Harmonic Mapping Problem in Geometric Function Theory and Minimal Surfaces

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April 23, 2015

CDT in Oxford

Coffee is proof that God likes mathematicians, wants us to prove theorems and to be happy (a paraphrase of Benjamin Franklin)

No coffee-no theorems. That is why coffeecolleagues often become coffeeholics

Jani Onninen

Planar Domains (plates and thin films)

Throughout this lecture $\mathbb{X}$ and $\mathbb{Y}$ are bounded domains in $\mathbb{R}^2 \simeq \mathbb{C}$. Although Riemann surfaces are not in the center of our discussion, the ideas really crystalize in a differential-geometric setting. Thus we suggest, as a possibility, to think of $\mathbb{X}$ and $\mathbb{Y}$ as Riemannian 2-manifolds (thin films).

The topics are about *Sobolev mappings*

$$h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y} \,, \qquad h \in \mathscr{W}^{1,2}_{\text{loc}}(\mathbb{X}, \mathbb{R}^2)$$

We have well defined linear tangent map

$$Dh(x) : \mathbf{T}_x\mathbb{X} \to \mathbf{T}_y\mathbb{Y}\,, \quad y = h(x)$$

at almost every point $x \in \mathbb{X}$, called $deformation\ gradient$.
In the Euclidean setting Dh is just a measurable function on $\mathbb{X}$ whose values are 2×2 -matrices, so we write $Dh(x) \in \mathbb{R}^{2\times 2}$.

The major differential quantity is the Jacobian determinant $J_h(x) = J(x,h) = \det\, Dh(x)$, which we assume to be nonnegative.

A homeomorphism $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ of Sobolev class $\mathscr{W}^{1,2}(\mathbb{X}, \mathbb{R}^2)$ between domains in $\mathbb{R}^2$ (possibly Riemannian 2 -manifolds) has *finite energy* if

$$\mathscr{E}[h] = \int_{\mathbb{X}} |Dh(x)|^2 \, \mathrm{d}x \geqslant 2 \int_{\mathbb{X}} J(x,h) \mathrm{d}x = 2|\mathbb{Y}|$$

is finite. Note that equality occurs for conformal deformations.

Traction Free Problem for simply connected plates reduces to the famous Riemann Mapping Theorem

The Riemann Moduli and Mapping Problem

The conformal type of a domain of connectivity $\ell > 2$ is determined by $3\ell - 6$ parameters (moduli of the domain); that is, two ℓ-connected domains are conformally equivalent if and only if they agree in all $3\ell - 6$ moduli. As for the bounded doubly connected domains, Schottky's Theorem (1877) asserts that:

A conformal mapping $h : \mathbb{A} = A(r, R) \xrightarrow{\text{onto}} A(r_*, R_*) = \mathbb{A}^*$
between circular annuli exists if and only if

$$\text{Mod}\,\mathbb{A}^* := \log \frac{R_*}{r_*} = \log \frac{R}{r} := \text{Mod}\,\mathbb{A}$$

One then defines conformal modulus for all doubly connected domains which, being next in the order of complexity (after simply connected case), are the subject of our examples.

G- Conformal Energy , $n = 2$

Let a -***distortion tensor*** $G = \mathbf{G}(x)$, $\det \mathbf{G}(x) \equiv 1$, be given. Then

$$\mathscr{E}_{\mathbf{G}}[h] = \int_{\mathbb{X}} \langle\, \mathbf{G}^{-1}(x) D^* h \,,\, D^* h \,\rangle \; \mathrm{d}x \geqslant 2 \int_{\mathbb{X}} J(x,h) \; \mathrm{d}x = 2|\mathbb{Y}|$$

Equality occurs if and only if h satisfies the Beltrami system

$$D^*h(x)\; Dh(x) = J(x,h)\mathbf{G}(x) \quad , \quad \mathbf{G} - \text{conformal solutions}$$

REMARK. G -conformal solutions may not exist, while the energy-minimal deformations often do exist.

The Dirichlet Energy Integral Versus Harmonic Mappings

Let $h = u + i\,v \;:\; \mathbb{X} \to \mathbb{C}$ **. The guiding example of the energy integrals is:**

$$\mathscr{D}[h] = \iint_{\mathbb{X}} |Dh|^2 \;=\; \iint_{\mathbb{X}} \left(\, |\nabla u|^2 + |\nabla v|^2 \, \right) \;<\; \infty$$

It is a persistent, popular and even published misconception that minimizing the Dirichlet integral is none other than a theory of harmonic mappings. Whereas, upon a little reflection on the admissible variations such view becomes well short of reality.

In Quest of Smallest Energy

It is certainly unrealistic to expect that the infimum energy will always be attained within homeomorphisms $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$; injectivity is often lost when passing to the weak limit of the minimizing sequence.

$$\boxed{\textbf{\textit{The loss of injectivity comes exactly in the locality where the weak limit of the minimizing sequence of homeomorphisms fails to be harmonic}}}$$

The best example is the collapsing phenomenon in the minimization of the Dirichlet integral for mappings between circular annuli.

