# Local pointwise second derivative estimates for strong solutions to the $\sigma_k$-Yamabe equation on Euclidean domains

Jonah A. J. Duncan*† and Luc Nguyen‡

**Abstract**

We prove local pointwise second derivative estimates for positive $W^{2,p}$ solutions to the $\sigma_k$-Yamabe equation on Euclidean domains, addressing both the positive and negative cases. Generalisations for augmented Hessian equations are also considered.

Keywords: second derivative estimates, $\sigma_k$-Yamabe equation.

MSC: 35B65, 35D35, 35J15, 35J60, 53C21.

## Contents



*Mathematical Institute, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, OX2 6GG, UK. Email: jonah.duncan@maths.ox.ac.uk.

†Supported by EPSRC grant number EP/L015811/1.

‡Mathematical Institute and St Edmund Hall, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, OX2 6GG, UK. Email: luc.nguyen@maths.ox.ac.uk.

# 1 Introduction

Let $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) be a domain. In this paper, we obtain local pointwise second derivative estimates for positive $W^{2,p}$ solutions to the equations

$$\sigma_k^{1/k}(A_u(x)) = f(x, u(x), \nabla u(x)) > 0, \quad \lambda(A_u(x)) \in \Gamma_k^+ \quad \text{for a.e. } x \in \Omega \tag{1.1$^+$}$$

and

$$\sigma_k^{1/k}(-A_u(x)) = f(x, u(x), \nabla u(x)) > 0, \quad \lambda(-A_u(x)) \in \Gamma_k^+ \quad \text{for a.e. } x \in \Omega. \tag{1.1$^-$}$$

Throughout the paper, $A_u$ denotes the symmetric matrix-valued function

$$A_u := \nabla^2 u - \frac{|\nabla u|^2}{2u} I,$$

where $I$ is the $n \times n$ identity matrix and $\sigma_k$ is the $k$'th elementary symmetric polynomial, defined on a symmetric matrix $A$ with eigenvalues $\lambda(A) = (\lambda_1, \dots, \lambda_n)$ by

$$\sigma_k(A) = \sigma_k(\lambda_1, \dots, \lambda_n) := \sum_{1 \leq i_1 < \dots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}.$$

Note that $\sigma_1(A)$ is the trace of $A$ and $\sigma_n(A)$ is the determinant of $A$. We also denote by $\Gamma_k^+$ the open convex cone

$$\Gamma_k^+ = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n : \sigma_j(\lambda_1, \dots, \lambda_n) > 0 \text{ for all } 1 \leq j \leq k\}.$$

It is well-known that the equations ($1.1^{\pm}$) are elliptic. Furthermore, $\sigma_k^{1/k}$ is a concave function on the set of symmetric matrices with eigenvalues in $\Gamma_k^+$.

The motivation behind ($1.1^{\pm}$) comes from conformal geometry: if $g_{ij} = u^{-2}\delta_{ij}$ is a metric conformal to the flat metric on a domain $\Omega \subset \mathbb{R}^n$, then $uA_u$ is the $(1,1)$-Schouten tensor of $g$, and the $\sigma_k$-Yamabe equation in the so-called positive/negative ($\pm$) case is given by

$$\sigma_k(\pm uA_u) = 1, \quad \lambda(\pm A_u) \in \Gamma_k^+, \quad u > 0. \tag{1.2}$$

The equations (1.2) and their counterparts on Riemannian manifolds were first studied by Viaclovsky in [**63**]. Since then, these equations have been addressed by various authors – for a partial list of references, see [**1–3, 8–12, 14, 16–19, 24, 27, 28, 31–33, 35, 39–41, 43, 44, 46, 47, 53, 54, 56, 64, 65**] in the positive case and [**13, 23, 25, 29, 30, 42, 45, 55**] in the negative case. When $k = 1$, these equations reduce to the original Yamabe equation. When $k \geq 2$, they are fully nonlinear and elliptic at a solution (although, a priori, not necessarily uniformly elliptic). Fully nonlinear elliptic equations involving eigenvalues of the Hessian were first considered by Caffarelli, Nirenberg and Spruck in [**6**].

A priori local first and second derivative estimates play an important role in the study of the $\sigma_k$-Yamabe equation, and were established in the positive case by Chen [**14**], Guan and Wang [**27**], Jin, Li and Li [**39**], Li and Li [**40**], Li [**43**] and Wang [**65**]. In the negative case, an a priori (global) $C^1$ estimate is proven by Gursky and Viaclovsky [**30**], but it is unknown whether a priori $C^2$ estimates hold. In this paper, we are concerned with the local regularity of positive $W^{2,p}$ solutions to the equations ($1.1^{\pm}$). More precisely, for $2 \leq k \leq n$ we derive local pointwise boundedness of second derivatives, provided $p > kn/2$ in the positive case and $p > (k+1)n/2$ in the negative case. To simplify the discussion, we do not include the case $k = 1$, in which the equations ($1.1^{\pm}$) are semilinear. We prove:

**Theorem 1.1.** *Let $\Omega$ be a domain in $\mathbb{R}^n$ ($n \geq 3$) and let $f \in C^{1,1}_{\mathrm{loc}}(\Omega \times (0,\infty) \times \mathbb{R}^n)$ be a positive function. Suppose that $2 \leq k \leq n$, $p > kn/2$ and $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ is a positive solution to ($1.1^+$). Then $u \in C^{1,1}_{\mathrm{loc}}(\Omega)$, and for any concentric balls $B_R \subset B_{2R} \Subset \Omega$ we have*

$$\|\nabla^2 u\|_{L^\infty(B_R)} \leq C,$$

*where $C$ is a constant depending only on $n, p, R, f$ and an upper bound for $\|\ln u\|_{W^{2,p}(B_{2R})}$.*

**Theorem 1.2.** *Let $\Omega$ be a domain in $\mathbb{R}^n$ ($n \geq 3$) and let $f \in C^{1,1}_{\mathrm{loc}}(\Omega \times (0,\infty) \times \mathbb{R}^n)$ be a positive function. Suppose that $2 \leq k \leq n$, $p > (k+1)n/2$ and $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ is a positive solution to ($1.1^-$). Then $u \in C^{1,1}_{\mathrm{loc}}(\Omega)$, and for any concentric balls $B_R \subset B_{2R} \Subset \Omega$ we have*

$$\|\nabla^2 u\|_{L^\infty(B_R)} \leq C,$$

*where $C$ is a constant depending only on $n, p, R, f$ and an upper bound for $\|\ln u\|_{W^{2,p}(B_{2R})}$.*

**Remark 1.3.** As noted above, it is unknown whether a priori $C^2$ estimates hold for solutions to the $\sigma_k$-Yamabe equation in the negative case. We also note that for the closely related $\sigma_k$-Loewner-Nirenberg problem, there exist locally Lipschitz but non-differentiable viscosity solutions – see [**45**]. As far as the authors are aware, Theorem 1.2 currently provides the only available local second derivative estimate for solutions to the $\sigma_k$-Yamabe equation in the negative case.

To put things in perspective, we note that our estimates in Theorem 1.1 are closely related to certain analytical aspects in the work of Chang, Gursky and Yang in [**12**]. In [**12**], under natural conformally invariant conditions on a Riemannian 4-manifold $(M^4, g_0)$, the authors established the existence of a metric in the conformal class $[g_0]$ whose Schouten tensor has eigenvalues in $\Gamma_2^+$. An important part of the proof in [**12**] was to obtain $W^{2,s}$ estimates for $4 < s < 5$ on smooth solutions to a one-parameter family of regularised $\sigma_2$-equations (see equation (A.1) in Appendix A) which are uniform with respect to the parameter. This was achieved by first obtaining a uniform $W^{1,4}$ estimate (see Theorem 3.1 in [**12**]), and subsequently carrying out an integrability improvement argument (see Sections 5 and 6 in [**12**]). With the $W^{2,s}$ estimate in hand, the authors then applied a heat flow argument to obtain the desired conformal metric.

**Remark 1.4.** A natural question to ask is whether the heat flow argument in [**12**] can be avoided by instead taking the regularisation parameter directly to zero. One application of Theorem 1.1 above and [**46, Proposition 5.3**] is that this can be achieved when $(M^4, g_0)$ is locally conformally flat. We refer the reader to Appendix A for the details.

Our work is also closely related to the work of Urbas in [**60**], where local pointwise second derivative estimates for $W^{2,p}$ solutions to the $k$-Hessian equation

$$\sigma_k^{1/k}(\nabla^2 u(x)) = f(x) > 0, \quad \lambda(\nabla^2 u(x)) \in \Gamma_k^+$$

were established on domains in $\mathbb{R}^n$. At the heart of Urbas' proof is also an integrability improvement argument, assuming an initial lower bound of $p > kn/2$ (see also [**15, 49, 58, 61, 62**]). By an application of Moser iteration, the $C^{1,1}_{\mathrm{loc}}$ estimate is then obtained. We note that Moser iteration has previously been utilised in the context of the $\sigma_k$-Yamabe equation to establish local boundedness of solutions, see for instance [**21, 22, 33**].

