Selected Topics from Analytical Foundations of Quasiconformal Mappings

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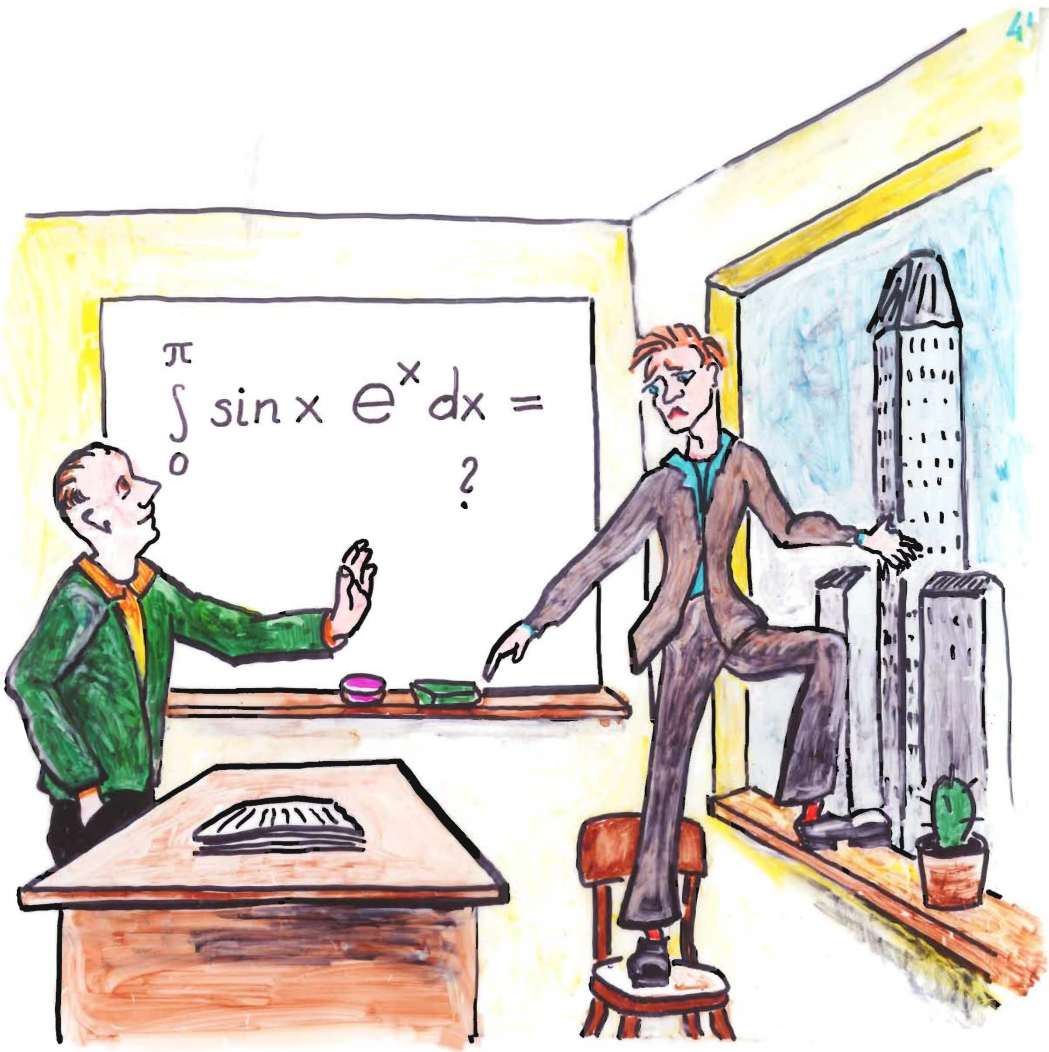


Don't jump, try
integration by parts

Quasiconformal Mappings

Geometric function theory in higher dimensions is based on all the analytic and geometric spirit of holomorphic functions. That is why, the mappings we shall discuss in this Section come as solutions to certain nonlinear elliptic systems of PDEs closely related to the Cauchy-Riemann equations. Modern approach makes use of the Hodge theory of differential forms. We shall put these beautiful equations into play later. The term *Sobolev mapping*, or weakly differentiable deformation of an open region $\Omega \subset \mathbb{R}^n$, refers to a vector field

$$f = (f^1, f^2, ..., f^n) : \Omega \longrightarrow \mathbb{R}^n \,, \qquad f \in \mathscr{W}^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$$