Existence of Energy-Minimal Homeomorphisms

(K. Astala, G. Martin, T.I. Arch. Rat. Mech. Anal. 2010)

THEOREM ABOUT ANNULI

An energy–minimal homeomorphism
$h:\ A(r,R) \xrightarrow{\text{onto}} A(r_*,R_*)$ ***between annuli exists iff***

$$\frac{R_*}{r_*} \geqslant \frac{1}{2}\left(\frac{R}{r}+\frac{r}{R}\right)$$

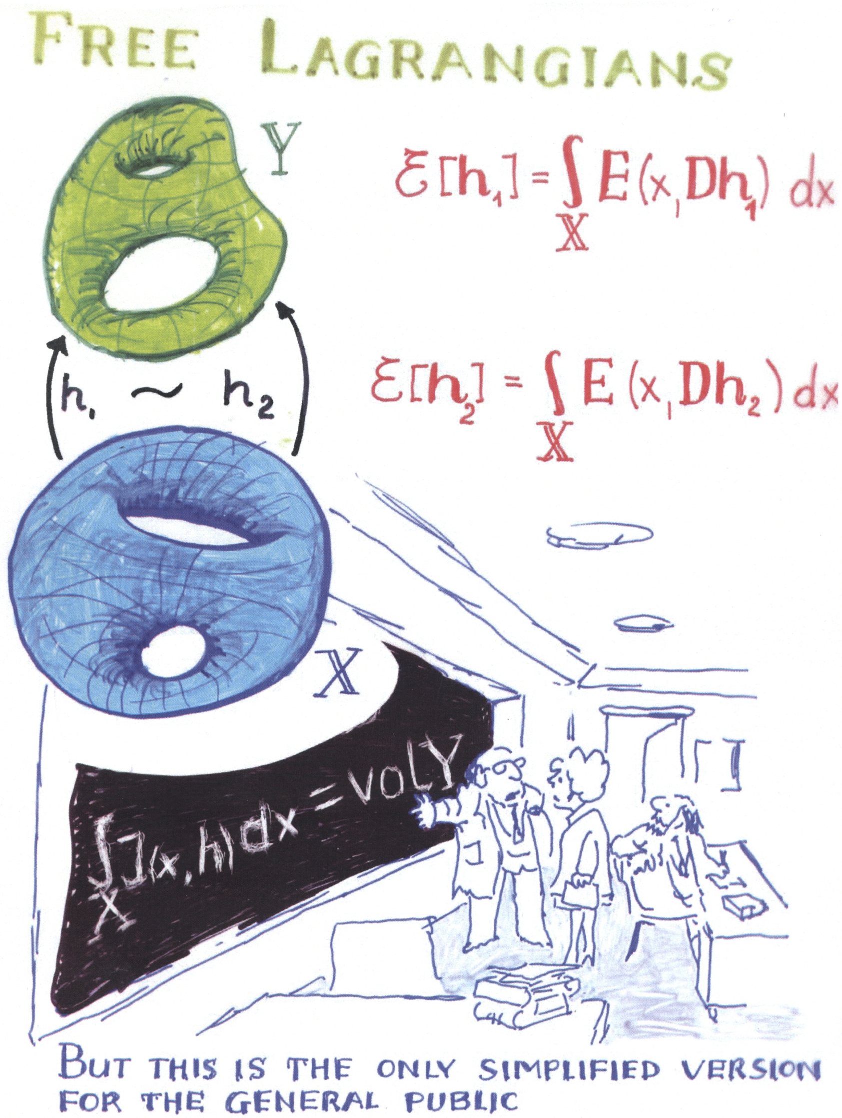
A straightforward proof of this (*so called Nitsche bound*) has been established *(J. Onninen and T.I. Memoirs of AMS, 2011)* via the concept of *free Lagrangians*.

Definition.

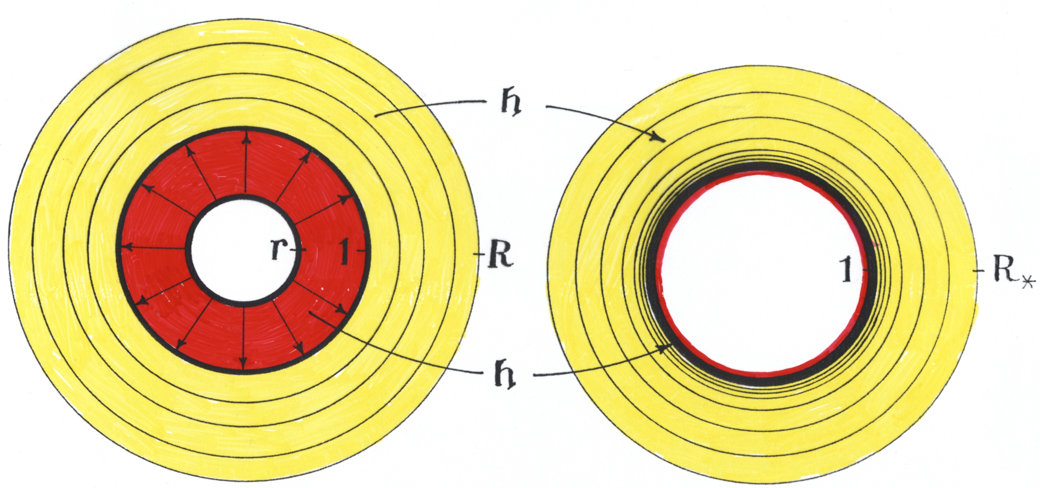
A free Lagrangian is a nonlinear differential n- form $\mathbf{L}(x,h,Dh)\mathrm{d}x$ defined on Sobolev homeomorphisms $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ whose integral depends only on the homotopy class of h, regardless of its boundary values.