We will in fact prove a more general version of Theorems 1.1 and 1.2, and consider an operator of the form

$$A_H[u] := \nabla^2 u - H[u] \tag{1.3}$$

in place of $\pm A_u$. Here, $H[u](x) = H(x, u(x), \nabla u(x))$ for a given matrix-valued function $H = H(x, z, \xi) \in C^{1,1}_{\mathrm{loc}}(\Omega \times \mathbb{R} \times \mathbb{R}^n\,; \mathrm{Sym}_n(\mathbb{R}))$, where $\mathrm{Sym}_n(\mathbb{R})$ denotes the space of real symmetric $n \times n$ matrices. Rather than (1.1$^\pm$), we consider the equation

$$\sigma_k^{1/k}\big(A_H[u](x)\big) = f(x, u(x), \nabla u(x)) > 0, \quad \lambda(A_H[u](x)) \in \Gamma_k^+ \quad \text{for a.e. } x \in \Omega, \tag{1.4}$$

where $f \in C^{1,1}_{\rm loc}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$.

It is clear that if $u$ satisfies (1.1$^+$) with $u \geq \frac{1}{C} > 0$, then $u$ satisfies (1.4) provided $H(x,z,\xi) = \frac{|\xi|^2}{2z} I$ for $z \geq \frac{1}{C}$. Likewise, if $u \in W^{2,p}_{\rm loc}(\Omega)$ is a solution to (1.1$^-$) with right hand side (RHS) $f$ and $u \geq \frac{1}{C} > 0$, then $w := -u \in W^{2,p}_{\rm loc}(\Omega)$ satisfies (1.4) with RHS $\tilde{f}(x,z,\xi) := f(x,-z,-\xi)$, provided $H(x,z,\xi) = \frac{|\xi|^2}{2z} I$ for $z \leq -\frac{1}{C}$. Therefore, for the purpose of obtaining Theorems 1.1 and 1.2, it will suffice to consider the case that $H$ is a multiple of the identity matrix:

**Theorem 1.5.** *Let $\Omega$ be a domain in $\mathbb{R}^n$ $(n \geq 3)$, $f \in C^{1,1}_{\rm loc}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ a positive function and $H \in C^{1,1}_{\rm loc}(\Omega \times \mathbb{R} \times \mathbb{R}^n\,;\,{\rm Sym}_n(\mathbb{R}))$. Suppose $2 \leq k \leq n$, $p \geq 1$ and $u \in W^{2,p}_{\rm loc}(\Omega)$ is a solution to* (1.4)*, and that one of the following conditions holds:*

1. *$H(x,z,\xi) = H_1(x,z)|\xi|^2 I$ with $H_1 \geq 0$ and $p > \frac{kn}{2}$,*
2. *$H(x,z,\xi) = H_2(x,z,\xi) I$ and $p > \frac{(k+1)n}{2}$.*

*Then $u \in C^{1,1}_{\rm loc}(\Omega)$, and for any concentric balls $B_R \subset B_{2R} \Subset \Omega$ we have*

$$\|\nabla^2 u\|_{L^\infty(B_R)} \leq C, \tag{1.5}$$

*where $C$ is a constant depending only on $n, p, R, f, H$ and an upper bound for $\|u\|_{W^{2,p}(B_{2R})}$.*

**Remark 1.6.** The constant $C$ in (1.5) depends only on $n, p, R$ and upper bounds for $\|u\|_{W^{2,p}(B_{2R})}$, $\|H\|_{C^{1,1}(\Sigma)}$ and $\|\ln f\|_{C^{1,1}(\Sigma)}$, where $\Sigma := \overline{B}_{2R} \times [-M, M] \times \overline{B}_M(0) \subset \Omega \times \mathbb{R} \times \mathbb{R}^n$ and $M \geq \|u\|_{C^1(\overline{B}_{2R})}$. Note that since $p > n$ in Theorem 1.5, an upper bound for $\|u\|_{W^{2,p}(B_{2R})}$ implies an upper bound for $\|u\|_{C^1(\overline{B}_{2R})}$, in light of the Morrey embedding theorem.

**Remark 1.7.** When $H \equiv 0$ and $f = f(x)$, Theorem 1.5 was proved in [**60, Theorem 1.6**].

The matrix $A_H[u]$ introduced in (1.3) is sometimes referred to as an augmented Hessian of $u$. The corresponding augmented Hessian equations have been extensively studied in recent years – see [**36–38**] and the references therein. In this vein, it is therefore of interest to generalise Theorem 1.5 to arbitrary $H \in C^{1,1}_{\rm loc}$. As we will see, the proof of Theorem 1.5 uses some favourable divergence structure in the case that $H$ is a multiple of the identity matrix. However, when $k = 2$, a similar divergence structure holds for general $H$ and we obtain the following:

**Theorem 1.8.** *Let $\Omega$ be a domain in $\mathbb{R}^n$ $(n \geq 3)$, $f \in C^{1,1}_{\rm loc}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ a positive function and $H \in C^{1,1}_{\rm loc}(\Omega \times \mathbb{R} \times \mathbb{R}^n\,;\,{\rm Sym}_n(\mathbb{R}))$. Suppose $p > \frac{3n}{2}$ and $u \in W^{2,p}_{\rm loc}(\Omega)$ is a solution to* (1.4) *with $k = 2$. Then $u \in C^{1,1}_{\rm loc}(\Omega)$, and for any concentric balls $B_R \subset B_{2R} \Subset \Omega$ we have*

$$\|\nabla^2 u\|_{L^\infty(B_R)} \leq C,$$

*where $C$ is a constant depending only on $n, p, R, f, H$ and an upper bound for $\|u\|_{W^{2,p}(B_{2R})}$.*

**Remark 1.9.** In [**36–38**] and the references therein, it is usually assumed that $H$ satisfies a so-called co-dimension one convexity condition, which is known to be a necessary and sufficient condition to obtain $C^1$ estimates – see [**50, 51, 57**]. We point out that we do *not* assume a co-dimension one convexity condition in our treatment of second derivative estimates (the exception is Case 1 of Theorem 1.5, where we have convexity in $\xi$).

Under a stronger assumption on $p$, we will also obtain an extension of Theorem 1.8 to the case $k \geq 3$ – see Section 6.

In adapting the methods of [**60**] to prove Theorems 1.5 and 1.8, we will need to deal with the term $H[u]$ which, whilst being of lower order in the definition of $A_H[u]$, creates new higher order terms in our estimates. Roughly speaking, the two terms which are formally problematic consist of:

**(i)** a contraction of the linearised operator

$$F[u]^{ij} := \frac{\partial \sigma_k(A_H[u])}{\partial (A_H[u])_{ij}} \tag{1.6}$$

with double difference quotients of $H[u]_{ij}$ (this arises as a result of taking difference quotients of (1.4) twice), and

**(ii)** the divergence of $F[u]^{ij}$ multiplied by a term formally of third order in $u$ (this arises after integrating by parts).

In [**60**], neither of these terms exist since $F[u]^{ij}$ is divergence-free when $H \equiv 0$. In the more general case that we are considering, it is unclear whether these third order terms have a favourable sign individually. However, we will estimate them so as to show that, when combined, they yield a cancellation phenomenon that ensures the overall higher order contribution is positive. For the estimates of the higher order terms arising from the divergence of $F[u]^{ij}$, see Lemmas 4.4 and 4.5, and for those arising from the double difference quotients of $H[u]$, see Lemma 4.10. For the resulting cancellation phenomena, see Corollaries 4.12, 4.13 and 4.14.

We close the introduction by noting that in Theorems 1.1 and 1.2, we do not know whether our lower bounds on $p$ to obtain $C^{1,1}_{\mathrm{loc}}$ regularity are sharp, and it would be interesting to determine the sharp lower bounds. In the case of the $k$-Hessian equation for $3 \leq k \leq n$, it is shown by Urbas in [**59**] that there exist $W^{2,p}$-strong solutions with $p < \frac{k(k-1)}{2}$ which fail to be $C^{1,\alpha}_{\mathrm{loc}}$ for any $\alpha > 1 - \frac{2}{k}$. Other lower bounds on $p$ leading to $C^{1,1}_{\mathrm{loc}}$ regularity for $k$-Hessian equations have been studied in [**15, 49, 58, 61, 62**], for instance.

The plan of the paper is as follows. We begin in Section 2 with an outline of the proof of Theorems 1.5 and 1.8. This prompts us to consider the divergence structure of the linearised operator, which we address in Section 3, and also motivate the estimates established from

Section 4 onwards. In Section 4 we carry out the main body of our integral estimates. In Section 5, we use these estimates and the Moser iteration technique to obtain the desired $C^{1,1}_{\rm loc}$ estimates, completing the proofs of Theorems 1.5 and 1.8. In Section 6, we give the aforementioned extension of Theorem 1.8 to the case $k \geq 3$.

