = d^2$, $z \in R$ is invariant for this system.

Outline of a "dynamical systems" proof of existence for the Falkner-Skan problem

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It is sufficient to prove existence for some $\varepsilon > 0$. Write as a system

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This is a "singular perturbation" problem, because when $\varepsilon = 0$ there is no solution.

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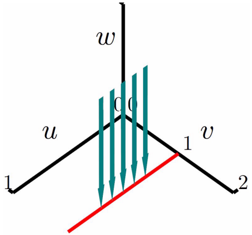

Thus there is an invariant plane.

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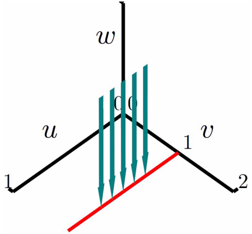
There is a family of solutions $u = c > 0$, $v = 1$, $w = de^{-ct}$.



Thus there is an invariant plane.
How does the picture change when $\varepsilon > 0$?

$$\dot{u} = \varepsilon v$$
$$\dot{v} = \varepsilon w$$
$$\dot{w} = -uw - \beta\left(1 - v^2\right)$$

The line $v = 1,\ w = 0$ is still invariant.

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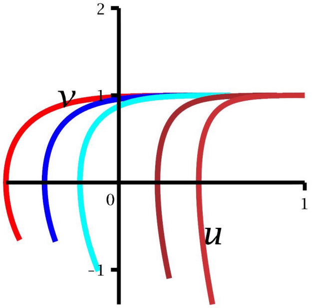
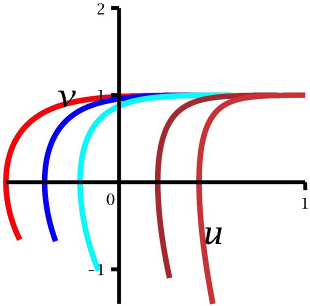
The line $v = 1, \; w = 0$ is still invariant. The general theory of "center manifolds" implies that the invariant plane found for $\varepsilon = 0$ becomes an invariant surface. Those solutions on this surface which enter $u > 0$ still tend to the invariant line as $t \to \infty$.



Projection of trajectories on the invariant surface onto $w = 0$



Projection of trajectories on the invariant surface onto $w = 0$

We must show that this manifold intersects the w - axis. ("This is the hard part.")

Swinnerton-Dyer and Sparrow: (1995)

"We learn (by personal communication) that a more geometric approach may be possible via Poincaré compactification, but we have yet to see all the details."

Proof using a "shooting method"

Initial value problem:

$$
\begin{aligned}
f''' + ff'' + \beta\left(1 - f'^2\right) &= 0 \\
f(0) = f'(0) &= 0 \\
f''(0) &= c
\end{aligned}
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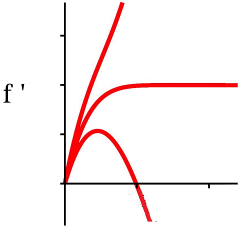
Graphs of f' for three values of c :



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\begin{aligned}
f''' + ff'' + \beta \left(1 - f'^2\right) &= 0 \\
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\end{aligned}
\tag{2}
$$

Lemma 1. *If f solves this initial value problem, then there is no η with $f''(\eta) = 1 - f'(\eta) = 0$.* (That is, the graph of f' is not tangent to the line $f' = 1$.)

$$f''' + ff'' + \beta\left(1 - f'^2\right) = 0 \qquad (2)$$

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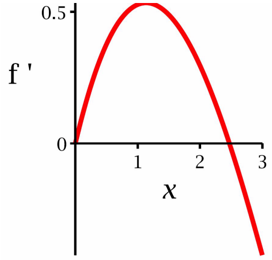
Proof: For any η_0 and f_0, the initial value problem (2) with $f(\eta_0) = f_0$, $f'(\eta_0) = 1$, $f''(\eta_0) = 0$ has the unique solution $f = f_0 + \eta - \eta_0$, with $f' = 1$. Hence $f'(0) \neq 0$, a contradiction

,

Lemma 2. *Let* $A = \{c > 0 \mid f'' < 0 \text{ before } f' = 1\}$. *Then* A *is an open subset of* R *and contains an interval* $(0, c_0]$ with $c_0 > 0$.