The idea is to find a point-wise sharp lower bound of the energy by means of a free Lagrangian (**truly a work of art**) so that for every homeomorphisms $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ it holds:

$$\mathscr{E}[h] \geqslant \int_{\mathbb{X}} \mathbf{L}(x,h,Dh)\mathrm{d}x = \inf \mathscr{E}$$

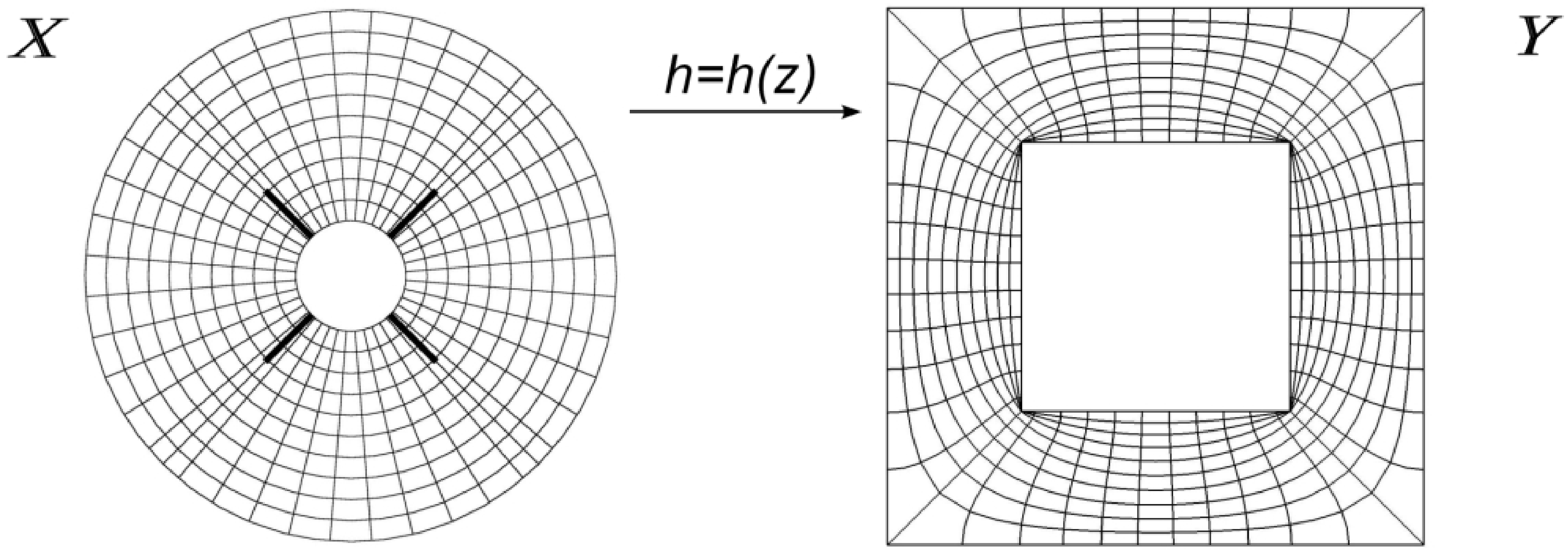

FREE LAGRANGIANS
Y
$\mathcal{E}[h_1] = \int_{\mathbb{X}} E(x, Dh_1)\, dx$
$h_1 \sim h_2$
$\mathcal{E}[h_2] = \int_{\mathbb{X}} E(x, Dh_2)\, dx$
X
$\int_X J(x, h)\, dx = \mathrm{vol}\, Y$
BUT THIS IS THE ONLY SIMPLIFIED VERSION
FOR THE GENERAL PUBLIC

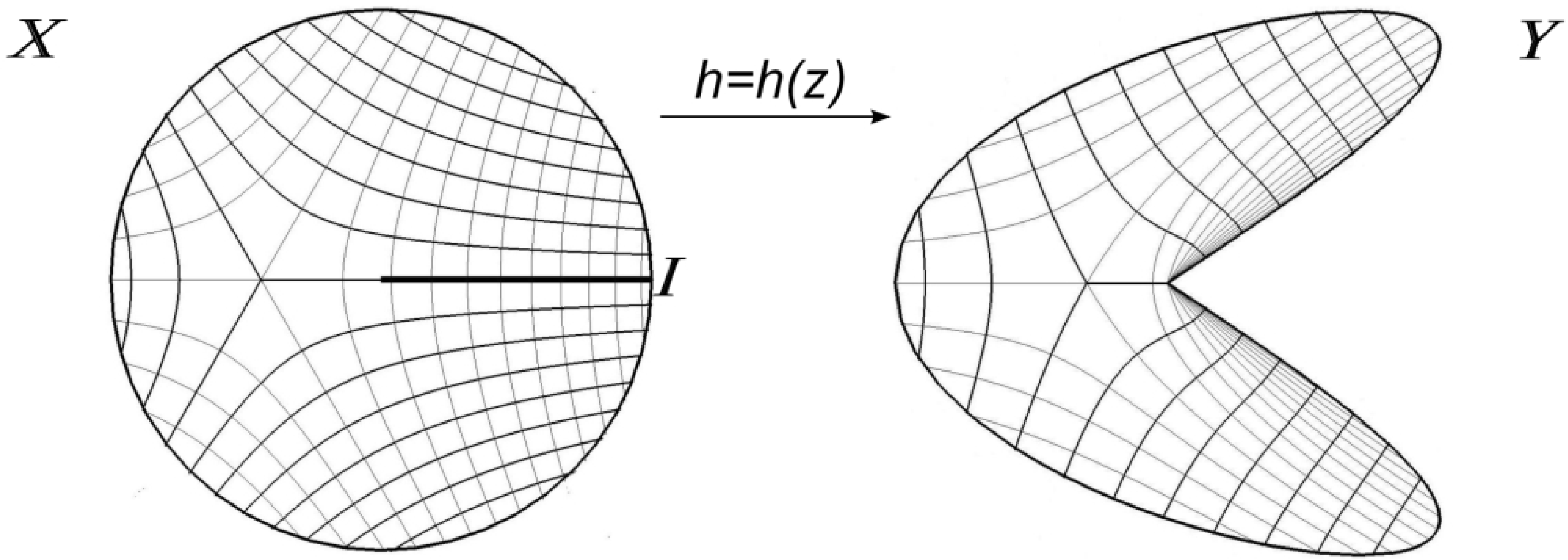
Below the Nitsche bound, $\frac{R_*}{r_*} < \frac{1}{2}\left(\frac{R}{r} + \frac{r}{R}\right)$, $r_* = 1$



$$\mathfrak{h}(z) = \begin{cases} \frac{z}{|z|}\,, & r < |z| < 1 \quad \left(\begin{array}{c}\text{collapsing into}\\ \text{concave boundary}\end{array}\right) \\ \frac{1}{2}\left(z + \frac{1}{\bar{z}}\right)\,, & 1 < |z| < R \quad \text{critical harmonic Nitsche map} \end{cases}$$