# 2 Outline of the proofs of Theorems 1.5 and 1.8

Our proofs of Theorems 1.5 and 1.8 use an integrability improvement argument, from which the $C^{1,1}_{\rm loc}$ estimate is obtained by the Moser iteration technique. In Case 1 of Theorem 1.5, we will obtain, for a solution $u \in W^{2,q+k-1}_{\rm loc}(\Omega)$ to (1.4) with $q > \frac{kn}{2} - k + 1$, the estimate

$$\left(\int_{B_{R+\rho}} (\Delta u + C_1)^{\beta q}\right)^{1/\beta} \leq \frac{Cq}{\rho^2} \int_{B_{R+3\rho}} (\Delta u + C_1)^{q+k-1}, \tag{2.1}$$

where $\rho \in (0, \frac{R}{3}]$, $\beta = \frac{kn}{kn-2k+2}$ and $C_1$ is a positive constant ensuring $\Delta u + C_1 \geq 1$ a.e. (see the paragraph after Remark 2.3 for the justification of the existence of $C_1$). Similarly, in Case 2 of Theorem 1.5 and in Theorem 1.8, we will obtain, for a solution $u \in W^{2,q+k}_{\rm loc}(\Omega)$ to (1.4) with $q > \frac{(k+1)n}{2} - k$, the estimate

$$\left(\int_{B_{R+\rho}} (\Delta u + C_1)^{\beta q}\right)^{1/\beta} \leq \frac{Cq}{\rho^2} \int_{B_{R+3\rho}} (\Delta u + C_1)^{q+k}, \tag{2.2}$$

now with $\beta = \frac{(k+1)n}{(k+1)n-2(k+1)+2}$. The estimates (2.1) and (2.2) then yield an improvement in integrability under the respective lower bounds on $q$, which can then be iterated to yield the desired $C^{1,1}_{\rm loc}$ estimates.[1]

In the rest of this section we explain how the estimates (2.1) and (2.2) are obtained. Due to the lack of regularity, we derive our estimates through taking difference quotients of the equation (1.4). For an index $l \in \{1, \ldots, n\}$ and increment $h \in \mathbb{R}\backslash\{0\}$, we recall the first order difference quotient $\nabla^h_l u(x) := h^{-1}(u(x + he_l) - u(x))$ and the second order difference quotient

$$\Delta^h_{ll} u(x) := \nabla^h_l(\nabla^{-h}_l u(x)) = \frac{u(x + he_l) - 2u(x) + u(x - he_l)}{h^2}. \tag{2.3}$$

We also denote

$$v_h(x) := \sum_{l=1}^n \Delta^h_{ll} u(x).$$

The above expressions are well-defined for $x \in \Omega_h := \{y \in \Omega : \mathrm{dist}(y, \partial\Omega) > |h|\}$.

[1]One might ask whether a reverse Hölder-type inequality for a single second derivative $\nabla_l\nabla_l u$, similar to (2.1) and (2.2), can be established. We have been unable to show this.

It is well-known (see, for instance, [**20, Lemma 7.23**]) that

$$\|\nabla_l^h u\|_{L^s(\Omega')} \leq \|\nabla_l u\|_{L^s(\Omega)} \quad \text{for all } s \geq 1 \text{ and } \Omega' \Subset \Omega \text{ s.t. } \operatorname{dist}(\Omega', \partial\Omega) > |h|. \tag{2.4}$$

It follows from (2.3) and (2.4) that there exists a constant $C = C(n)$ such that

$$\|v_h\|_{L^s(\Omega')} \leq C\|\nabla^2 u\|_{L^s(\Omega)} \quad \text{for all } s \geq 1. \tag{2.5}$$

We will also use the following fact – see Appendix B for a proof:

**Lemma 2.1.** *Suppose $u \in W^{2,s}(\Omega)$ for some $s \geq 1$. Then $v_h \to \Delta u$ in $L^s_{\mathrm{loc}}(\Omega)$ as $h \to 0$.*

We assume now that both the increment $h$ and our solution $u$ are fixed, and write $v$ as shorthand for $v_h$. Taking difference quotients of the equation $\sigma_k^{1/k}(A_H[u](x)) = f[u](x) := f(x, u(x), \nabla u(x))$ and appealing to the concavity of $\sigma_k^{1/k}$ in $\Gamma_k^+$, we will derive (at the start of Section 4) the pointwise estimate

$$\sum_l k(f[u])^{k-1}\Delta_{ll}^h f[u] \leq F[u]^{ij}\nabla_i\nabla_j v - \sum_l F[u]^{ij}\Delta_{ll}^h(H[u])_{ij} \quad \text{a.e. in } \Omega_h. \tag{2.6}$$

Here, $F[u]^{ij} = \partial\sigma_k(A_H[u])/\partial(A_H[u])_{ij}$ is the linearised operator.

**Remark 2.2.** In (2.6), and from this point onwards, summation notation is employed *only over repeated indices which appear in both upper and lower positions.* Positioning of indices is purely to indicate whether summation convention is being utilised; since we are working with the Euclidean metric, we are free to raise and lower indices at will. For instance, $A_{ij}$, $A^i_j$, $A^j_i$ and $A^{ij}$ all denote the $(i,j)$-entry of a symmetric matrix $A$. Similarly, we do not distinguish between the derivatives $\nabla^i$ and $\nabla_i$ when using index notation.

**Remark 2.3.** Since $u$ is fixed, we write $f[u], H[u], A_H[u], F[u]^{ij}$ etc. to emphasise that these are to be considered as functions of $x$. If it is clear from the context (e.g. if there are no derivatives involved), we will simply write $f, H, A_H, F$ etc.

The estimates (2.1) and (2.2) are derived by testing (2.6) against suitable test functions. First fix a ball $B_{2R} \Subset \Omega_h$. Since $\lambda(A_H) \in \Gamma_2^+$ is equivalent to $\operatorname{tr}(A_H) = \Delta u - \operatorname{tr}(H) > 0$ and $\sigma_2(A_H) > 0$, there exists a constant $C_1 \geq 0$ (depending on an upper bound for $\|H\|_{C^0(\Sigma)}$ - see Remark 1.6) for which $\Delta u + C_1 \geq 1$ and $|\nabla^2 u| \leq \Delta u + C_1$ a.e. in $B_{2R}$. We define $\tilde{v} := v + C_1$, and for a small parameter $\delta > 0$ (that we eventually take to zero) we denote

$$Q_\delta := \big((\tilde{v}^+)^2 + \delta^2\big)^{1/2}.$$

For $\rho \in (0, \frac{R}{3}]$ we also let $\eta \in C_c^\infty(B_{R+2\rho})$ be a standard non-negative cutoff function. Testing (2.6) against $\eta Q_\delta^{q-1}$ (where $q > 1$) then yields

$$\sum_l \int_{B_{R+2\rho}} k\eta Q_\delta^{q-1} f^{k-1}\Delta_{ll}^h f[u] \leq \int_{B_{R+2\rho}} \eta Q_\delta^{q-1} F^{ij}\nabla_i\nabla_j\tilde{v} - \sum_l \int_{B_{R+2\rho}} \eta Q_\delta^{q-1} F^{ij}\Delta_{ll}^h(H[u])_{ij} \tag{2.7}$$

for all $q > 1$ and $u \in W^{2,q+k-1}_{\rm loc}(\Omega) \cap W^{1,\infty}_{\rm loc}(\Omega)$ solving (1.4).

For ease of outlining our argument, let us suppose that $f = f(x,z)$ (the general case $f = f(x,z,\xi)$ will only require minor changes - see Section 5.3). Then the integrand on the left hand side (LHS) of (2.7) is a lower order term, whereas the integrands on the RHS of (2.7) involve higher order terms, formally of fourth and third order in the limit $h \to 0$, and thus need to be treated.

In Section 4, we integrate by parts in the first integral on the RHS of (2.7), using a result of Section 3 that tells us $\nabla_i F[u]^{ij}$ is a regular distribution belonging to $L^{(q+k-1)/(k-1)}_{\rm loc}(\Omega)$ if $u \in W^{2,q+k-1}_{\rm loc}(\Omega) \cap W^{1,\infty}_{\rm loc}(\Omega)$. After taking $\delta \to 0$ and carrying out some further calculations (see Lemmas 4.2 and 4.3), we will obtain the estimate

$$\begin{aligned}\frac{q-1}{Cq^2}\int_{B_{R+\rho}} f^k \frac{\big|\nabla\big((\tilde{v}^+)^{q/2}\big)\big|^2}{\mathrm{tr}(A_H)} + \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\nabla_i F[u]^{ij}\nabla_j \tilde{v} \\ + \sum_l \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}F^{ij}\Delta^h_{ll}(H[u])_{ij} \\ \leq \frac{C}{\rho^2}\bigg(\int_{B_{R+2\rho}}(\tilde{v}^+)^{q+k-1} + \int_{B_{R+3\rho}}(\Delta u + C_1)^{q+k-1}\bigg),\end{aligned} \tag{2.8}$$

where $C$ is a constant independent of $h, q$ and $\rho$.

