3. Layers and spikes in a reaction-diffusion equation

Angenent, Mallet-Paret, and Peletier (1987) considered

$$u_t = \varepsilon^2 u_{xx} + u(1-u)(u-\phi(x)), \quad 0 < x < 1, \ t > 0$$
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Also studied by Ai, Chen and Hastings in 2006 and by Matano and others. With Dirichlet conditions it was considered in R^n by Dancer and Yan (2003).

This problem arises, for example, in population genetics when a population is spread over $(0, 1)$, with no flux across the boundary, and where each individual is AA, Aa, or aa. The fraction of alleles of type A across the population at point x and time t is denoted by $u(x, t)$.

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$$\text{sign } u_t = \text{sign } (u - \phi(x)),$$

so as t increases,

$$\begin{aligned} u(x,t) &\to 0 \text{ if } u(x,0) < \phi(x) \\ u(x,t) &\to 1 \text{ if } u(x,0) > \phi(x). \end{aligned}$$

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But dispersion (similar to diffusion in its effect) may change this.

Steady state solutions ($u_t = 0$) also satisfy an ode,

$$\varepsilon^2 u'' + u(1-u)(u-\phi(x)) = 0. \tag{1}$$

$$u'(0) = u'(1) = 0. \tag{2}$$

$$0 < \phi(x) < 1. \tag{3}$$

It is useful to consider the case where ϕ is constant. Thus,

$$\varepsilon^2 u'' + u\left(1-u\right)\left(u-\kappa\right) = 0, \tag{4}$$

where $\kappa \in \left(0,1\right)$.

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Angenent, Mallet-Paret, and Peletier (AMP) found all of the solutions to (1)-(2) which are stable steady states of the corresponding reaction-diffusion pde . These solutions can have single layers (defined below) near the points in $[0, 1]$ where $\phi = \frac{1}{2}$.

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However the set of all solutions (stable or not) is considerably more complicated, and may (depending on ϕ) include solutions with multiple layers clustered near some of the points where $\phi = \frac{1}{2}$, and also single or multiple spikes near critical points of ϕ.

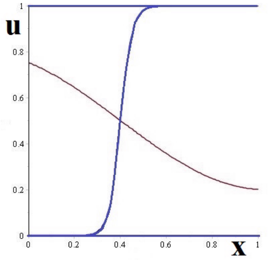
As an illustrative special case we will assume that

$$\phi' < 0$$

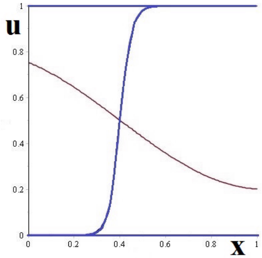
on $[0, 1]$ and for some $x_0 \in (0, 1)$,

$$\phi(x_0) = \frac{1}{2}.$$

Theorem: A,M,P: *For sufficiently small* $\varepsilon > 0$ *there are exactly three stable solutions:* $u = 0$, $u = 1$, *and one which is increasing with most of the increase occurring near* x_0. (This forms a "layer".)



red: graph of ϕ; blue: stable solutions



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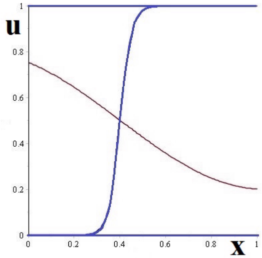
Why is there a third solution?



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Why is there a third solution?

We can motivate a search for another solution for small ε by using the calculus of variations, since (1) is the Euler-Lagrange equation for a minimization problem.

$$f(x,u) = u(1-u)(u-\phi(x))$$
$$F(x,u) = \int_0^u f(x,s)\,ds$$

Minimize

$$I_\varepsilon(u) = \int_0^1 \left(\frac{1}{2}\varepsilon^2 u'^2 - F(x,u)\right) dx$$