SQUEEZENG PHENOMENA

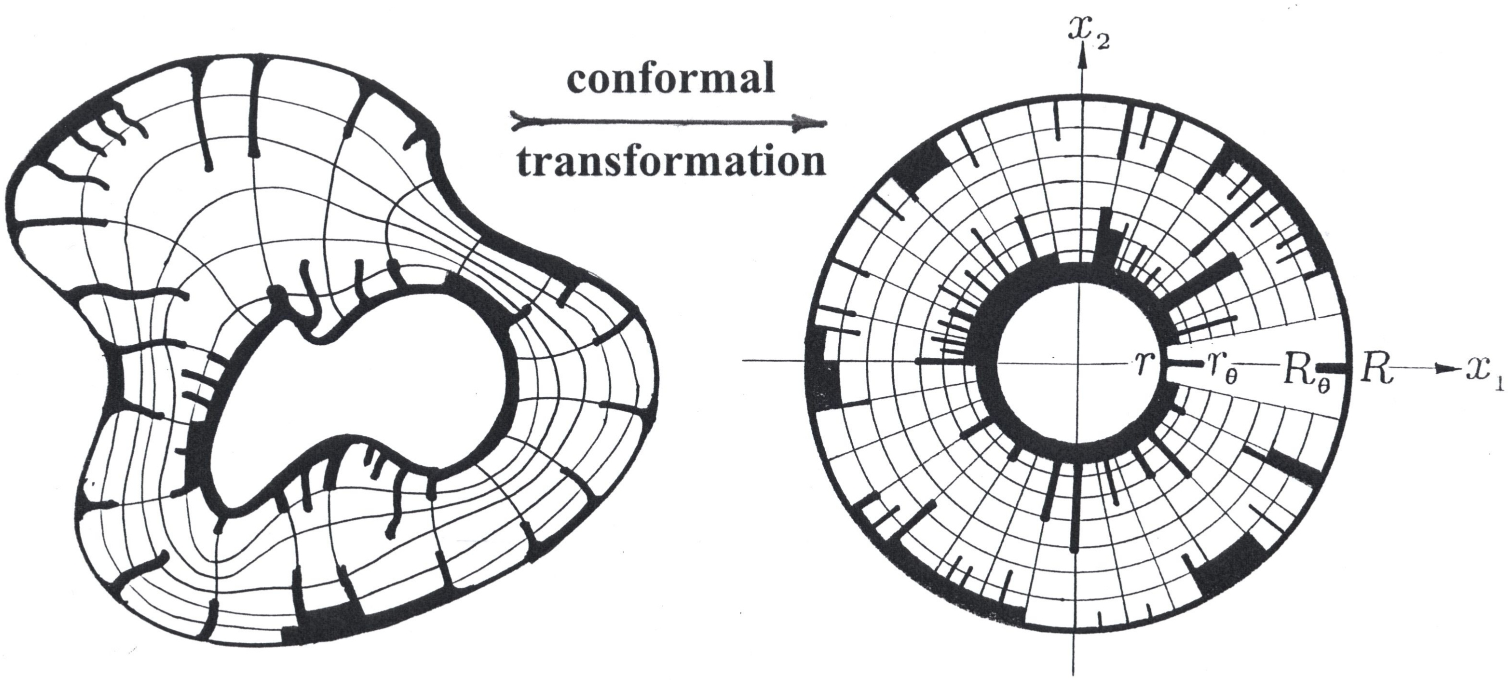




X
h=h(z)
I
Y

HOPF LAPLACE EQUATION

$$\frac{\partial}{\partial \bar{z}}\left(h_z \overline{h_{\bar{z}}}\right) = 0$$



Slits (cracks) propagate along vertical trajectories of the Hopf differential $h_z\overline{h_{\bar z}}\,\mathrm{d}z \otimes \mathrm{d}z$ and terminate inside the domain. They are necessary to lower the conformal modulus. Outside the cracks the energy-minimal map is a harmonic diffeomorphism.

Proof of the Theorem About Annuli

Let $h : \mathbb{A} \to \mathbb{A}^*$ be a homeomorphism between annuli in the Sobolev class $\mathscr{W}^{1,2}(\mathbb{A}, \mathbb{A}^*)$. We view h as a complex valued function. Let us recall the formulas

$$|Dh|^2 = |h_N|^2 + |h_T|^2 \tag{1}$$

and

$$\det Dh = \operatorname{Im}\left(h_T\, \overline{h_N}\right) \leqslant |h_N|\, |h_T| \tag{2}$$

The expanding pair, $\frac{R}{r} \leqslant \frac{R^*}{r^*}$. In other words the target annulus is conformally fatter than the domain. We find a unique number $\omega \leqslant 0$ such that

$$\frac{R}{r} = \frac{R_* + \sqrt{R_*^2 - \omega}}{r_* + \sqrt{r_*^2 - \omega}} \tag{3}$$