Whilst the first integral on the LHS of (2.8) is a favourable positive higher order term, the other two integrals on the LHS (which we denote by $(\mathrm{I}_2)_h$ and $(\mathrm{I}_3)_h$, respectively) involve higher order terms which are, a priori, of unknown sign. Treating $(\mathrm{I}_2)_h$ and $(\mathrm{I}_3)_h$ is the most technical part of our proof.

Now, if we momentarily assume sufficiently high regularity on $u$, say $u \in W^{2,q+2k-1}_{\rm loc}(\Omega) \cap W^{1,\infty}_{\rm loc}(\Omega)$ ($q > 1$), the issue of dealing with $(\mathrm{I}_2)_h$ and $(\mathrm{I}_3)_h$ is largely simplified. As will be detailed in the proof of Theorem 6.1, one may apply the Cauchy inequality to each of the integrands and absorb the resulting third order terms into the positive term on the LHS of (2.8). Under the stated integrability assumption, this crude estimation is sufficient to show

$$\frac{q-1}{q^2}\int_{B_{R+\rho}} f^k \frac{\big|\nabla\big((\tilde{v}^+)^{q/2}\big)\big|^2}{\mathrm{tr}(A_H)} \leq \frac{C}{\rho^2}\bigg(\int_{B_{R+2\rho}}(\tilde{v}^+)^{q+2k-1} + \int_{B_{R+3\rho}}(\Delta u + C_1)^{q+2k-1}\bigg).$$

An estimate analogous to (2.1) and (2.2) can then be obtained, assuming $q > kn - 2k + 1$.

The difficulty is to therefore deal with $(\mathrm{I}_2)_h$ and $(\mathrm{I}_3)_h$ under the *weaker* integrability assumptions of Theorems 1.5 and 1.8. At this point, we make the distinction between the various cases. In each case, we estimate $(\mathrm{I}_2)_h$ and $(\mathrm{I}_3)_h$ so as to produce a cancellation phenomenon when combined, leaving only lower order terms; see Lemmas 4.4 and 4.5 for the estimates on $(\mathrm{I}_2)_h$, Lemma 4.10 for the estimates on $(\mathrm{I}_3)_h$, and Corollaries 4.12, 4.13 and 4.14 for the resulting cancellations. It will then follow from (2.8) that, in Case 1 of Theorem

1.5 with the relaxed assumption $u \in W^{2,q+k-1}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ $(q > 1)$, we have the estimate

$$\frac{q-1}{q^2}\int_{B_{R+\rho}} f^k \frac{\big|\nabla\big((\tilde{v}^+)^{q/2}\big)\big|^2}{\mathrm{tr}(A_H)} \leq \frac{C}{\rho^2}\bigg(\int_{B_{R+2\rho}} (\tilde{v}^+)^{q+k-1} + \int_{B_{R+3\rho}} (\Delta u + C_1)^{q+k-1}\bigg). \tag{2.9}$$

Similarly, in the remaining cases with $u \in W^{2,q+k}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ $(q > 1)$, we will obtain

$$\frac{q-1}{q^2}\int_{B_{R+\rho}} f^k \frac{\big|\nabla\big((\tilde{v}^+)^{q/2}\big)\big|^2}{\mathrm{tr}(A_H)} \leq \frac{C}{\rho^2}\bigg(\int_{B_{R+2\rho}} (\tilde{v}^+)^{q+k} + \int_{B_{R+3\rho}} (\Delta u + C_1)^{q+k}\bigg). \tag{2.10}$$

To obtain (2.1) from (2.9) (resp. (2.2) from (2.10)), we proceed as follows (the details can be found in Section 5). We first obtain an integral estimate for $\big|\nabla\big((\tilde{v}^+)^{q/2}\big)\big|^2$, to which we can apply the Sobolev inequality. We then justify taking the limit $h \to 0$ and impose the lower bound $q + k - 1 > \frac{kn}{2}$ (resp. $q + k > \frac{(k+1)n}{2}$), from which we obtain (2.1) (resp. (2.2)).

# 3 Divergence structure of the linearised operator $F[u]^{ij}$

In this section we derive a divergence formula for the linearised operator $F[u]^{ij}$ (defined in (1.6)), which we will use at various stages of our proof.

We note that in the case that $A_H[u] = \nabla^2 u$ or $A_H[u] = A_u$, the divergence properties of $F[u]^{ij}$ are well-documented (for smooth $u$). In the former case, $F[u]^{ij}$ is divergence-free with respect to the flat metric (see [**52**]), and in the latter case, $u^{1-k}F[u]^{ij}$ is divergence-free with respect to the conformal metric $g_{ij} = u^{-2}\delta_{ij}$ (see [**63**]). For related discussions, see also [**4, 5, 26, 34, 53**].

For $A \in \mathrm{Sym}_n(\mathbb{R})$ and $1 \leq k \leq n$, define the $k$'th Newton tensor of $A$ inductively by

$$T_k(A) := \sigma_k(A)I - T_{k-1}(A)A, \quad T_0(A)^{ij} := \delta^{ij}. \tag{3.1}$$

It is well-known (see [**52**]) that

$$\frac{\partial \sigma_k(A)}{\partial A_{ij}} = T_{k-1}(A)^{ij} \tag{3.2}$$

and

$$\mathrm{tr}(T_k(A)) = (n-k)\sigma_k(A), \tag{3.3}$$

and moreover $T_{k-1}(A)^{ij}$ is positive definite when $\lambda(A) \in \Gamma_k^+$ (see [**6**]). In particular, by (1.6) and (3.2), $F[u]^{ij} = T_{k-1}(A_H[u])^{ij}$.

**Lemma 3.1.** *Let $\Omega \subset \mathbb{R}^n$ be a domain and $u \in C^3(\Omega)$. Then for $H \in C^1(\Omega \times \mathbb{R} \times \mathbb{R}^n\,; \mathrm{Sym}_n(\mathbb{R}))$ and $2 \leq k \leq n$,*

$$\nabla_i F[u]^{ij} = \sum_{p=1}^{k-1}(-1)^{p+1}T_{k-p-1}(A_H)^{ab}\Big(\nabla_a(H[u])_b^c - \nabla^c(H[u])_{ab}\Big)(A_H^{p-1})_c^j =: V[u]^j. \tag{3.4}$$

*Moreover, if* $H(x,z,\xi) = H_2(x,z,\xi)I$*, then*

$$\nabla_i F[u]^{ij} = -(n-k+1)\nabla_i (H_2[u])\, T_{k-2}(A_H)^{ij}. \tag{3.5}$$

*Proof.* The identity (3.4) will follow once we show that for $1 \le k \le n-1$,

$$\nabla_i T_k(A_H[u])^{ij} = \sum_{p=1}^{k} (-1)^{p+1} T_{k-p}(A_H)^{ab} \Big(\nabla_a (H[u])_b^c - \nabla^c (H[u])_{ab}\Big)(A_H^{p-1})_c^j \quad \text{in } \Omega. \tag{3.6}$$

Similarly, (3.5) will follow once we show that for $1 \le k \le n-1$ and $H(x,z,\xi) = H_2(x,z,\xi)I$,

$$\nabla_i T_k(A_H[u])^{ij} = -(n-k)\nabla_i (H_2[u])\, T_{k-1}(A_H)^{ij} \quad \text{in } \Omega. \tag{3.7}$$

To this end, we take the divergence of both sides in (3.1), which yields

$$\begin{aligned}
\nabla_i T_k(A_H[u])^{ij} &= \nabla^j \sigma_k(A_H[u]) - \nabla_i \big(T_{k-1}(A_H[u])^{il} (A_H[u])_l^j\big) \\
&= \frac{\partial \sigma_k(A_H)}{\partial (A_H)_{il}} \nabla^j (A_H[u])_{il} - \nabla_i (T_{k-1}(A_H[u]))^{il} (A_H)_l^j - T_{k-1}(A_H)^{il} \nabla_i (A_H[u])_l^j \\
&\overset{(3.2)}{=} T_{k-1}(A_H)^{il} \big(\nabla^j (A_H[u])_{il} - \nabla_i (A_H[u])_l^j\big) - \nabla_i (T_{k-1}(A_H[u]))^{il} (A_H)_l^j \\
&= T_{k-1}(A_H)^{il} \big(\nabla_i (H[u])_l^j - \nabla^j (H[u])_{il}\big) - \nabla_i \big(T_{k-1}(A_H[u])\big)^{il} (A_H)_l^j.
\end{aligned} \tag{3.8}$$

Then (3.6) is readily seen by applying (3.8) iteratively.