whose coordinates lie in the Sobolev space $\mathscr{W}^{1,1}_{\mathrm{loc}}(\Omega)$. Thus, we can speak of the differential matrix

$$\mathfrak{D}f \;=\; \begin{bmatrix} \dfrac{\partial f^1}{\partial x_1} & \dfrac{\partial f^1}{\partial x_2} & \cdots & \dfrac{\partial f^1}{\partial x_n} \\ \\ \dfrac{\partial f^2}{\partial x_1} & \dfrac{\partial f^2}{\partial x_2} & \cdots & \dfrac{\partial f^2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial f^n}{\partial x_1} & \dfrac{\partial f^n}{\partial x_2} & \cdots & \dfrac{\partial f^n}{\partial x_n} \end{bmatrix} \;\in\; \mathscr{L}^1_{\mathrm{loc}}(\Omega,\ \mathbb{R}^{n\times n}) \tag{1}$$

We say that f is orientation preserving if its Jacobian determinant

$$\mathcal{J}(x,f) \;=\; \det \mathfrak{D}f(x) \;=\; \begin{vmatrix} \frac{\partial f^1}{\partial x_1} & \frac{\partial f^1}{\partial x_2} & \cdots & \frac{\partial f^1}{\partial x_n} \\ \\ \frac{\partial f^2}{\partial x_1} & \frac{\partial f^2}{\partial x_2} & \cdots & \frac{\partial f^2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f^n}{\partial x_1} & \frac{\partial f^n}{\partial x_2} & \cdots & \frac{\partial f^n}{\partial x_n} \end{vmatrix} \;\geqslant\; 0 \qquad (2)$$

is nonnegative almost everywhere. The operator norm of the differential;

$$\|\,\mathfrak{D}f(x)\| \;=\; \max\{\;|\,\mathfrak{D}f(x)\,\xi\,|; \quad |\xi| = 1\;\}$$

represents the magnitude of the infinitesimal deformation of 1-dimensional objects. On the other hand, the n-form

$$J(x, f)\,\mathrm{d}x = df^1 \wedge ... \wedge df^n$$

represents an infinitesimal change of volume at the point $x \in \Omega$. **Definition**

We want these two infinitesimal deformations to be comparable point-wise at almost every point.

A Sobolev mapping $f : \Omega \longrightarrow \mathbb{R}^n$ is said to have finite distortion if:

$\imath)$ **Its Jacobian determinant is locally integrable.**

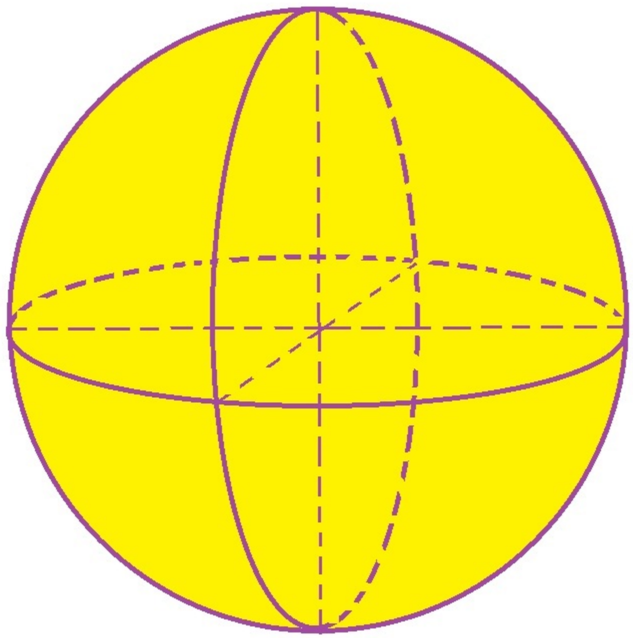
$\imath\imath)$ **There is a measurable function $K = K(x) \geqslant 1$,**