Without loss of generality we may assume that

$$R = R_* + \sqrt{R_*^2 - \omega} \quad \text{and} \quad r = r_* + \sqrt{r_*^2 - \omega} \tag{4}$$

We begin with the inequality

$$\left(\frac{|h|\,|h_N|}{\sqrt{|h|^2 - \omega}} - |h_T| \right)^2 \geqslant 0 \tag{5}$$

Equivalently,

$$\begin{aligned}|Dh|^2 &\geqslant \frac{-\omega}{|h|^2-\omega}\,|h_N|^2+\frac{2\,|h|}{\sqrt{|h|^2-\omega}}\,|h_N|\;|h_T|\\ &\geqslant \frac{-\omega}{|h|^2-\omega}\left(\,|h|_N\,\right)^2+\frac{2\,|h|}{\sqrt{|h|^2-\omega}}\;|h_N|\;|h_T|\\ &= -\omega\left\{\left[\log\left(|h|+\sqrt{|h|^2-\omega}\right)\right]_N\right\}^2+\frac{2\,|h|}{\sqrt{|h|^2-\omega}}\;|h_N|\;|h_T|\end{aligned}$$

Here we have used an elementary fact that $|\,h_N\,| \geqslant |\,|h|_N\,|$, equality occurs if and only if $\frac{h_N}{h}$ is a real valued function. Let us integrate this point-wise estimate. For the second term, we have

$$\int_{\mathbb{A}}\frac{2\,|h|}{\sqrt{|h|^2-\omega}}\;|h_N|\;|h_T| \geqslant \int_{\mathbb{A}}\frac{2\,|h|}{\sqrt{|h|^2-\omega}}\,J(z,h)\mathrm{d}z = 4\pi\int_{r_*}^{R_*}\frac{\tau}{\sqrt{\tau^2-\omega}}\,d\tau$$

The first term is estimated by Hölder's inequality

$$\begin{aligned}
\int_{\mathbb{A}} \|Dh\|^2 &\geqslant -\omega \left[\int_{\mathbb{A}} \left[\log\left(|h| + \sqrt{|h|^2 - \omega} \right) \right]_N \frac{dx}{|x|} \right]^2 \cdot \left(\int_{\mathbb{A}} \frac{dx}{|x|^2} \right)^{-1} + \\
&\quad + 4\pi \int_{r_*}^{R_*} \frac{\tau}{\sqrt{\tau^2 - \omega}}\, d\tau \\
&= -\omega \left| 2\pi \log \frac{R_* + \sqrt{R_*^2 - \omega}}{r_* + \sqrt{r_*^2 - \omega}} \right|^2 \left(2\pi \log \frac{R}{r} \right)^{-1} + \\
&\quad + 2\pi \left[\tau \sqrt{\tau^2 - \omega} + \tau \log(\tau + \sqrt{\tau^2 - \omega}) \right]_{\tau = r_*}^{R_*} \\
&= 2\pi R_* \sqrt{R_*^2 - \omega} - 2\pi r_* \sqrt{r_*^2 - \omega} \qquad (6)
\end{aligned}$$

Elementary inspection reveals that equality holds for the Nitsche mapping

$$h(z) = \frac{1}{2}\left(z + \frac{\omega}{\bar{z}}\right)$$

The contracting pair, $\frac{R_*}{r_*} < \frac{R}{r} \leqslant \frac{R_*}{r_*} + \sqrt{\frac{R_*^2}{r_*^2} - 1}$ **(optional).**

In this case the target annulus is conformally thinner than the domain, but not too thin. We express this condition by using the equation (5), where this time $0 < \omega \leqslant r_*^2$. Again we may assume that the relations at (4) hold. The same inequality (5) can be rewritten as

$$\|Dh\|^2 \geqslant \omega \left|\frac{h_T}{h}\right|^2 + 2\,|h_N|\;|h_T|\,\sqrt{1 - \frac{\omega}{|h|^2}} \tag{7}$$

As before, we apply Hölder's inequality to the first term, which together with a similar estimate of the second term (via change of variables) the yields

$$\int_{\mathbb{A}} \|Dh\|^2 \geqslant \omega \left(\int_{\mathbb{A}} \frac{|h_T|}{|x|\,|h|} \right)^2 \cdot \left(\int_{\mathbb{A}} \frac{\mathsf{d}x}{|x|^2} \right)^{-1} + 4\pi \int_{r_*}^{R_*} \sqrt{\tau^2 - \omega}\, d\tau \tag{8}$$

Next, using polar coordinates (or Fubini's Theorem) we split the first integral as to obtain a bound $\geqslant \left(\int_r^R \frac{\mathsf{d}t}{t} \right) \times \left| \int_{|z|=t} \mathsf{d}(\arg h) \right|$. The estimates give

$$\begin{aligned} \int_{\mathbb{A}} \|Dh\|^2 &\geqslant \omega \left(2\pi \log \frac{R}{r} \right)^2 \cdot \left(2\pi \log \frac{R}{r} \right)^{-1} + 4\pi \int_{r_*}^{R_*} \sqrt{\tau - \omega}\, d\tau \\ &= 2\pi\omega \log \frac{R}{r} + 2\pi \Big[\tau\sqrt{\tau^2 - \omega} - \omega \log(\tau + \sqrt{\tau^2 - \omega}) \Big]_{\tau = r_*}^{R_*} \end{aligned} \tag{9}$$

In view of (**??**) it holds

$$\int_{\mathbb{A}} \|Dh\|^2 \geqslant 2\pi R_* \sqrt{R_*^2 - \omega} - 2\pi r_* \sqrt{r_*^2 - \omega} \tag{10}$$

