Selected Topics from Analytical Foundations of Quasiconformal Mappings

Tadeusz Iwaniec (Syracuse University)

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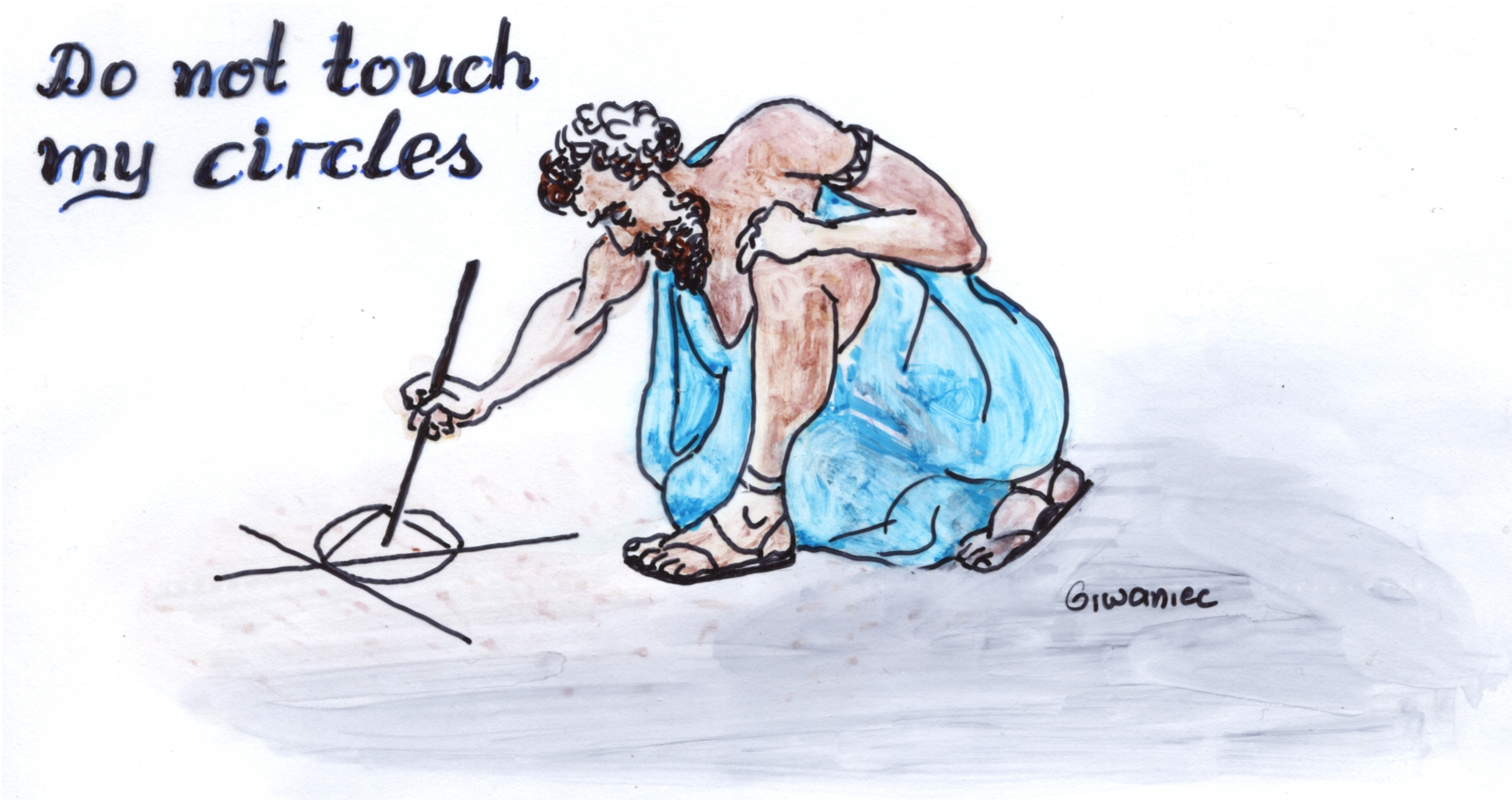
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Beltrami Equation

$$f_{\overline{z}}(z) = \mu(z) f_z(z) \qquad (1)$$

$$|\mu(z)| \leqslant k < 1 \qquad \operatorname{supp} \mu \Subset \mathbb{C}$$

Principal Solution $f : \mathbb{C} \xrightarrow{\text{onto}} \mathbb{C}$

$$f(z) = z + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots , \quad z \sim \infty$$

$$f(z) = z + \mathcal{C}\omega$$

$$\omega \in \mathcal{L}^p(\mathbb{C}) , \quad p > 2 , \quad \operatorname{supp} \omega \Subset \mathbb{C}$$