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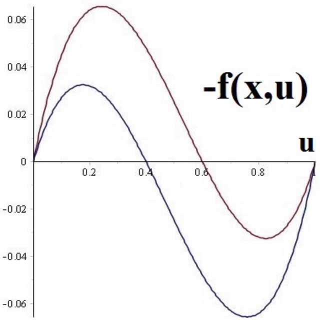

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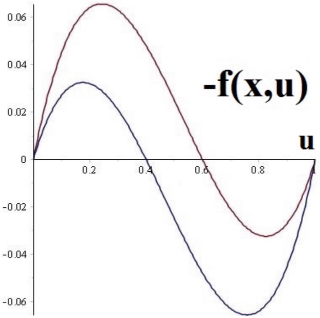

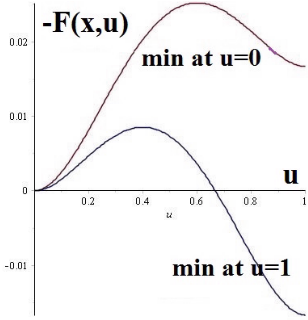

$$I_\varepsilon(u) = \int_0^1 \left(\frac{1}{2}\varepsilon^2 u'^2 - F(x,u)\right) dx$$

But $\phi > \frac{1}{2}$ on $[0, x_0)$ and $\phi < \frac{1}{2}$ on $(x_0, 1]$. Hence the second term in I_u is minimized by taking u close to 0 in $(0, x_0)$ and close to 1 in $(x_0, 1)$.

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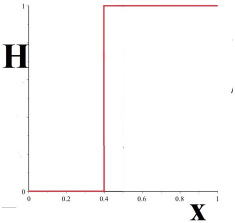
Suppose that v_n is a sequence of smooth functions tending to the Heaviside function $H(x - x_0)$.



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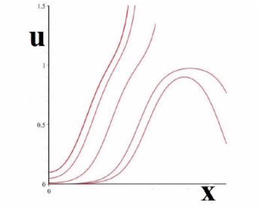
Dancer and Yan used calculus of variations arguments in R^n to construct most, but not all, of the solutions we will describe.

Second Motivation: Numerical “Shooting”

$$u(0) = \alpha,\, u'(0) = 0$$



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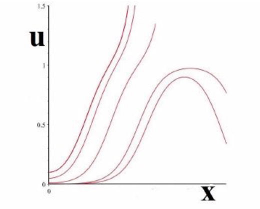

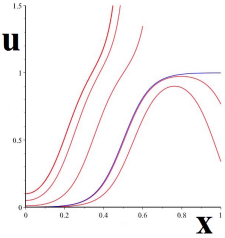

PROOF OF EXISTENCE USING SUB- AND SUPER- SOLUTIONS (A,M,P)

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*Definition (**non-standard**): A C^2 strong subsolution of* (1)-(2) *is a function $u_1 \in C^2([0,1])$ such that*

$$\text{(i)}\quad u_1'(0) = u_1'(1) = 0$$

$$\text{(ii)}\ \varepsilon^2 u_1''(x) + f(x, u_1(x)) > 0 \textit{ at each } x \in [0,1].$$

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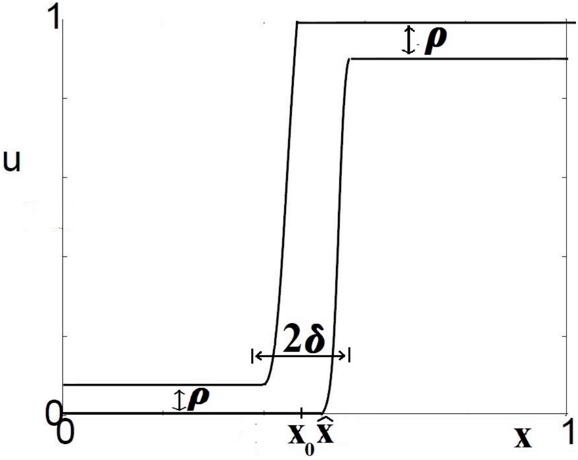
We start, however, by defining two solutions which are not sub- or super-solutions, but are the starting points in defining sequences of each. Recall that $\phi(x_0) = \frac{1}{2}$. Choose small positive numbers δ and ρ.