We now turn to (3.7), for which we apply an induction argument on $k$ using (3.8). The base case $k = 1$ is clear. We suppose that for some $k \ge 2$ we have the identity

$$\nabla_i T_{k-1}(A_H[u])^{ij} = -(n-k+1)\nabla_i (H_2[u])\, T_{k-2}(A_H)^{ij}, \tag{3.9}$$

and we show that (3.7) then follows. First observe that, by (3.9) and the fact $H_{ij} = H_2 \delta_{ij}$, (3.8) simplifies to

$$\begin{aligned}
\nabla_i T_k(A_H[u])^{ij} &= \nabla_i (H_2[u]) T_{k-1}(A_H)^{ij} - \nabla^j (H_2[u])\, \mathrm{tr}(T_{k-1}(A_H)) \\
&\quad + (n-k+1)\nabla_i (H_2[u]) (T_{k-2}(A_H) A_H)^{ij}.
\end{aligned} \tag{3.10}$$

After substituting (3.1) and (3.3) into the last term and the penultimate term in (3.10), respectively, we arrive at (3.7). □

Note that $V[u]^j$ (defined in (3.4)) contains at most second order derivatives of $u$. As a consequence, $\nabla_i F[u]^{ij}$ is a regular distribution for $u \in W^{2,q+k-1}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$. More precisely, we have:

**Lemma 3.2.** *Let $\Omega \subset \mathbb{R}^n$ be a domain and $u \in W^{2,q+k-1}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ with $q > 1$ and $2 \le k \le n$. Then for $H \in C^{0,1}_{\mathrm{loc}}(\Omega \times \mathbb{R} \times \mathbb{R}^n; \mathrm{Sym}_n(\mathbb{R}))$ and $\varphi \in W^{1,s}_0(\Omega; \mathbb{R}^n)$, $s := \frac{q+k-1}{q}$, we have*

$$\int_\Omega F[u]^{ij} \nabla_i \varphi_j = -\int_\Omega V[u]^j \varphi_j, \tag{3.11}$$

*where $V[u]^j$ is defined in* (3.4). *In particular, $\nabla_i F[u]^{ij} = V[u]^j \in L^{(q+k-1)/(k-1)}_{\mathrm{loc}}(\Omega)$ and*

$$\big|\nabla_i F[u]^{ij}\big| \le C\big(1 + |\nabla^2 u|^{k-1}\big) \quad \text{a.e. in } B_{2R}, \tag{3.12}$$

*where $C$ is a constant depending on an upper bound for $\|H\|_{C^{0,1}(\Sigma)}$.*

*Proof.* It is clear that $u \in W^{2,q+k-1}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ implies $V[u]^j \in L^{(q+k-1)/(k-1)}_{\mathrm{loc}}(\Omega)$. Since $\frac{1}{s} + \frac{k-1}{q+k-1} = 1$, it suffices to prove (3.11) for $\varphi \in C^\infty_0(\Omega; \mathbb{R}^n)$. Let $u_{(m)} \in C^3(\Omega)$ be such that $u_{(m)} \to u$ in $W^{2,q+k-1}_{\mathrm{loc}}(\Omega)$. Then by (3.4), we have for each $m \in \mathbb{N}$ the identity $\nabla_i F[u_{(m)}]^{ij} = V[u_{(m)}]^j$, and it follows that

$$\int_\Omega F[u_{(m)}]^{ij} \nabla_i \varphi_j = -\int_\Omega V[u_{(m)}]^j \varphi_j. \tag{3.13}$$

Now, since $u_{(m)} \to u$ in $W^{2,q+k-1}_{\mathrm{loc}}(\Omega)$, we have both $F[u_{(m)}] \to F[u]$ and $V[u_{(m)}] \to V[u]$ in $L^{(q+k-1)/(k-1)}_{\mathrm{loc}}(\Omega)$. In particular, we can take $m \to \infty$ in (3.13) to get (3.11). The estimate (3.12) follows from the definition of $V[u]^j$. □

# 4 Main estimates

## 4.1 Initial integral estimates: isolating higher order terms

The following lemma provides the starting point for our integral estimates:

**Lemma 4.1.** *Suppose $f \in C^0(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ is positive, $H \in C^0(\Omega \times \mathbb{R} \times \mathbb{R}^n; \mathrm{Sym}_n(\mathbb{R}))$ and $u$ is a solution to* (1.4). *Then for fixed $h$,*

$$\sum_l k f^{k-1} \Delta^h_{ll} f[u] \le F^{ij} \nabla_i \nabla_j v - \sum_l F^{ij} \Delta^h_{ll} (H[u])_{ij} \quad \text{a.e. in } \Omega_h. \tag{4.1}$$

*Proof.* The proof follows [**60**], with some adjustments. For $A \in \mathrm{Sym}_n(\mathbb{R})$, let $G^{ij}(A) = \partial \sigma_k^{1/k}(A)/\partial A_{ij} = k^{-1} \sigma_k(A)^{(1-k)/k} F^{ij}(A)$, and denote $G^{ij} := G^{ij}(A_H[u])$. Fix $l \in \{1, \dots, n\}$ and $h \in \mathbb{R} \backslash \{0\}$. Then there exists a set $S_{h,l} \subset \Omega_h$ with $\mathcal{L}(\Omega_h \backslash S_{h,l}) = 0$ (where $\mathcal{L}$ is the Lebesgue measure) such that $\lambda(A_H[u](x)), \lambda(A_H[u](x \pm h e_l)) \in \Gamma_k^+$ for all $x \in S_{h,l}$. By concavity of $\sigma_k^{1/k}$ in $\Gamma_k^+$, it follows that for $x \in S_{h,l}$ we have

$$\sigma_k^{1/k}\big(A_H[u](x \pm h e_l)\big) - \sigma_k^{1/k}(A_H[u](x)) \le G^{ij}(x) \big(A_H[u](x \pm h e_l) - A_H[u](x)\big)_{ij}. \tag{4.2}$$

Adding the two equations in (4.2), dividing through by $h^2$ and summing over $l$, we have

$$\sum_l \Delta_{ll}^h \sigma_k^{1/k}\big(A_H[u](x)\big) \leq \sum_l G^{ij}(x)\Delta_{ll}^h\big(A_H[u](x)\big)_{ij} \quad \text{for all } x \in S_h := \bigcap_{l=1}^n S_{h,l}, \tag{4.3}$$

with $S_h$ clearly satisfying $\mathcal{L}(\Omega_h \backslash S_h) = 0$. Substituting the definition of $G^{ij}$ into (4.3) and recalling that $A_H[u] = \nabla^2 u - H[u]$, we obtain

$$\sum_l k\sigma_k^{\frac{k-1}{k}}(A_H)\Delta_{ll}^h \sigma_k^{1/k}(A_H[u]) \leq \sum_l F^{ij}\Delta_{ll}^h\big(\nabla^2 u - H[u]\big)_{ij} \quad \text{in } S_h. \tag{4.4}$$

Substituting the equation $\sigma_k^{1/k}(A_H) = f$ into the LHS of (4.4), and commuting difference quotients with derivatives on the RHS of (4.4), we arrive at (4.1). □

As outlined in Section 2, we proceed to derive a series of integral estimates by multiplying (4.1) by suitable test functions and integrating by parts using the divergence structure proved in Lemma 3.2. Recall that for a fixed increment $h > 0$, we defined $v(x) = \sum_l \Delta_{ll}^h u(x)$, and that we fixed a ball $B_{2R} \Subset \Omega_h$ and a constant $C_1$ (depending on an upper bound for $\|H\|_{C^0(\Sigma)}$) such that $\Delta u + C_1 \geq 1$ and $|\nabla^2 u| \leq \Delta u + C_1$ a.e. in $B_{2R}$. The existence of such a constant is guaranteed by the assumption $\lambda(A_H) \in \Gamma_2^+$. We then defined $\tilde{v} = v + C_1$, and for a small parameter $\delta > 0$ (that we eventually take to zero) we defined $Q_\delta = \big((\tilde{v}^+)^2 + \delta^2\big)^{1/2}$. For $\rho \in (0, \frac{R}{3}]$, we also fix a cutoff function $\eta \in C_c^\infty(B_{R+2\rho})$ satisfying $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B_{R+\rho}$ and $|\nabla^l \eta| \leq C(n)\rho^{-l}$ for $l = 1, 2$.

Suppose $u \in W_{\text{loc}}^{2,q+k-1}(\Omega) \cap W_{\text{loc}}^{1,\infty}(\Omega)$ $(q > 1)$ is a solution to (1.4). Multiplying (4.1) by $\eta Q_\delta^{q-1}$ and integrating over the domain $B_{R+2\rho}$, we see

$$\sum_l \int_{B_{R+2\rho}} k\eta Q_\delta^{q-1} f^{k-1}\Delta_{ll}^h f[u] \leq \int_{B_{R+2\rho}} \eta Q_\delta^{q-1} F^{ij}\nabla_i\nabla_j \tilde{v} - \sum_l \int_{B_{R+2\rho}} \eta Q_\delta^{q-1} F^{ij}\Delta_{ll}^h (H[u])_{ij}, \tag{4.5}$$

which is just the estimate (2.7) in Section 2, repeated here for convenience.