Cauchy Transform $\mathcal{C} : \mathcal{L}^p(\mathbb{C}) \to \mathcal{W}^{1,p}(\mathbb{C})$

$$(\mathcal{C}\omega)(z) = \frac{1}{\pi} \iint_{\mathbb{C}} \frac{\omega(\tau)\ \mathsf{d}\tau}{z - \tau} =$$

$$\left[\frac{1}{\pi} \iint_{\mathbb{C}} \omega(\tau)\ \mathsf{d}\tau\right] \cdot \frac{1}{z} + \frac{c_2}{z^2} + \cdots$$

$$\frac{\partial}{\partial \overline{z}} \circ \mathcal{C} \ = \ \mathsf{Id} \ : \ \mathcal{L}^p(\mathbb{C}) \to \mathcal{L}^p(\mathbb{C})$$

For a partial differential operator with constant coefficients, we have

$$\mathfrak{D} \circ \mathcal{C} = \mathcal{C} \circ \mathfrak{D}$$

$$|\,\mathcal{C}\omega(z_1) - \mathcal{C}\omega(z_2)\,| \leqslant C_p \,|\, z_1 - z_2\,|^{1-2/p} \ \|\omega\|_p$$

Beurling Transform $\mathcal{S} : \mathcal{L}^p(\mathbb{C}) \to \mathcal{L}^p(\mathbb{C})$

$$(\mathcal{S}\omega)(z) = -\frac{1}{\pi}\iint_{\mathbb{C}} \frac{\omega(\tau)\, \mathsf{d}\tau}{(z - \tau)^2} = \frac{\partial}{\partial z}\, \mathcal{C}\omega$$

$$\mathcal{S}\phi \;=\; \mathcal{C}\phi_z \quad \text{for } \phi \in \mathcal{W}^{1,p}_{\circ}(\mathbb{C})$$

$$\mathcal{S} \,:\, \mathcal{L}^2(\mathbb{C}) \xrightarrow{\text{onto}} \mathcal{L}^2(\mathbb{C})$$

This is an isometry (proof in class). Hence by *Riesz Thorin Convexity Theorem*

$$S_p = \|\mathcal{S} : \mathcal{L}^p(\mathbb{C}) \to \mathcal{L}^p(\mathbb{C})\|_p = 1 + \mathcal{O}(|p-2|)$$

CONJECTURE

$$S_p = p - 1 \quad \text{for } p \geqslant 2$$

$$S_p = \frac{1}{p-1} \quad \text{for } 1 < p \leqslant 2$$

The operator S provides an important $\mathscr{L}^p$- transition between two homotopy classes of elliptic PDEs

Proposition 1 The operator

$$I - \mu\mathcal{S} : \mathcal{L}^p(\mathbb{C}) \to \mathcal{L}^p(\mathbb{C})$$

is invertible whenever $kS_p < 1$

Proposition 2 The operator

$$I - \mu\mathcal{S} : \mathcal{W}^{1,p}(\mathbb{C}) \to \mathcal{W}^{1,p}(\mathbb{C})$$

is invertible whenever $kS_p < 1$ and $\mu \in \mathcal{C}_0^\infty(\mathbb{C})$ the proof in class

Proposition 3.

Let $kS_p < 1$ and $\mu \in \mathcal{C}_0^\infty(\mathbb{C})$. Consider the (unique) principal solution of the Beltrami equation $f_{\overline{z}} = \mu(z) f_z$. Then $f \in \mathcal{C}^1(\mathbb{C})$ and $J_f > 0$.

Proof (differentiate the equation)

$f_{z\overline{z}} = \mu f_{zz} + \mu_z f_z$. Denote by $F = f_z$

$F_{\overline{z}} = \mu F_z + \mu_z F$. Look for $F = e^\sigma$

$\sigma_{\overline{z}} = \mu \sigma_z + \mu_z$ (here are rigorous arguments)

Consider the equation $\sigma_{\overline{z}} = \mu \sigma_z + \mu_z$, for $\sigma \in \mathcal{W}^{1,p}(\mathbb{C})$

$\sigma = \mathcal{C}\phi = \frac{c}{z} +$ higher powers of $\frac{1}{z}$,

where $\phi = \mu\, \mathcal{S}\phi + \mu_z$,

$\phi = (I = \mu \mathcal{S})^{-1} \mu_z \in \mathcal{W}^{1,p}_{\circledcirc}(\mathbb{C})$. Denote by

$$\mathfrak{F} = z + \mathcal{C}(\mu e^{\sigma}) \in \mathcal{C}^1(\mathbb{C})$$

$$\mathfrak{F}_{\overline{z}} = \mu e^{\sigma} \qquad \mathfrak{F}_z = 1 + \mathcal{S}(\mu e^{\sigma})$$

$$e^\sigma - 1 \in \mathcal{W}^{1,p}(\mathbb{C})$$
$$e^\sigma - 1 = \frac{\partial}{\partial \overline{z}}\, \mathcal{C}(e^\sigma - 1) = \mathcal{C}(e^\sigma - 1)_{\overline{z}} =$$
$$\mathcal{C}(\mu e^\sigma)_z = \mathcal{S}(\mu e^\sigma)$$

Hence

$$e^\sigma = 1 + \mathcal{S}(\mu e^\sigma) = \mathfrak{F}_z \qquad \mathfrak{F}_{\overline{z}} = \mu \mathfrak{F}_z$$