Then choose certain functions $\underline{u} < \bar{u}$ as shown below. They are constant except in certain intervals contained in $[x_0 - \delta, x_0 + \delta]$, where they solve (1) exactly. This is only possible for sufficiently small ε.



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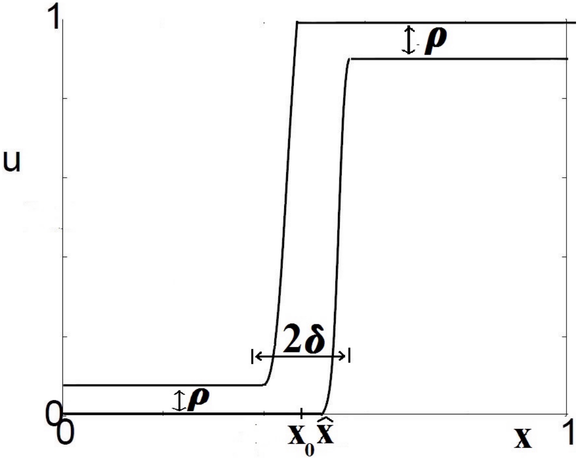

The functions $\bar{u}$ and $\underline{u}$ are continuous but not smooth.

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Suppose that $\underline{u} \leq v \leq \bar{u}$ on $[0,1]$ and consider the linear boundary value problem

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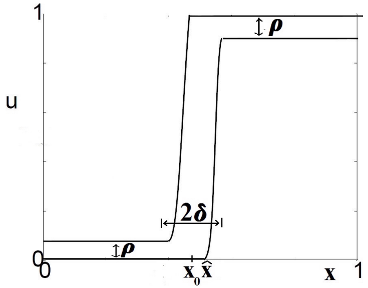
has only the solution $u = 0$. Hence the inhomogeneous problem (L) has a unique solution u. Let $Tv = u$.



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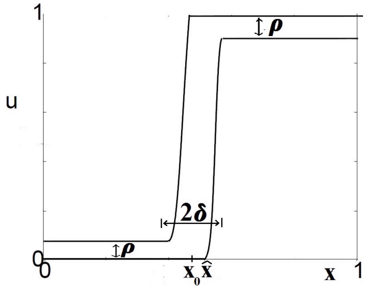

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We will show that u_1 is a strong C^2 subsolution of (1) - (2).

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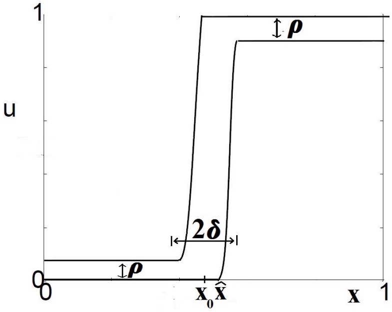

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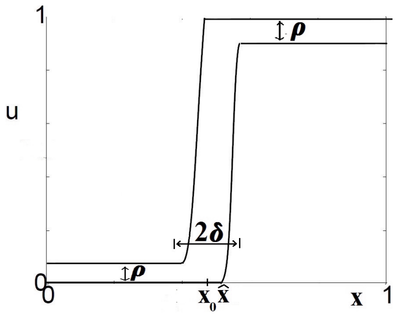
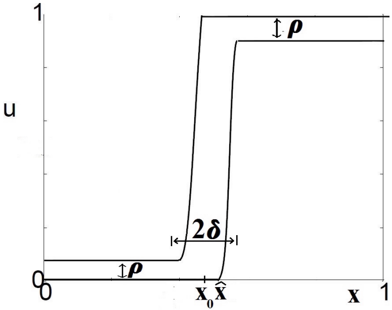