Again equality holds for the Nitsche mapping

$$h(z) = \frac{1}{2}\left(z + \frac{\omega}{\bar{z}}\right)$$

The borderline case. Taking $\omega = r_*^2$ we obtain what is called the critical Nitsche map with $R = R_* + \sqrt{R_*^2 - r_*^2}$ and $r = r_*$,

$$h^*(z) = \frac{1}{2}\left(z + \frac{r_*^2}{\bar{z}}\right) \tag{11}$$

The conformal energy of the critical Nitsche map equals:

$$\int_{\mathbb{A}} \|Dh^*\|^2 = 2\pi R_* \sqrt{R_*^2 - r_*^2} \tag{12}$$

We are now in a position to consider the case:

Below the lower Nitsche bound. $\frac{R_*}{r_*} + \sqrt{\frac{R_*^2}{r_*^2} - 1} < \frac{R}{r}$. The target annulus $\mathbb{A}^*$ is too thin, a portion of $\mathbb{A}$ has to be hammered flat to the inner circle of $\mathbb{A}^*$. We can certainly assume that $R = R_* + \sqrt{R_*^2 - r_*^2}$ which, together with the hypothesis of this case, yields $r < r_*$.

For every permissible map $h : \mathbb{A} \to \mathbb{A}^*$ we still have the estimate (9) with

$\omega = r_*^2$. Hence

$$\begin{aligned}
\int_{\mathbb{A}} \|Dh\|^2 \quad &\geqslant \quad 2\pi r_* \log \frac{R}{r} + 2\pi \left[\tau \sqrt{\tau^2 - r_*} - r_* \log(\tau + \sqrt{\tau^2 - r_*}) \right]_{\tau = r_*}^{R_*} \\
&= \quad 2\pi R_* \sqrt{R_*^2 - r_*^2} \, + \, 2\pi r_*^2 \log \frac{r_*}{r} \\
&= \quad \mathscr{E}_{h^*} + \mathscr{E}_g \, , \qquad g(z) = r_* \frac{z}{|z|} \qquad (13)
\end{aligned}$$

Note that the additional term

$$\mathscr{E}_g = \int_{r<|z|<r_*} \|Dg\|^2 = 2\pi r_*^2 \log \frac{r_*}{r} \qquad (14)$$

is the conformal energy of the hammering map $g : \mathbb{A}(r, r_*) \to \mathbb{S}^1_{r_*}$.

EXISTENCE THEOREM

Theorem. (*Koh, Kovalev, Onninen, T.I.* Invent. Math. 2011) **Among all homeomorphisms** $h\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ **between bounded doubly connected domains such that**

$$\operatorname{Mod} \mathbb{X} \leqslant \operatorname{Mod} \mathbb{Y}$$

there exists an energy-minimal harmonic diffeomorphism.

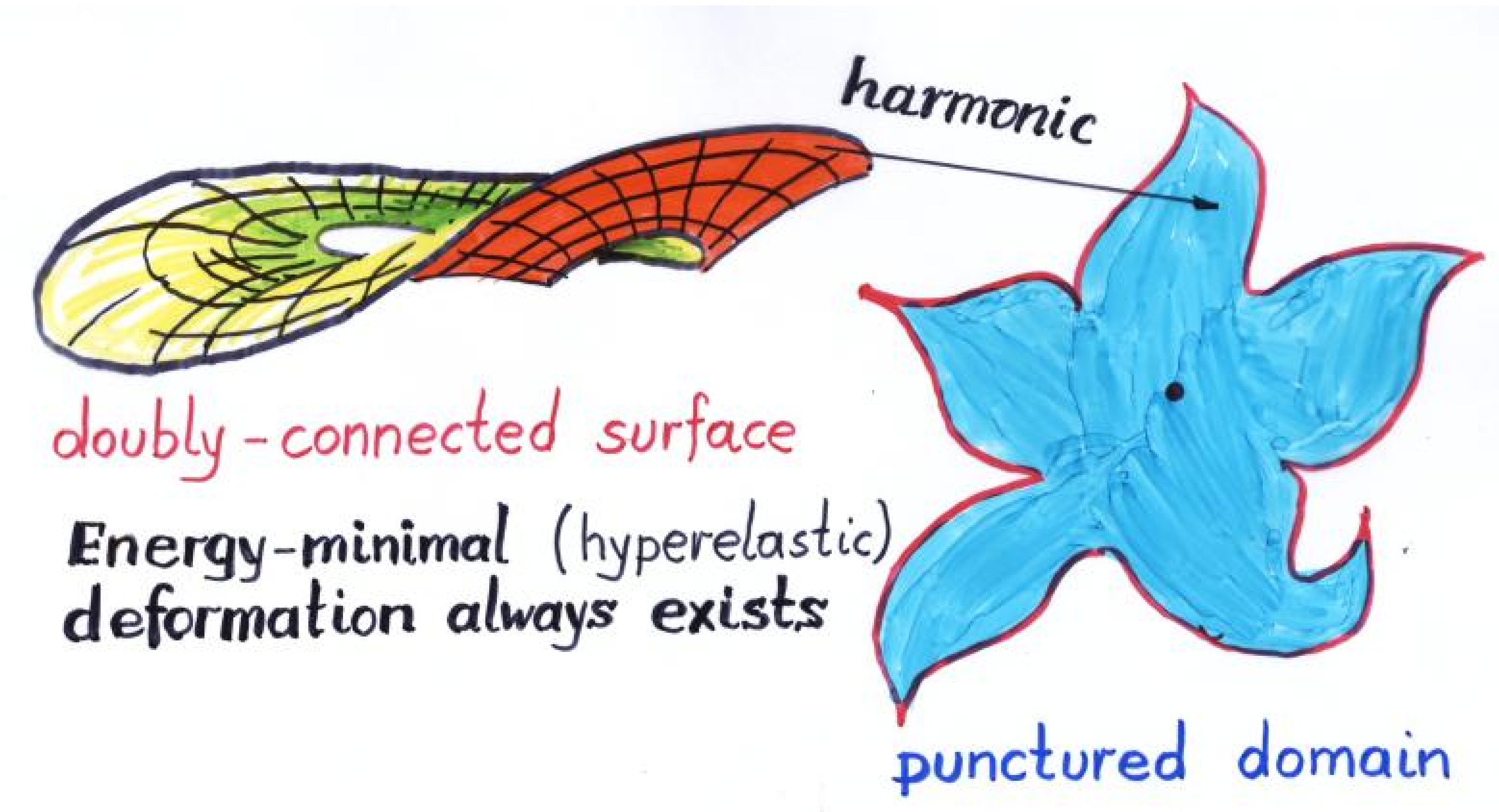
(*unique up to conformal authomorphisms of* $\mathbb{X}$)