$\mathfrak{F}$ is a principal solution, thus equal to f .

$$J_f = |f_z|^2 - |f_{\overline{z}}|^2 = |e^{2\sigma}|(1 - |\mu|^2) > 0$$

COROLLARY. For $\mu \in \mathcal{C}_0^\infty(\mathbb{C})$ the principal solution

$f : \widehat{\mathbb{C}} \xrightarrow{\text{onto}} \widehat{\mathbb{C}}$

is a $\mathcal{C}^1$- diffeomorphisms and $f(\infty) = \infty$. The inverse map $g = g(w) = f^{-1}(w)$ is a principal solution of the equation

$$g_{\overline{w}} = -\mu(g(w))\,\overline{g_w}$$

by chain rule

Uniform Hölder Estimates

$$|f(z_1) - f(z_2)| \leqslant$$
$$|z_1 - z_2| + C_p(k)\,|z_1 - z_2|^{1-2/p}$$

$$|g(w_1) - g(w_2)| \leqslant$$
$$|w_1 - w_2| + C_p(k)\,|w_1 - w_2|^{1-2/p}$$

Hence

$$|z_1 - z_2| = |g(w_1) - g(w_2)| \leqslant$$

$$|w_1 - w_2| + C_p(k)\,|w_1 - w_2|^{1-2/p} =$$

$$|f(z_1) - f(z_2)| + C_p(k)\,|f(z_1) - f(z_2)|^{1-2/p}$$

$$==========================$$

$$|z_1 - z_2| \leqslant$$
$$|f(z_1) - f(z_2)| + C_p(k)\,|f(z_1) - f(z_2)|^{1-2/p}$$

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Approximation

Let $\mu^j \to \mu$, almost everywhere in $\mathbb{C}$,

$\mu^j \in \mathcal{C}_0^\infty(\mathbb{C})$, $|\mu^j(z)| \leqslant k < 1$ (convolution method).

$f^j_{\bar z} = \mu^j f^j_z$ principal solutions

$$f^j = z + \mathcal{C}\omega^j \quad , \quad \omega^j = \mu^j + \mu^j \mathcal{S}\omega^j$$

$f_{\bar z} = \mu\, f_z$ principal solution

$$f = z + \mathcal{C}\omega \quad , \quad \omega = \mu + \mu\mathcal{S}\omega$$

$$\omega^j - \omega = \mu^j - \mu \; + \mu^j \mathcal{S}\omega^j - \mu \mathcal{S}\omega =$$

$$(\mu^j - \mu) \; + \mu^j \left[\mathcal{S}(\omega^j - \omega)\right] \; + (\mu^j - \mu)\mathcal{S}\omega$$

$$\|\omega^j - \omega\|_p \leqslant \|\mu^j - \mu\|_p \; + \; kS_p\|\omega^j - \omega\|_p \; +$$

$$\|(\mu^j - \mu)\mathcal{S}\omega\|_p$$

Hence

$$(1 - kS_p)\,\|\omega^j - \omega\|_p \leqslant$$

$$\|\mu^j - \mu\|_p \; + \; \|(\mu^j - \mu)\mathcal{S}\omega\|_p \; \longrightarrow 0$$

Thus f^j converge to f uniformly (point-wise suffices)

In Conclusion

$$|z_1 - z_2| \leqslant$$
$$|f^j(z_1) - f^j(z_2)| + C_p(k)\, |f^j(z_1) - f^j(z_2)|^{1-2/p}$$
$$\downarrow$$
$$|f(z_1) - f(z_2)| + C_p(k)\, |f(z_1) - f(z_2)|^{1-2/p}$$

This yields that $f(z_1) \neq f(z_2)$ whenever $z_1 \neq z_2$. Thus $f : \mathbb{C} \overset{\text{onto}}{\longrightarrow} \mathbb{C}$ is a homeomorphism.

Measurable Riemann Mapping Theorem

Let Ω and Ω' be bounded simply connected domains and μ - a measurable Beltrami coefficient such that $|\mu(z)| \leqslant k < 1$ almost everywhere in Ω. Then the Beltrami equation

$$f_{\overline{z}} = \mu(z)\, f_z$$

admits a homeomorphic solution $f : \Omega \xrightarrow{\text{onto}} \Omega'$ in the Sobolev class

$\mathcal{W}^{1,p}_{\mathsf{loc}}(\Omega, \mathbb{C}) \subset \mathcal{C}^{\alpha}_{\mathsf{loc}}(\Omega)$

General Elliptic Systems

$$f_{\overline{z}} = \mu(z)\, f_z + \nu(z)\, \overline{f_z}$$
$$|\mu(z)| + |\nu(z)| \leqslant k < 1$$

$$f_{\overline{z}} = \mu(z, f)\, f_z \, + \, \nu(z, f)\, \overline{f_z}$$

$$f_{\overline{z}} = \mathcal{H}(z, f,\, f_z)$$
$$|\mathcal{H}(z, f, \xi) - \mathcal{H}(z, f, \zeta)| \leqslant k|\xi - \zeta|$$

Every Riemann surface is conformally flat (locally)




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