Suppose, for example, that $u_1(\tilde{x}) < 0$ for some $\tilde{x} \in (0, \hat{x})$. Then $\underline{u}(\tilde{x}) = 0$ and so from the ode defining u_1,

$$\varepsilon^2 u_1''(\tilde{x}) = \lambda u_1(\tilde{x}) < 0. \tag{8}$$



Suppose, for example, that $u_1(\tilde{x}) < 0$ for some $\tilde{x} \in (0, \hat{x})$. Then $\underline{u}(\tilde{x}) = 0$ and so from the ode defining u_1,

$$\varepsilon^2 u_1''(\tilde{x}) = \lambda u_1(\tilde{x}) < 0. \tag{8}$$

It follows that $u_1 < 0$ and $u_1'' < 0$ to the left of $\tilde{x}$, implying that $u_1'(0) > 0$. This contradicts a boundary condition satisfied by u_1.

With similar maximal principle type arguments we can show that

$$0 < \underline{u} < u_1 < u_2 < ... < \bar{u}.$$

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It is then not hard to show that

$$u = \lim_{i \to \infty} u_i$$

exists and is the desired increasing solution to (1)-(2).

PROOF OF EXISTENCE USING SHOOTING

Ai, Chen, Hastings, (2006) (See also Ai and Hastings, 2002).

$$\varepsilon^2 u'' = u\,(1-u)\,(\phi\,(x) - u)$$

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It follows that if α is sufficiently small and positive then $u'_\alpha(1) > 0$. Similarly, if $1 - \alpha$ is sufficiently small and positive then $u'_\alpha(1) < 0$.

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It can be further shown (~ 1 paragraph proof on request!) that if we choose the smallest possible α^*, then $u'_{\alpha^*} > 0$ on $(0, 1)$.

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$$\lim_{\varepsilon \to 0} (y(x^* + \varepsilon s), z(x^* + \varepsilon s)) = (Y(s), Z(s))$$

uniformly on $0 \leq s \leq s_1$.

$$Y' = Z$$
$$Z' = Y\left(1 - Y\right)\left(\phi\left(x^{*}\right) - Y\right)$$

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$$\frac{1}{2}Y'^2 + F\left(x^*, Y\right) = C$$

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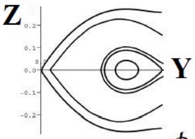

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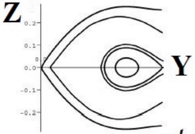

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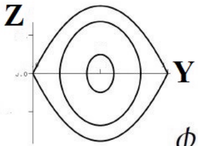

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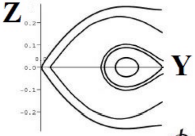

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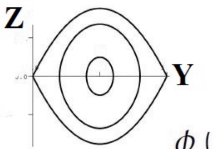

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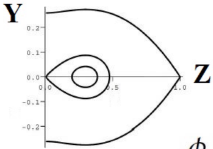

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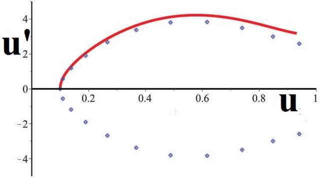
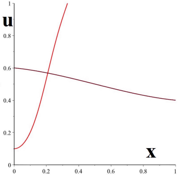
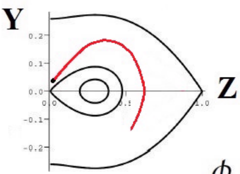




Dotted curve is in (Y, Z) phase plane when $x^* < x_0$.



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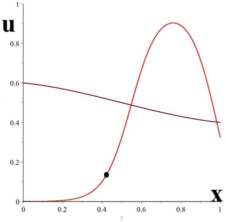

Shooting argument: If $u > 1$ then $u'' > 0$.