We are now in a position to prove our first integral estimate. In what follows, let

$$J_h^{(s)} := \int_{B_{R+2\rho}} (\tilde{v}^+)^s + \int_{B_{R+3\rho}} (\Delta u + C_1)^s.$$

Roughly speaking, if $u \in W_{\text{loc}}^{2,s}(\Omega)$ then $J_h^{(s)}$ should be interpreted as a lower order term, and terms bounded by $J_h^{(s)}$ are consequently considered ‘good terms’.

We will first address the case $f = f(x,z)$ for simplicity and postpone the more general case until Section 5.3. The relevant equation is therefore

$$\sigma_k^{1/k}\big(A_H[u](x)\big) = f(x, u(x)) > 0, \quad \lambda(A_H[u](x)) \in \Gamma_k^+ \quad \text{for a.e. } x \in \Omega. \tag{4.6}$$

Throughout Section 4, unless otherwise stated, $C$ will denote a generic positive constant which may vary from line to line, depending only on $n, R, f, H$ and an upper bound for $\|u\|_{W^{1,\infty}(B_{2R})}$. In particular, $C$ is independent of $h$, $q$ and $\rho$, and any norm of $\nabla^2 u$. In addition, we will often use the inequalities $\Delta u + C_1 \geq 1$ and $|\nabla^2 u| \leq \Delta u + C_1$ without explicit reference.

**Lemma 4.2.** *Suppose $f \in C^{1,1}_{\mathrm{loc}}(\Omega \times \mathbb{R})$ is positive, $H \in C^{0,1}_{\mathrm{loc}}(\Omega \times \mathbb{R} \times \mathbb{R}^n\,;\mathrm{Sym}_n(\mathbb{R}))$ and $u \in W^{2,q+k-1}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ ($q > 1$) is a solution to (4.6). Then for $R > 0$ with $B_{2R} \Subset \Omega$, $\rho \in (0, \frac{R}{3}]$ and $|h|$ sufficiently small, we have*

$$(q-1)\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-2}F^{ij}\nabla_i\tilde{v}\nabla_j\tilde{v} + \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\nabla_i F[u]^{ij}\nabla_j\tilde{v} + \sum_l \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}F^{ij}\Delta^h_{ll}(H[u])_{ij} \leq C\rho^{-2}J_h^{(q+k-1)}. \quad (4.7)$$

*Proof of Lemma 4.2.* Appealing to Lemma 3.2 with $\varphi_j = \eta Q_\delta^{q-1}\nabla_j\tilde{v}$, and noting that

$$\nabla_i\varphi_j = Q_\delta^{q-1}\nabla_i\eta\nabla_j\tilde{v} + (q-1)\tilde{v}^+Q_\delta^{q-3}\nabla_i\tilde{v}\nabla_j\tilde{v} + \eta Q_\delta^{q-1}\nabla_i\nabla_j\tilde{v},$$

we have

$$\int_{B_{R+2\rho}} F^{ij}\Big(Q_\delta^{q-1}\nabla_i\eta\nabla_j\tilde{v} + (q-1)\tilde{v}^+Q_\delta^{q-3}\nabla_i\tilde{v}\nabla_j\tilde{v} + \eta Q_\delta^{q-1}\nabla_i\nabla_j\tilde{v}\Big) = -\int_{B_{R+2\rho}} \eta Q_\delta^{q-1}\nabla_i F[u]^{ij}\nabla_j\tilde{v}. \quad (4.8)$$

Rearranging (4.8) to get the desired integration by parts formula for $\int_{B_{R+2\rho}} \eta Q_\delta^{q-1}F^{ij}\nabla_i\nabla_j\tilde{v}$, and substituting this back into (4.5), we obtain

$$(q-1)\int_{B_{R+2\rho}} \eta\tilde{v}^+Q_\delta^{q-3}F^{ij}\nabla_i\tilde{v}\nabla_j\tilde{v} + \int_{B_{R+2\rho}} \eta Q_\delta^{q-1}\nabla_i F[u]^{ij}\nabla_j\tilde{v} + \sum_l \int_{B_{R+2\rho}} \eta Q_\delta^{q-1}F^{ij}\Delta^h_{ll}(H[u])_{ij} \leq -\int_{B_{R+2\rho}} Q_\delta^{q-1}F^{ij}\nabla_i\eta\nabla_j\tilde{v} - \sum_l \int_{B_{R+2\rho}} k\eta Q_\delta^{q-1}f^{k-1}\Delta^h_{ll}f[u]. \quad (4.9)$$

We now take $\delta \to 0$ in (4.9), using Fatou's lemma for the first integral (which is positive) and the dominated convergence theorem elsewhere (which is justified since $q > 1$). This yields

$$
\begin{aligned}
(q-1)\int_{B_{R+2\rho}} \eta(\tilde v^+)^{q-2}F^{ij}\nabla_i\tilde v\nabla_j\tilde v + \int_{B_{R+2\rho}} \eta(\tilde v^+)^{q-1}\nabla_i F[u]^{ij}\nabla_j\tilde v \\
+\sum_l \int_{B_{R+2\rho}} \eta(\tilde v^+)^{q-1}F^{ij}\Delta^h_{ll}(H[u])_{ij} \leq -\int_{B_{R+2\rho}} (\tilde v^+)^{q-1}F^{ij}\nabla_i\eta\nabla_j\tilde v \\
-\sum_l \int_{B_{R+2\rho}} k\eta(\tilde v^+)^{q-1}f^{k-1}\Delta^h_{ll}f[u]. \qquad (4.10)
\end{aligned}
$$

To conclude the proof of Lemma 4.2, we must bound the RHS of (4.10) from above by $C\rho^{-2}J_h^{(q+k-1)}$. We begin with the first integral on the RHS of (4.10). Appealing again to Lemma 3.2, now with $\varphi_j = \frac{1}{q}(\tilde v^+)^q\nabla_j\eta$ and $\nabla_i\varphi_j = (\tilde v^+)^{q-1}\nabla_i\tilde v\nabla_j\eta + \frac{1}{q}(\tilde v^+)^q\nabla_i\nabla_j\eta$, we have

$$
\int_{B_{R+2\rho}} F^{ij}\Big((\tilde v^+)^{q-1}\nabla_i\eta\nabla_j\tilde v + \frac{1}{q}(\tilde v^+)^q\nabla_i\nabla_j\eta\Big) = -\frac{1}{q}\int_{B_{R+2\rho}} (\tilde v^+)^q\nabla_i F[u]^{ij}\nabla_j\eta.
$$

Therefore,

$$
\begin{aligned}
\bigg|\int_{B_{R+2\rho}} (\tilde v^+)^{q-1}F^{ij}\nabla_i\eta\nabla_j\tilde v\bigg| &\leq \bigg|\frac{1}{q}\int_{B_{R+2\rho}} (\tilde v^+)^q F^{ij}\nabla_i\nabla_j\eta\bigg| + \bigg|\frac{1}{q}\int_{B_{R+2\rho}} (\tilde v^+)^q\nabla_i F[u]^{ij}\nabla_j\eta\bigg| \\
&\leq \frac{C}{\rho^2}\int_{B_{R+2\rho}} (\tilde v^+)^q|F| + \frac{C}{\rho}\int_{B_{R+2\rho}} (\tilde v^+)^q\big|\mathrm{div}\, F[u]\big|. \qquad (4.11)
\end{aligned}
$$

Recalling $|F| \leq C(\Delta u + C_1)^{k-1}$ and applying Hölder's inequality to the penultimate integral in (4.11), we see that $\int_{B_{R+2\rho}}(\tilde v^+)^q|F| \leq CJ_h^{(q+k-1)}$. The final integral in (4.11) satisfies the same estimate, since $|\,\mathrm{div}\, F[u]| \leq C(\Delta u + C_1)^{k-1}$ by (3.12).