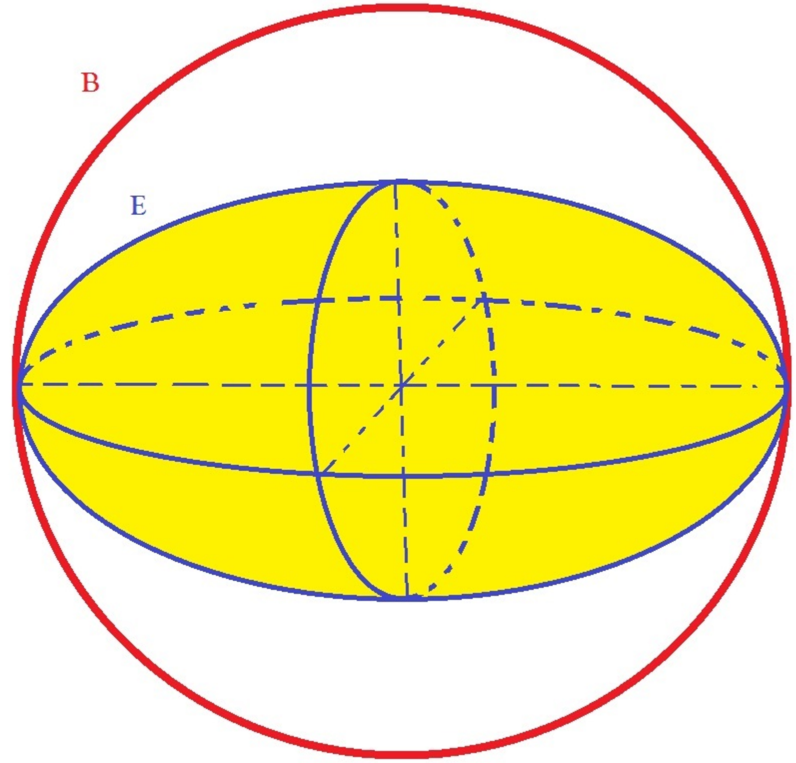
finite almost everywhere, such that:

$$J(x, f) \leqslant \| \mathfrak{D} f(x) \|^n = K(x)\, J(x, f)$$

Geometrically, the distortion inequality tells us how the linear differential map deforms spheres into ellipsoids

$$\mathfrak{D}f(x) : \mathbb{R}^n \longrightarrow \mathbb{R}^n \qquad \frac{\mathrm{vol}\ \mathbf{B}}{\mathrm{vol}\ \mathbf{E}} = \frac{\|\mathfrak{D}f(x)\|^n}{J(x,f)} = K(x) < \infty$$





There is no distortion when $K(x) \equiv 1$, which results in conformal mappings and the equation: $\|\mathfrak{D}f(x)\|^n = J(x,f)$. Equivalently, we have the n-dimensionsl (nonlinear) Cauchy-Riemann system

$$\mathfrak{D}^*f(x) \cdot \mathfrak{D}f(x) = J(x,f)^{\frac{2}{n}}\,\mathbf{Id}$$

Using Frobenius norm of a matrix

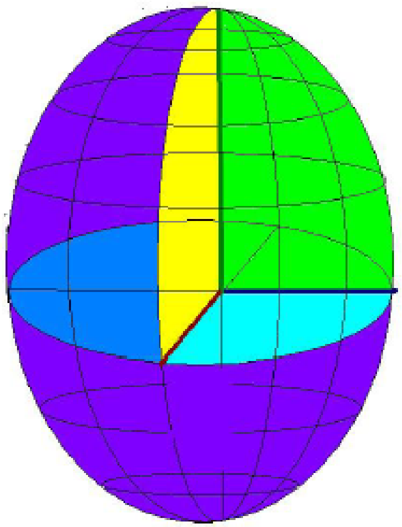
$$|\mathfrak{D}f(x)|^2 \stackrel{\text{def}}{=\!=} \mathsf{Trace}[\mathfrak{D}^*f(x) \cdot \mathfrak{D}f(x)]$$

we write it as:

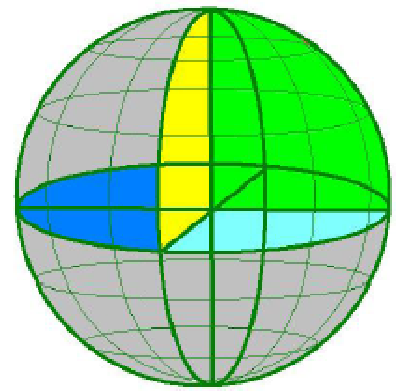
$$|\mathfrak{D}f(x)|^n = n^{n/2} J(x,f)$$

Quasiconformal Deformation

$$h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}\,, \quad y = h(x)$$



$$Dh(x) \quad : \quad \mathbf{T}_x\mathbb{X} \quad \xrightarrow{\text{onto}} \quad \mathbf{T}_y\mathbb{Y}$$



$$\{\xi \in \mathbf{T}_x\mathbb{X}\ ;\ \langle G(x)\xi\,,\,\xi\rangle\ = constant\,\}$$