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Let

$$\Lambda = \left\{\alpha > 0 \mid u_\alpha' > 0 \text{ on } [0,1] \text{ or else } u_\alpha' > 0 \text{ on } (0,\hat{x}] \text{ and } u_\alpha(\hat{x}) = 1.\right\}$$

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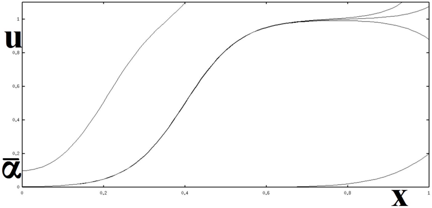
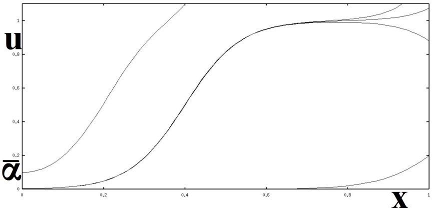


$\bar{\alpha} \in \Lambda$ if ε is sufficiently small

But for fixed small ε, some $\alpha \in (0, \bar{\alpha})$ are not in Λ.



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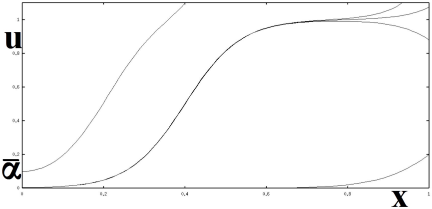

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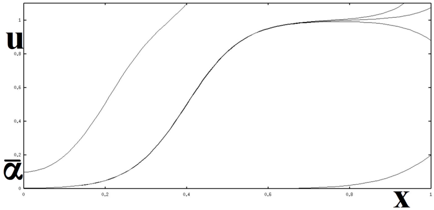

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Then $u_*'(\alpha^*) = 0$.

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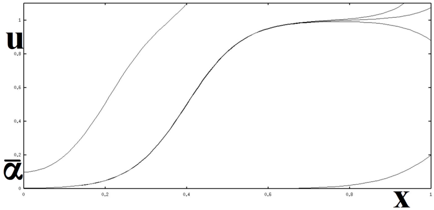

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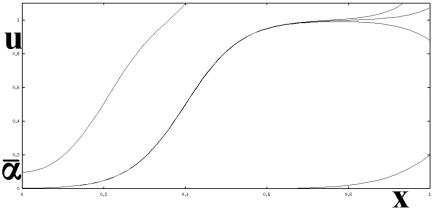
Then $u'_* (\alpha^*) = 0$.

Proof by contradiction.

But for fixed small ε, some $\alpha \in (0, \bar{\alpha})$ are not in Λ.



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Proof by contradiction. Main difficulty: Eliminate an inflection point of u_{α^*} at $\hat{x} \in (0, 1)$.

If $u'(\hat{x}) = u''(\hat{x}) = 0$, then

$$\begin{aligned} u(\hat{x}) &= \phi(\hat{x}) \\ \varepsilon^2 u'''(\hat{x}) &= u(\hat{x})(1 - u(\hat{x}))\, \phi'(\hat{x}) < 0, \end{aligned}$$

which means that u_{α^*} is decreasing in a neighborhood of $\hat{x}$.

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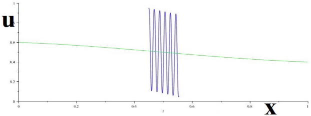
Hence α^* cannot lie on the boundary of Λ.

Multilayered solutions (also obtained by Dancer and Yan)



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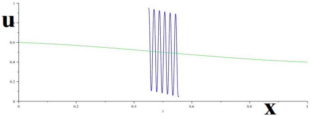
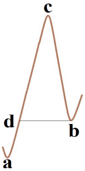


Lemma: *If $\phi' < 0$ on $[0,1]$ then successive minima and successive maxima decrease. If $\phi' > 0$ then they increase.* (J.B.McLeod)

Proof. Suppose that $u(a)$ and $u(b)$ are successive minima and $u(b) \geq u(a)$.