It remains to estimate the second term on the RHS of (4.10). Keeping in mind that $f = f(x,z) \in C^{1,1}_{\mathrm{loc}}(\Omega\times\mathbb{R})$, we apply Hölder's inequality followed by (2.5) to obtain

$$
\begin{aligned}
\bigg|\sum_l\int_{B_{R+2\rho}} k\eta(\tilde v^+)^{q-1}f^{k-1}\Delta^h_{ll}f[u]\bigg| &\leq C\bigg(\int_{B_{R+2\rho}} (\tilde v^+)^q\bigg)^{\frac{q-1}{q}}\bigg(\int_{B_{R+2\rho}}\bigg|\sum_l \Delta^h_{ll}f[u]\bigg|^q\bigg)^{\frac{1}{q}} \\
&\overset{(2.5)}{\leq} C\bigg(\int_{B_{R+2\rho}} (\tilde v^+)^q\bigg)^{\frac{q-1}{q}}\bigg(\int_{B_{R+3\rho}}\big|\Delta f[u]\big|^q\bigg)^{\frac{1}{q}} \leq CJ_h^{(q)}. \qquad (4.12)
\end{aligned}
$$

This conclude the proof. $\square$

To clear up notation, we denote the three integrals on the LHS of (4.7) involving higher order terms by

$$\begin{aligned}(\mathrm{I}_1)_h &:= (q-1)\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-2}F^{ij}\nabla_i\tilde{v}\nabla_j\tilde{v},\\ (\mathrm{I}_2)_h &:= \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\nabla_i F[u]^{ij}\nabla_j\tilde{v} \quad\text{and}\\ (\mathrm{I}_3)_h &:= \sum_l \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}F^{ij}\Delta^h_{ll}(H[u])_{ij}.\end{aligned}$$

The terms $(\mathrm{I}_1)_h$, $(\mathrm{I}_2)_h$ and $(\mathrm{I}_3)_h$ will be considered in turn. In Section 4.2, we prove an estimate for $(\mathrm{I}_1)_h$. In Section 4.3.1, we estimate $(\mathrm{I}_2)_h$ in the case that $H$ is a multiple of the identity, and in Section 4.3.2 we estimate $(\mathrm{I}_2)_h$ for general $H$ when $k=2$. The estimate for $(\mathrm{I}_3)_h$ in the general case is slightly involved, so for illustrative purposes we first address the simpler case when $H(x,z,\xi) = H_1(x,z)|\xi|^2 I$ with $H_1 \geq 0$, which includes the $\sigma_k$-Yamabe equation in the positive case. This is done in Section 4.4.1. The estimate for $(\mathrm{I}_3)_h$ in the general case is proved in Section 4.4.2. In the process, we will prove the cancellation phenomenon between $(\mathrm{I}_2)_h$ and $(\mathrm{I}_3)_h$ alluded to earlier – see Corollaries 4.12, 4.13 and 4.14.

## 4.2 A pointwise lower bound for $F[u]^{ij}\nabla_i\tilde{v}\nabla_j\tilde{v}$

The term $F^{ij}\nabla_i\tilde{v}\nabla_j\tilde{v}$ in $(\mathrm{I}_1)_h$ can be bounded in the same way as in [**60**] (see equation (3.6) therein). We reproduce the argument here for the reader's convenience.

**Lemma 4.3.** *Suppose $f \in C^0(\Omega\times\mathbb{R})$ is positive, $H \in C^0(\Omega\times\mathbb{R}\times\mathbb{R}^n\,;\mathrm{Sym}_n(\mathbb{R}))$ and $u$ is a solution to (4.6). Then for $q>0$,*

$$(v^+)^{q-2}F^{ij}\nabla_i\tilde{v}\nabla_j\tilde{v} \geq \frac{4f^k}{q^2}\frac{\big|\nabla\big((\tilde{v}^+)^{q/2}\big)\big|^2}{\Delta u - \mathrm{tr}(H)} \quad \text{a.e. in } \Omega_h. \tag{4.13}$$

*In particular, for $R>0$ with $B_{2R}\Subset\Omega$, $\rho\in(0,\frac{R}{3}]$, $q>1$ and $|h|$ sufficiently small, we have*

$$(\mathrm{I}_1)_h \geq \frac{q-1}{Cq^2}\int_{B_{R+\rho}} f^k\frac{\big|\nabla\big((\tilde{v}^+)^{q/2}\big)\big|^2}{\Delta u - \mathrm{tr}(H)}. \tag{4.14}$$

*Proof.* Denote by $\mathcal{M}_k^+ \subset \mathrm{Sym}_n(\mathbb{R})$ the set of symmetric matrices $M$ with $\lambda(M)\in\Gamma_k^+$. For $1\leq l\leq n$, denote by $F^{ij}_{(l)}(A)$ the matrix with entries $\partial\sigma_l(A)/\partial A_{ij}$. Using the concavity of $\sigma_k(A)/\sigma_{k-1}(A)$ on $\mathcal{M}_k^+$, we have

$$\frac{F^{ij}_{(k)}(A)}{\sigma_k(A)} \geq \frac{F^{ij}_{(k-1)}(A)}{\sigma_{k-1}(A)} \quad \text{for all } A\in\mathcal{M}_k^+ \tag{4.15}$$

(see e.g. [**48, 60**]). Applying (4.15) inductively, it follows that

$$\frac{F^{ij}_{(k)}(A)}{\sigma_k(A)} \geq \cdots \geq \frac{F^{ij}_{(1)}(A)}{\sigma_1(A)} = \frac{\delta^{ij}}{\mathrm{tr}(A)} \quad \text{for all } A \in \mathcal{M}^+_k. \tag{4.16}$$

Taking $A = A_H[u]$ in (4.16), where $u$ is a solution to (4.6), we obtain

$$\frac{F[u]^{ij}(x)}{f^k[u](x)} \geq \frac{\delta^{ij}}{\Delta u(x) - \mathrm{tr}(H[u](x))} \quad \text{for a.e. } x \in \Omega,$$

from which (4.13) is readily seen. The estimate (4.14) then follows from properties of $\eta$. □

## 4.3 Integral estimates for $\nabla_i F[u]^{ij}\nabla_j \tilde{v}$

In this section we obtain estimates for the quantity $(\mathrm{I}_2)_h = \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\nabla_i F[u]^{ij}\nabla_j\tilde{v}$. The case in which $H$ is a multiple of the identity matrix will be dealt with first, in Section 4.3.1. The case for general $H$ when $k = 2$ will then be addressed in Section 4.3.2.

### 4.3.1 The case $H = H_2(x, z, \xi)I$

In this section we prove the following two lemmas:

**Lemma 4.4.** *Suppose $f \in C^0(\Omega \times \mathbb{R})$ is positive, $H \in C^{1,1}_{\mathrm{loc}}(\Omega \times \mathbb{R} \times \mathbb{R}^n\,;\mathrm{Sym}_n(\mathbb{R}))$ with $H(x, z, \xi) = H_1(x, z)|\xi|^2 I$, and that $u \in W^{2,q+k-1}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ ($q > 1$) is a solution to (4.6). Then for $R > 0$ with $B_{2R} \Subset \Omega$, $\rho \in (0, \frac{R}{3}]$ and $|h|$ sufficiently small, we have*

$$(\mathrm{I}_2)_h \geq -\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\,\mathrm{tr}(F)\frac{\partial(H_1|\xi|^2)}{\partial\xi_a}[u]\nabla_a\tilde{v} - C\rho^{-1}J_h^{(q+k-1)}. \tag{4.17}$$

**Lemma 4.5.** *Suppose $H \in C^{1,1}_{\mathrm{loc}}(\Omega \times \mathbb{R} \times \mathbb{R}^n\,;\mathrm{Sym}_n(\mathbb{R}))$ with $H(x, z, \xi) = H_2(x, z, \xi)I$, and that $u \in W^{2,q+k}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ ($q > 1$). Then for $R > 0$ with $B_{2R} \Subset \Omega$, $\rho \in (0, \frac{R}{3}]$ and $|h|$ sufficiently small, we have*

$$(\mathrm{I}_2)_h \geq -\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\,\mathrm{tr}(F)\frac{\partial H_2}{\partial\xi_a}[u]\nabla_a\tilde{v} - C\rho^{-1}J_h^{(q+k)}. \tag{4.18}$$

**Remark 4.6.** Note that in Lemma 4.5, we do not assume that $u$ solves (4.6). In contrast, the weaker integrability assumption in Lemma 4.4 relies on both the fact that $u$ solves (4.6) *and* that $H_2$ depends quadratically on $\nabla u$.

**Remark 4.7.** The first term on the RHS of (4.17) and (4.18) will later be shown to cancel with a term arising from our estimate for $(\mathrm{I}_3)_h$.

*Proof of Lemmas 4.4 and 4.5.* The proof consists of three steps. In Step 1, we prove a preliminary estimate assuming only $u \in W^{2,q+k-1}_{\rm loc}(\Omega) \cap W^{1,\infty}_{\rm loc}(\Omega)$ and $H = H_2(x,z,\xi)I$, but we do not assume at this point that $u$ necessarily solves (4.6). Only in Steps 2 and 3 will we appeal to the specific hypotheses of Lemmas 4.4 and 4.5.

Our starting point is the following expression for $(\mathrm{I}_2)_h$, which follows from (3.5):

$$(\mathrm{I}_2)_h = -(n-k+1)\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\nabla_j(H_2[u])\, T_{k-2}(A_H)^{ij}\nabla_i\tilde{v}.$$

***Step 1:*** In this step, we show that for every $u \in W^{2,q+k-1}_{\rm loc}(\Omega) \cap W^{1,\infty}_{\rm loc}(\Omega)$,

$$(\mathrm{I}_2)_h \geq -\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\,\mathrm{tr}(F)\frac{\partial H_2}{\partial \xi_a}[u]\nabla_a\tilde{v} - \frac{n-k+1}{q}\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^q F^i_a\frac{\partial^2 H_2}{\partial\xi_a\partial\xi_b}[u](A_H)_{ib} - C\rho^{-1}J_h^{(q+k-1)}. \tag{4.19}$$

Note that the first integral on the RHS of (4.19) is the desired term seen in (4.17) and (4.18).