$D^*h(x) \cdot Dh(x) \ = J(x,h)^{2/n}\,\mathbf{G}(x)$ - ***Beltrami Equation***

The n-harmonic equations

Conformal mappings have the smallest possible n-harmonic energy among all mappings with given boundary values, that is:

$$\mathscr{E}[f] \stackrel{\text{def}}{=\!=} \int_\Omega |\mathfrak{D}f(x)|^n \,\mathrm{d}x \;=\; n^{n/2}\int_\Omega J(x,f)\,\mathrm{d}x$$

$$=\; n^{n/2}\int_\Omega J(x,h)\,\mathrm{d}x \;=\; \int_\Omega |\mathfrak{D}h(x)|^n\,\mathrm{d}x \;= \mathscr{E}[h]$$

whenever $h \in\; f + \mathscr{W}_\circ^{1,n}(\Omega, \mathbb{R}^n)$.

In particular, conformal mappings solve the n-harmonic system

$$\mathfrak{D}^*\Big(|\mathfrak{D}f(x)|^{n-2}\,\mathfrak{D}f(x)\,\Big) =\; 0 \tag{3}$$

where

$$\mathfrak{D}^* \;=\; \mathrm{Div}: \quad \mathscr{D}'(\Omega, \mathbb{R}^{n\times n}) \longrightarrow \; \mathscr{D}'(\Omega, \mathbb{R}^n)$$

In dimension $n = 2$ the real part $u = \Re f$ and the imaginary part $v = \Im f$ of a conformal mapping are harmonic conjugate functions. For this reason we refer to the components of a conformal mapping $f = (f^1, f^2, ..., f^n)$) as *n-harmonic conjugate functions*.

In fact, the n-harmonic system (3) for conformal mappings can be uncoupled so that each component solves the scalar n-harmonic equation

$$\mathrm{div}\, |\nabla u|^{n-2}\, \nabla u \;=\; 0\,, \quad \text{for} \quad u = f^i, \qquad i = 1, ..., n \tag{4}$$

More generally, mappings of finite distortion solve an $\mathbf{A}$-harmonic system

$$\mathfrak{D}^*\, \mathbf{A}(x, \mathfrak{D}f) \;=\; 0\,, \quad \mathbf{A}(x, \mathfrak{X}) = \Big\langle\, \mathrm{G}(x)\mathfrak{X} \mid \mathfrak{X} \,\Big\rangle^{\frac{n-2}{2}} \mathrm{G}(x)\mathfrak{X}\,, \quad \mathfrak{X} \in \mathbb{R}^{n\times n}$$

The matrix function $\mathbf{G} = \mathbf{G}(x)$ is called the distortion tensor of f. It is a symmetric positive definite matrix of determinant one. As before, this system can be uncoupled so that each component $u = f^i, \;\; i = 1, ..., n\,,$ of a mapping with finite distortion solves a scalar **A**- harmonic equation:

$$\operatorname{div} \mathbf{A}(x, \nabla u) \;=\; 0\,, \qquad\qquad \mathbf{A}(x, \xi) = \Big\langle\, \mathbf{G}(x)\xi \;\mid \xi \,\Big\rangle^{\frac{n-2}{2}} \mathbf{G}(x)\xi\;, \qquad \xi \in \mathbb{R}^n \tag{5}$$

This analogy with conformal mappings extends further by noticing that outside the zero points of f the function $u = \log|f(x)|$ is **A**-harmonic as well.

The Beltrami Dirac equation

We shall formulate more equations for quasiregular mappings. Let $\wedge^\ell = \wedge^\ell\mathbb{R}^n$, $\ell = 0, 1, \ldots, n$, denote the space of ℓ-covectors in $\mathbb{R}^n$. The ℓ-forms on $\Omega \subset \mathbb{R}^n$ are simply functions $\gamma :\ \Omega \to \wedge^\ell\mathbb{R}^n$. The symbols

$$\mathscr{C}^\infty(\Omega, \wedge^\ell), \qquad \mathscr{L}^p(\Omega, \wedge^\ell), \qquad \mathscr{W}^{1,p}(\Omega, \wedge^\ell)$$

etc. for corresponding subspaces of ℓ-forms on a domain $\Omega \subset \mathbb{R}^n$ are self-explanatory. The exterior derivative

$$d : \mathscr{C}^\infty(\Omega, \wedge^{\ell-1}) \to \mathscr{C}^\infty(\Omega, \wedge^\ell)$$

and its formal adjoint

$$d^\star :\ \mathscr{C}^\infty(\Omega, \wedge^\ell) \to \mathscr{C}^\infty(\Omega, \wedge^{\ell-1})$$