Mappings of Smallest Mean Distortion
$\mathscr{L}^1$-variant of the Teichmüller map

$$2\,\|K_f\|_{\mathscr{L}^1(\mathbb{Y})} \;=\; \iint_{\mathbb{Y}} \frac{|Df|^2}{\det Df} \quad \left(= \iint_{\mathbb{X}} |Dh|^2\right)$$

Let $\mathbb{X}$ and $\mathbb{Y}$ be bounded doubly connected domains in $\mathbb{C}$ such that $\operatorname{Mod}\mathbb{X} \leqslant \operatorname{Mod}\mathbb{Y}$. Among all homeomorphisms $f\colon \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ there exists, unique up to a conformal change of variables in $\mathbb{X}$, mapping of smallest $\mathscr{L}^1$-norm of the distortion.

Doubly connected membrane in $\mathbb{R}^3$



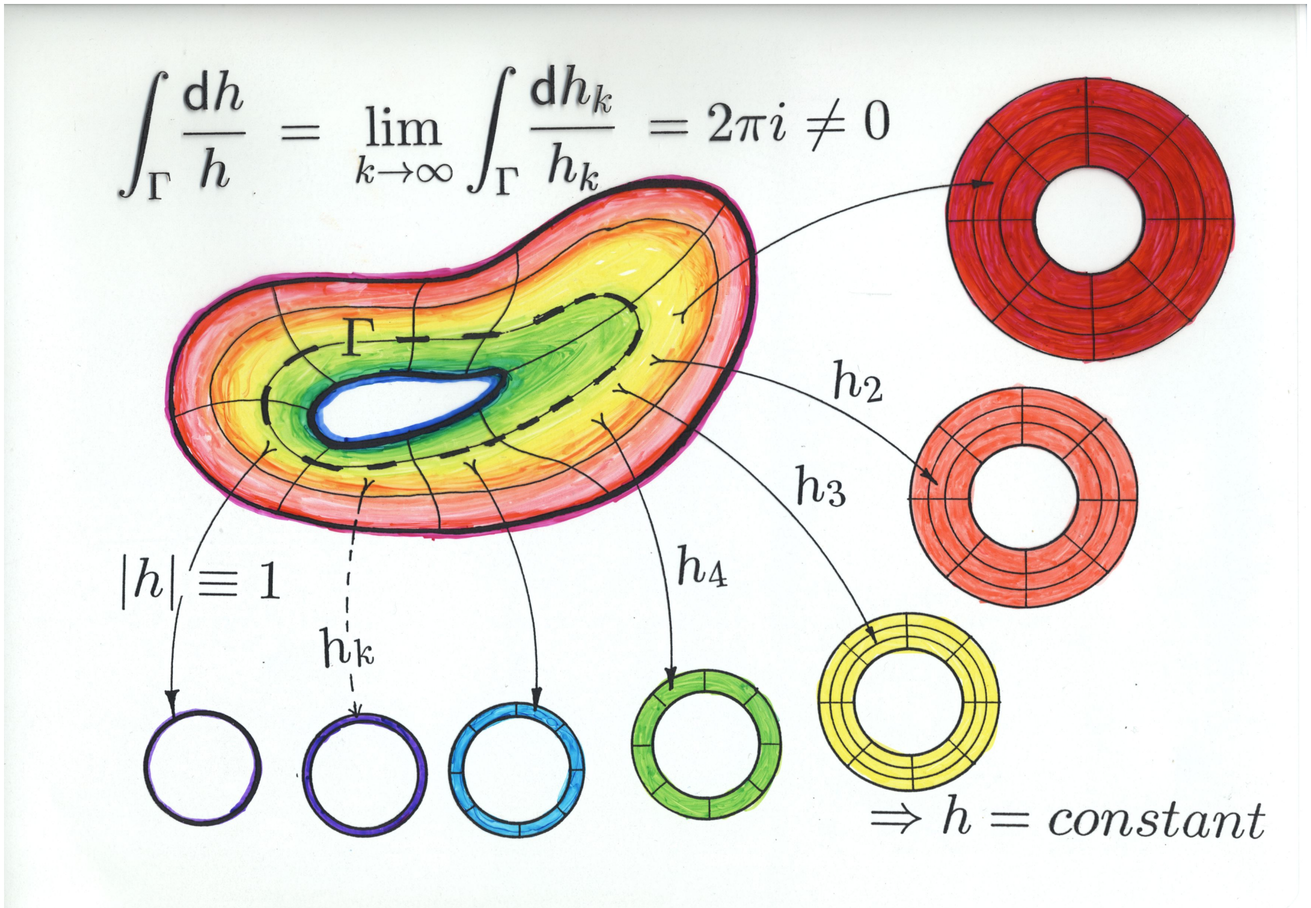
The Nitsche Conjecture

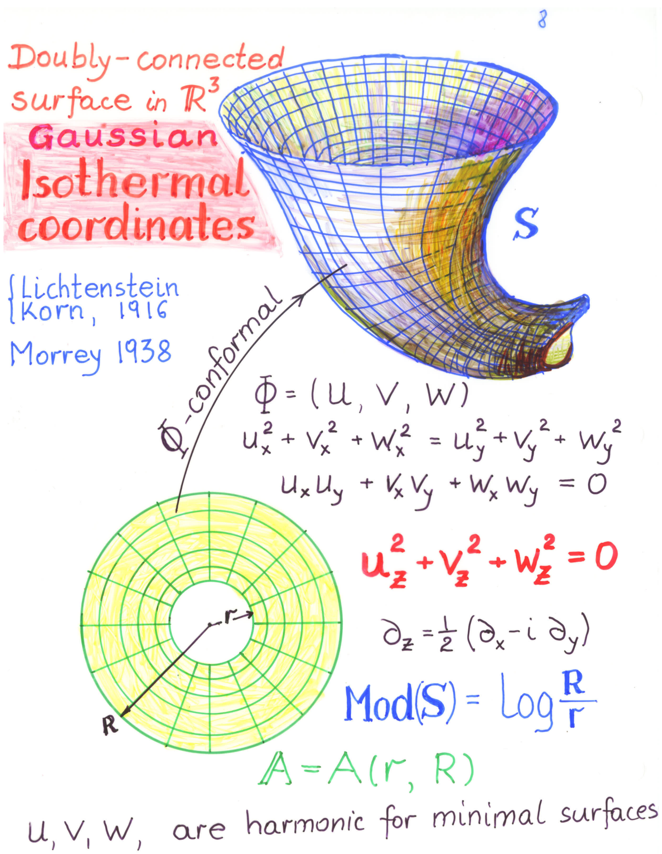
In the above example the energy-minimal deformation fails to be harmonic homeomorphism. Actually there is no harmonic homeomorphism at all. In the early 1960's German-American mathematician Johannes C.C. Nitsche raised a question of existence of harmonic homeomorphisms between annuli. This fascinating problem is deeply rooted in the theory of doubly connected minimal surfaces. Nitsche's conjecture, which is now a theorem (*L. Kovalev, J. Onninen, T.I.*, JAMS 2011), asserts that

Harmonic homeomorphisms do exist iff :

$$\frac{R_*}{r_*} \geqslant \frac{1}{2}\left(\frac{R}{r} + \frac{r}{R}\right)$$

The Normal Family Argument *(target annulus cannot be too thin)*



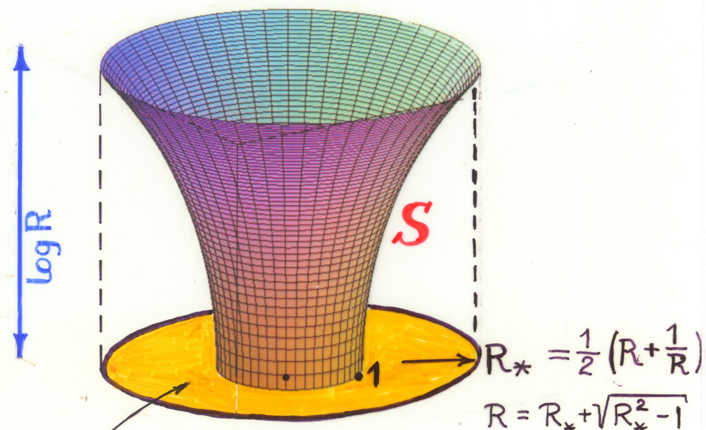