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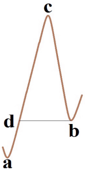
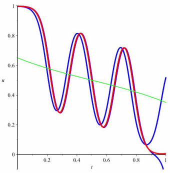


$$
\begin{aligned}
0 &= \int_d^b (\varepsilon^2 u' u'' + u' f(x,u))dx = -\frac{\varepsilon^2}{2} u'(d)^2 + \int_d^b u' f(x,u)\,dx \\
&\leq \int_{u(d)}^{u(c)} f(x_-(u), u)\,du + \int_{u(c)}^{u(d)} f(x_+(u), u)\,du \\
&= \int_{u(d)}^{u(c)} u(1-u)\left(\phi(x_+(u)) - \phi(x_-(u))\right) du \\
&< 0.
\end{aligned}
$$

COROLLARY: *Given N, if ε is sufficiently small then there are multilayer solutions with $1, 2, ..., N$ layers.* Reason: As α changes, layers disappear one by one. At each α where a layer disappears, $u'_{\alpha}(1) = 0$.



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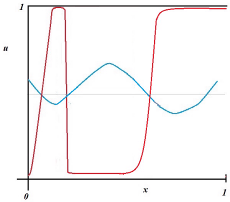

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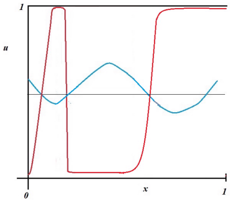

Theorem (AMP): *If $\phi - \frac{1}{2}$ has n zeros in $(0, 1)$, then there are F_{n+3} stable solutions, where F_m is the m^{th} Fibonacci number* $(1,1,2,3,5,8,...)$

Assume that $\phi' \neq 0$ whenever $\phi = \frac{1}{2}$. As $\varepsilon \longrightarrow 0$, with the number of layers fixed, all interior layers tend to zeros of $\phi(x) - \frac{1}{2}$ as $\varepsilon \longrightarrow 0$.

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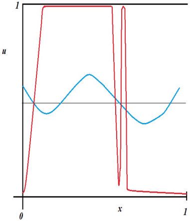
If $\phi(\hat{x}) = \frac{1}{2}$ and $\phi'(\hat{x}) < 0$, then multiple layers congregating at $\hat{x}$ must start high and end low.



The first layer is a "stable layer", but this solution is not stable.

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$$Q(u, u', x) = \left[\varepsilon^2 u'^2 + 2F_1(x, u)\right] u(1 - u)$$

and

$$F_1(x, u) = \begin{cases} \int_1^u f(x, s)\, ds \text{ if } 0 \le x \le x_0 \\ \int_0^u f(x, s)\, ds \text{ if } x_0 < x \le 1 \end{cases}.$$

Recall:

$$f(x, s) = s(1 - s)(s - \phi(x))$$

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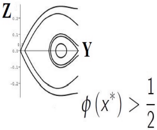
Hence $F_1 \geq 0$ and $F_1\left(x_0, \frac{1}{2}\right) = 0$.

Consider again the case of a decreasing ϕ, with $\phi(x_0) = \frac{1}{2}$. For small ε, if there is a layer at $x^* < x_0$, then $(u(x), u'(x))$ must be close to a phase curve for the rescaled system at x^*.



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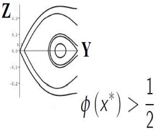

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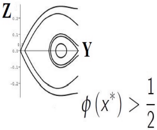

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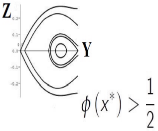

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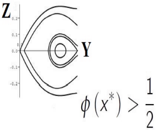

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Other Solutions

A spike is a segment of a solution containing exactly two crossings of ϕ. In this case, the solution has successive minima, or maxima, at about the same level. Since maxima increase where $\phi' < 0$ and decrease where $\phi' > 0$, spikes can only occur near minima or maxima of ϕ.

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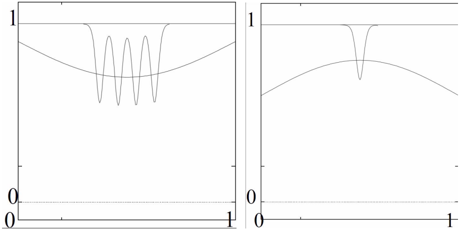

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Uniqueness and stability of the increasing layer solution.