are the fundamental differential operators on forms. Here

$$\star : \wedge^{\ell}\mathbb{R}^n \to \wedge^{n-\ell}\mathbb{R}^n$$

stands for the Hodge star duality operator.
More natural domains for d and $d^{\star}$ will become perfectly clear in the sequel.
One more space of concern to us consists of harmonic fields

$$\mathscr{H}^{\ell}(\mathbb{R}^n) \;=\; \{\; \theta \in \mathscr{C}^{\infty}(\Omega, \wedge^{\ell}), \quad d\theta = d^{\star}\theta = 0 \;\}, \qquad\qquad \ell = 1, 2, ..., \; n-1$$

Associated with a quasiconformal mapping $f \in \mathscr{W}^{1,n}(\Omega, \mathbb{R}^n)$ is the pullback of ℓ-forms

$$f^{\#} : \mathscr{C}^{\infty}(\Omega, \wedge^{\ell}) \to \mathscr{L}^{p}(\Omega, \wedge^{\ell}) \;, \qquad\qquad \text{with} \;\; p = \frac{n}{\ell}$$

and the pullback of $(n-\ell)$-forms.

$$f^{\#} : \mathscr{C}^{\infty}(\Omega, \wedge^{n-\ell}) \to \mathscr{L}^{q}(\Omega, \wedge^{n-\ell}) \ , \qquad \text{with } \ q = \frac{n}{n-\ell}$$

Let us fix an arbitrary harmonic field $\theta \in \mathcal{H}^{\ell}(\mathbb{R}^n)$.
To simplify matters we only consider

$$\theta = dy_1 \wedge ... \wedge dy_\ell \, , \qquad \text{so} \qquad \star\, \theta = \ dy_{\ell+1} \wedge ... \wedge dy_n$$

The pullbacks are closed forms on Ω

$$f^{\#}(\theta) \ = \ df^1 \wedge ... \wedge df^\ell \, , \qquad d\ f^{\#}(\theta) \ = 0$$

$$f^{\#}(\star\, \theta) \ = \ \ df^{\ell+1} \wedge ... \wedge df^n \, , \qquad d\ f^{\#}(\star\, \theta) \ = 0$$

We can express them (at least locally) as:

$$f^{\#}(\theta) \;=\; du \;\in \mathscr{L}^{p}(\Omega, \wedge^{\ell})\,, \qquad\qquad d^{\star}u = 0$$

$$\star f^{\#}(\star\,\theta) \;=\; d^{\star}v \;\in \mathscr{L}^{q}(\Omega, \wedge^{\ell}), \qquad\qquad dv = 0$$

Now, the differential forms $u \in \mathscr{W}^{1,p}(\Omega, \wedge^{\ell-1})$ and $v \in \mathscr{W}^{1,\,q}(\Omega, \wedge^{\ell+1})$, $p + q = p \cdot q$, will be the unknowns in our equations. They are viewed as generalizations of the real and imaginary part of a holomorphic function. First we take into consideration a conformal mapping $f \in W^{1,\ell}(\Omega, \mathbb{R}^n)$ in even dimension $n = 2\ell$. The equations obtained for u and v are extremely simple:

$$du \;=\; d^{\star}v \qquad \text{and} \qquad d^{\star}u \;=\; dv \;=\; 0$$

In particular, both u and v are harmonic forms, consequently $\mathscr{C}^\infty$-smooth. It hardly matters how we choose the underlying harmonic field $\theta \in \mathscr{H}^\ell(\mathbb{R}^n)$; our equations depend purely upon the differential $\mathfrak{D}f$. We indulge ourselves by putting on stage some simple ones. To this effect, we represent the differential $\mathfrak{D}f(x) : \mathbb{T}_x\Omega \to \mathbb{T}_y\Omega'$ by a square $2\ell \times 2\ell$ -matrix, and infer the following relations:

$$\mathfrak{D}f(x) = \begin{bmatrix} A(x) & B(x) \\ C(x) & D(x) \end{bmatrix}, \qquad \begin{cases} \det A &=& \det D \\ \det B &=& (-1)^\ell \det C \end{cases}$$

where A, B, C and D are the $\ell \times \ell$ -submatrices.
Notice the resemblance to the Cauchy-Riemann system in $\mathbb{R}^2$.

$$\mathfrak{D}f(x) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}, \qquad \begin{cases} u_x &=& v_y \\ u_y &=& -v_x \end{cases}$$