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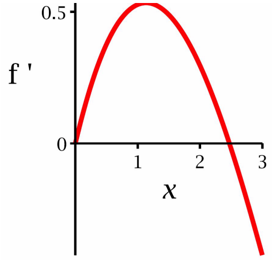
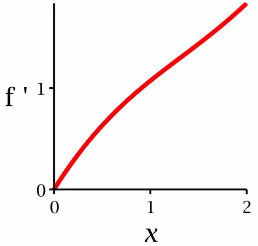


Proof: If $c = 0$ then $f''(0) = 0$, $f'''(0) = -1$ and so f'' turns negative immediately. Thus, for any $\eta_0 > 0$, if $c = 0$, then $f''(\eta_0) < 0$ and $f' < 1$ on $[0, \eta_0]$. Solutions depend continuously on c, so fixing η_0, if c is sufficiently small then $f''(\eta_0) < 0$ and $f' < 0$ on $[0, \eta_0]$. This continuity also shows that A is open.

Lemma 3. *Let* $B = \{c > 0 \mid f' > 1 \text{ before } f'' = 0\}$. *Then* B *is also open, and contains all sufficiently large* c.



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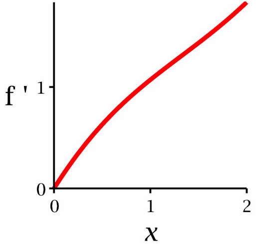

Proof: Multiply by $e^{\int_0^\eta f}$ to get

$$\left(f'' e^{\int_0^\eta f}\right)' = \beta e^{\int_0^\eta f}\left(f'^2 - 1\right) \geq -\beta e^{\int_0^\eta f}.$$

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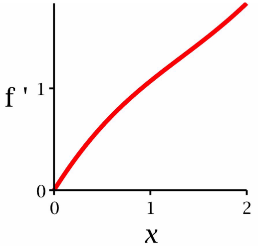

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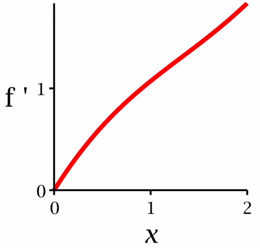

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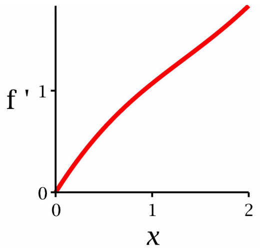

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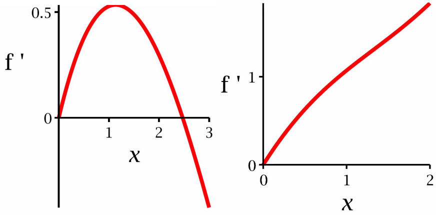
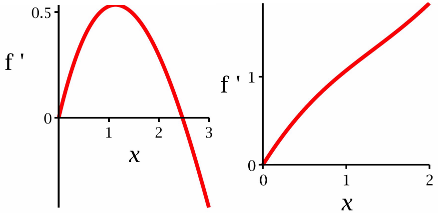


A B

Lemma 3 implies that A is bounded. Let $c^* = \sup A$. Then $c^* \notin A \cup B$, since both of these sets are open and they are disjoint.



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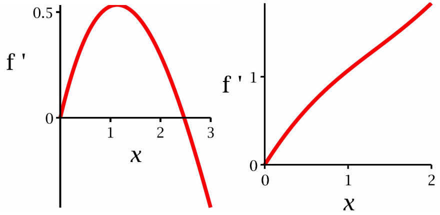

A *B*

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This limit can only be 1, for if $1 - f'^2 \to \delta > 0$, then for large η, $\left(f'' e^{\int_0^\eta f ds}\right)' \leq -\frac{\delta}{2} \beta e^{\int_0^\eta f ds}$, implying that f'' becomes negative, a contradiction. This completes the proof.