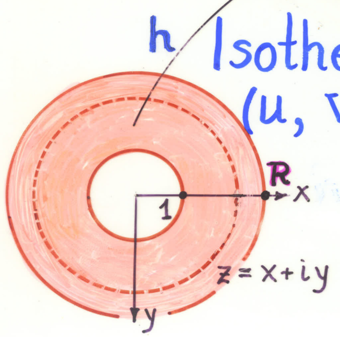
Minimal surfaces are fundamental forms in nature, typically arising as thin elastic films (e.g., as the bubbles that form when a wire frame is dipped into a soap solution).

Catenoid of Greatest Conformal Modulus

Critical Nitsche Map

$$h(z) = \frac{1}{2}\left(z + \frac{1}{z}\right); \quad 1 < |z| < R$$





Isothermal parameters

(u, v, w); $h = u + iv$

$$\log |z| = w$$

$$\log R = \mathrm{Mod}(S)$$

The catenoid is an extremal design configuration for numerous problems in geometry, physics, and engineering. To cite an instance of this, in modern turbines, say as used in power plants or jet engines, the rotor has the shape of a catenoid, for it is thermodynamically the most efficient shape.

Physical Significance of the Nitsche Conjecture

Suppose that we are given two wire loops that form a frame of a soap bubble. As the components of the frame are moved apart, the tension in the spanned surface (in mathematical terms, the conformal modulus of the surface) increases until it reaches a critical point where the bubble pops.

I am delighted by the invitation to the University of Oxford, to contribute to the PDE/CDT Spring Retreat. It has been a pleasure chatting with you. Certainly you will be successful, some even famous. Remember one of the greatest sayings:

It is nice to be important,
but it is more important to be nice

Thank you for listenning
see you at the Mathematical Institute
(April 17-May 22).

Tadeusz