When the rows and columns are permuted in $\mathfrak{D}f(x)$, more linear relations between the $\ell \times \ell$ -minors are obtained. In much the same way the equations describing quasiconformal mappings in dimension $n = 2\ell$, when lifted to the exterior ℓ -forms, give rise to linear relations with measurable coefficients between the $\ell \times \ell$ -minors of the matrix $\mathfrak{D}f(x)$.

$$d^\star v \;=\; \mathbf{A}(x)\, du\,, \qquad d^\star u = dv = 0\ , \qquad \det \mathbf{A}(x) \;\equiv\; 1$$

$$\mathbf{A}(x) \in \textsf{\textbf{Hom}}(\wedge^\ell \mathbb{R}^n,\ \wedge^\ell \mathbb{R}^n) \qquad - \text{symmetric positive definite}$$

Actually $\mathbf{A}(x)$ is the $\ell' s$ exterior power of $\mathbf{G}(x)$ (Cauchy-Green Tensor) ℓ subdeterminants. There is always further to go. Without getting into details we regard u and v as functions valued in the whole Grassmann algebra

$$\Lambda \;=\; \Lambda\mathbb{R}^n = \bigoplus_{\ell=0}^{n}\ \wedge^\ell \mathbb{R}^n$$

There we have elliptic *Dirac operators*

$$\overline{\partial} \;=\; d \,-\, d^\star \qquad \text{and} \qquad \partial \;=\; d \,+\, d^\star, \qquad \text{defined in } \mathscr{W}^{1,p}(\Omega, \Lambda)$$

These equations reduce to a single one for the differential form $\mathscr{F} = u + v$. We call it the *Beltrami-Dirac equation*:

$$\overline{\partial}\mathscr{F} \;=\; \mu(x)\,\partial\mathscr{F}\,, \qquad\qquad \text{where} \quad \mathscr{F} \;=\; u \,+\, v$$

$$\mu(x) \;=\; \frac{I - \mathbf{A}(x)}{I + \mathbf{A}(x)} :\; \wedge^{\ell}\mathbb{R}^{n} \to \wedge^{\ell}\mathbb{R}^{n}, \qquad \|\mu(x)\| \leqslant k < 1$$

In many respects this system seems to be an excellent generalization of the familiar complex Beltrami equation. The whole program is similar in spirit to the study of complex functions in the plane. Continuing this analogy, let us take a look at some relations between the ℓ-forms du and $d^{\star}v$, where ℓ is not necessarily half of the dimension. In this case du and $d^{\star}v$ are coupled nonlinearly:

$$d^\star v \;=\; \mathbf{A}(x, du), \quad \text{in conformal case,} \quad d^\star v \;=\; |\, du\,|^{p-2}\, du\;, \quad p = \frac{n}{\ell}$$

Here $\mathbf{A}:\ \Omega \times \wedge^\ell \to \wedge^\ell$ is a given monotone map. We refer to these PDEs as $Hodge\ system$. The pair of differential forms

$$(\, du,\ d^\star v\,) \in \mathscr{L}^p(\Omega, \wedge^\ell) \times \mathscr{L}^q(\Omega, \wedge^\ell), \qquad p + q = p \cdot q$$

will be called $\mathbf{A}$ -conjugate fields. These equations have led us to a nonlinear Hodge-DeRham theory on manifolds with boundary. But it would take us a bit afield to present this theory here. The conjugate fields du and $d^\star v$ can be uncoupled by differentiating the Hodge system. It results in the second order $\mathbf{A}$-harmonic and $\mathbf{B}$-harmonic systems

$$d^\star \mathbf{A}(x, du) \;=\; 0 \qquad \text{and} \qquad d\, \mathbf{B}(x, d^\star v) \;=\; 0$$

where $\mathbf{B}(x, \quad) : \wedge^\ell \mathbb{R}^n \to \wedge^\ell \mathbb{R}^n$ is inverse to $\mathbf{A}(x, \quad) : \wedge^\ell \mathbb{R}^n \to \wedge^\ell \mathbb{R}^n$.
In conformal case

$$d^\star\Big(|du|^{\,p-2}\, du \Big) \;=\; 0, \qquad p = \frac{n}{\ell}$$

$$d\left(|d^\star v|^{\,q-2} d^\star v \right) \;=\; 0, \qquad q = \frac{n}{n-\ell}$$

The second order $\mathbf{A}$-harmonic and $\mathbf{B}$-harmonic equations are certainly interesting from the nonlinear potential theory perspective. However, they seem to be a far less geometric than the first order Beltrami-Dirac system.