$$\varepsilon^2 u'' = u\left(1-u\right)\left(u-\phi\left(x\right)\right) = 0 \qquad (9)$$
$$u'\left(0\right) = u'\left(1\right) = 0$$

$$\phi'\left(x\right) < 0$$
$$\phi\left(0\right) > \frac{1}{2},\ \phi\left(1\right) < \frac{1}{2}. \qquad (10)$$

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Again assuming that

$$u_\alpha(0) = \alpha, u'_\alpha(0) = 0,$$

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Again assuming that

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we showed that there was an $\alpha = \alpha^*$ such that $u'_{\alpha^*}(1) = 0$. We don't expect this solution to be unique, even among non-trivial increasing solutions, because of the possibility of solutions with boundary layers. We will outline a proof that is it unique among all solutions close to the given solution.

To show local uniqueness it is sufficient to show that at any α close to α^* such that $u'_\alpha(1) = 0$,

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Let

$$v(x) = \frac{\partial u_\alpha(x)}{d\alpha}.$$

We wish to show that $v'(1) > 0$.

The function v satisfies the linearized equation

$$\varepsilon^2 v'' + f_u(x, u_\alpha(x))\, v = 0$$

with initial conditions

$$v(0) = 1, v'(0) = 0.$$

Lemma 18.10: *For sufficiently small* $\varepsilon > 0$, $v > 0$ *on* $[0, 1]$ *and* $v'(1) > 0$.

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Brief idea of proof (all we can give here; see book). We use the function $w = u'_\alpha$, which satisfies the equation

$$\varepsilon^2 w'' + f_u\left(x, u_\alpha\right) w = -f_x\left(x, u_a\right)$$

with

$$w\left(0\right) = 0, w'\left(0\right) = u''_{\alpha_1}\left(0\right).$$

Thus we are comparing the solution of the homogeneous linear problem

$$\begin{aligned} \varepsilon^2 v'' + f_u\left(x, u_{\alpha_1}(x)\right) v &= 0 \\ v(0) = 1, v'(0) &= 0 \end{aligned}$$

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But $w = u'_{\alpha_1}$, about which we know a lot. It is positive, zero at each endpoint, and has a spike in the middle which we can estimate. Unfortunately, further details are too complicated to present here. This kind of analysis arises fairly often in this area of ode's.

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Definition: *A steady state solution* $u^*(x)$ *is asymptotically stable if for each* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that if*

$$\max_{0 \leq x \leq 1} |u(x, 0) - u^*(x)| < \delta$$

for $0 \leq x \leq 1$, *then*

$$\max_{0 \leq x \leq 1} |u(x, t) - u^*(x, t)| < \varepsilon$$

for all $t > 0$ *and*

$$\lim_{t \to \infty} \max_{0 \leq x \leq 1} |u(x, t) - u^*(x, t)| = 0.$$

The linearization of this problem is

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By linearized stability, we mean that all solutions U satisfy

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uniformly on $0 \leq x \leq 1$. It can be shown that solutions can be written as Fourier series of the form

$$U(x, t) = \Sigma_{n=1}^{\infty} c_n e^{\lambda_n t} V_n(x)$$

where each λ_n and V_n is an eigenvalue and eigenfunction of the problem

$$\varepsilon^2 V'' + (f_u(x, u^*(x)) - \lambda)\, V = 0$$
$$V'(0) = V'(1) = 0.$$

We need to show that all eigenvalues have negative real parts.

In considering uniqueness we looked at this problem with $\lambda = 0$, and stated (without much proof) that if $V(0) = 1$, $V'(0) = 0$, then $v > 0$ on $[0, 1]$ and $v'(0) > 0$. The Sturm comparison theorem implies that if $\lambda > 0$ then these relations still hold, and so λ is not an eigenvalue. This result implies stability.

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The Morse index of a steady state solution is the number of positive eigenvalues for the corresponding eigenvalue problem. Suppose that n is the total number of oscillations of some solution u (sign changes of $u - \frac{1}{2}$). Then for sufficiently small ε the Morse index of u is equal to $n - n_{sl} - n_{ss}$, where n_{sl} is the number of stable layers.
The proof is long and complex. This result had been conjectured privately by Matano.