First observe that by the chain rule,

$$\begin{aligned}(\mathrm{I}_2)_h = &-(n-k+1)\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\frac{\partial H_2}{\partial x^j}[u]\, T_{k-2}(A_H)^{ij}\nabla_i\tilde{v}\\ &-(n-k+1)\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\frac{\partial H_2}{\partial z}[u]\, T_{k-2}(A_H)^{ij}\nabla_j u\nabla_i\tilde{v}\\ &-(n-k+1)\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\frac{\partial H_2}{\partial \xi_a}[u]\, T_{k-2}(A_H)^{ij}\nabla_j\nabla_a u\,\nabla_i\tilde{v}.\end{aligned} \tag{4.20}$$

Denote the top two lines of the RHS of (4.20) collectively by $L_1$, and the bottom line by $L_2$. Recalling that $\nabla_j\nabla_a u = H_2\delta_{ja} + (A_H)_{ja}$ and, in view of (3.1) and (3.3), that

$$\big(T_{k-2}(A_H)A_H\big)_{ia} = -F_{ia} + \frac{1}{n-k+1}\,\mathrm{tr}(F)\delta_{ia}, \tag{4.21}$$

we have

$$\begin{aligned}L_2 = &-(n-k+1)\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\frac{\partial H_2}{\partial \xi_a}[u]\, T_{k-2}(A_H)_{ia}H_2\nabla^i\tilde{v}\\ &-(n-k+1)\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\frac{\partial H_2}{\partial \xi_a}[u]\,\big(T_{k-2}(A_H)A_H\big)_{ia}\nabla^i\tilde{v}\\ \overset{(4.21)}{=} &-(n-k+1)\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\frac{\partial H_2}{\partial \xi_a}[u]\, T_{k-2}(A_H)_{ia}H_2\nabla^i\tilde{v}\\ &+(n-k+1)\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\frac{\partial H_2}{\partial \xi_a}[u]\, F_{ia}\nabla^i\tilde{v} - \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\,\mathrm{tr}(F)\frac{\partial H_2}{\partial \xi_a}[u]\,\nabla_a\tilde{v}.\end{aligned}$$

Substituting this identity for $L_2$ into (4.20) yields

$$(\mathrm{I}_2)_h = L_1 - (n-k+1)\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\frac{\partial H_2}{\partial \xi_a}[u]\, T_{k-2}(A_H)_{ia} H_2 \nabla^i \tilde{v} \\ + (n-k+1)\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\frac{\partial H_2}{\partial \xi_a}[u]\, F_{ia}\nabla^i\tilde{v} - \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\,\mathrm{tr}(F)\frac{\partial H_2}{\partial \xi_a}[u]\,\nabla_a \tilde{v}. \tag{4.22}$$

We claim that the terms on the top line of the RHS of (4.22) are bounded from below by $-C\rho^{-1}J_h^{(q+k-1)}$. Indeed, as $T_{k-2}(A_H)^{ij} = \partial\sigma_{k-1}(A_H)/\partial A_{ij}$, by Lemma 3.2 we have $|\nabla_i T_{k-2}(A_H[u])^{ij}| \leq C(\Delta u + C_1)^{k-2}$. It is also clear that $|T_{k-2}(A_H)^{ij}| \leq C(\Delta u + C_1)^{k-2}$. Thus, after integrating by parts using Lemma 3.2 and applying Hölder's inequality, the lower bound for these terms follows.

To estimate the penultimate integral in (4.22), we integrate by parts using Lemma 3.2 and apply the identity

$$\nabla_i\bigg(\frac{\partial H_2}{\partial \xi_a}[u](x)\bigg) = \bigg(\frac{\partial^2 H_2}{\partial \xi_a \partial \xi_b}[u](x)\bigg)\big((A_H)_{ib} + H_{ib}\big) + \bigg(\frac{\partial^2 H_2}{\partial z \partial \xi_a}[u](x)\bigg)\nabla_i u(x) + \frac{\partial^2 H_2}{\partial x^i \partial \xi_a}[u](x).$$

After an application of Hölder's inequality, this gives

$$\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\frac{\partial H_2}{\partial \xi_a}[u]\, F_{ia}\nabla^i\tilde{v} \geq -\frac{1}{q}\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^q F_a^i \frac{\partial^2 H_2}{\partial \xi_a \partial \xi_b}[u]\,(A_H)_{ib} - C\rho^{-1}J_h^{(q+k-1)},$$

from which (4.19) follows.

***Step 2:*** In this step we prove Lemma 4.5. Indeed, for $u \in W^{2,q+k}_{\mathrm{loc}}(\Omega)\cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ (not necessarily solving (4.6)) we have the estimate

$$-\frac{n-k+1}{q}\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^q F_a^i \frac{\partial^2 H_2}{\partial \xi_a \partial \xi_b}[u]\,(A_H)_{ib} \geq -C\int_{B_{R+2\rho}} (\tilde{v}^+)^q |F||A_H| \geq -CJ_h^{(q+k)},$$

with the last inequality following once again from the estimate $|F| \leq C(\Delta u + C_1)^{k-1}$ and Hölder's inequality. Substituting this into (4.19) then yields the desired estimate (4.18).

***Step 3:*** In this step we prove Lemma 4.4. Since we assume in this case that $H_2(x,z,\xi) = H_1(x,z)|\xi|^2$ and that $u$ solves (4.6), rather than estimating as in Step 2 we observe

$$F_a^i\frac{\partial^2 H_2}{\partial \xi_a \partial \xi_b}[u](A_H)_{ib} = 2H_1F_a^i\delta^{ab}(A_H)_{ib} = 2H_1F_a^i(A_H)_i^a = 2H_1k\sigma_k(A_H) = 2H_1kf^k. \tag{4.23}$$

Substituting (4.23) into the second integral in (4.19), we arrive at (4.17). □

#### 4.3.2 The case $k = 2$ for general $H$

In this section we obtain an estimate in the case $k = 2$ analogous to (4.17) and (4.18). We do not assume that $H$ is a multiple of the identity and, as in Lemma 4.5, we do not assume that $u$ solves (4.6):

**Lemma 4.8.** *Suppose $H \in C^{1,1}_{\mathrm{loc}}(\Omega\times\mathbb{R}\times\mathbb{R}^n\,;\,\mathrm{Sym}_n(\mathbb{R}))$, $k = 2$ and $u \in W^{2,q+2}_{\mathrm{loc}}(\Omega)\cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ ($q > 1$). Then for $R > 0$ with $B_{2R} \Subset \Omega$, $\rho \in (0, \frac{R}{3}]$ and $|h|$ sufficiently small, we have*

$$
\begin{aligned}
(\mathrm{I}_2)_h \geq \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\frac{\partial H^{ij}}{\partial \xi_a}[u]\,\nabla_i\nabla_a u\,\nabla_j\tilde{v} - \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\frac{\partial\,\mathrm{tr}(H)}{\partial \xi_a}[u]\ \mathrm{tr}(A_H)\nabla_a\tilde{v} \\
- C\rho^{-1}J_h^{(q+2)}. \qquad (4.24)
\end{aligned}
$$

**Remark 4.9.** The first two terms on the RHS of (4.24) will later be shown to cancel with a term arising from our estimate for $(\mathrm{I}_3)_h$ (cf. Remark 4.7).

*Proof of Lemma 4.8.* As $k = 2$, we have $\nabla_i F[u]^{ij} = \nabla_i H[u]^{ij} - \nabla^j\,\mathrm{tr}(H[u])$ (by (3.4)) and $\nabla^j\nabla^a u = \mathrm{tr}(A_H)\delta^{ja} - F^{ja} - H^{ja}$. It follows that

$$
\begin{aligned}
(\mathrm{I}_2)_h &= \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\big(\nabla_i H[u]^{ij} - \nabla^j\,\mathrm{tr}(H[u])\big)\nabla_j\tilde{v} \\
&= \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\bigg(\frac{\partial H^{ij}}{\partial \xi_a}[u]\,\nabla_i\nabla_a u - \frac{\partial\,\mathrm{tr}(H)}{\partial \xi^a}[u]\,\nabla^j\nabla^a u\bigg)\nabla_j\tilde{v} \\
&\quad + \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\bigg(\frac{\partial H^{ij}}{\partial x^i}[u] + \frac{\partial H^{ij}}{\partial z}[u]\,\nabla_i u - \frac{\partial\,\mathrm{tr}(H)}{\partial x_j}[u] - \frac{\partial\,\mathrm{tr}(H)}{\partial z}[u]\,\nabla^j u\bigg)\nabla_j\tilde{v} \\
&= \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\bigg(\frac{\partial H^{ij}}{\partial \xi_a}[u]\,\nabla_i\nabla_a u - \frac{\partial\,\mathrm{tr}(H)}{\partial \xi^a}[u]\ \mathrm{tr}(A_H)\delta^{ja}\bigg)\nabla_j\tilde{v} \\
&\quad + \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\bigg(\frac{\partial H^{ij}}{\partial x^i}[u] + \frac{\partial H^{ij}}{\partial z}[u]\,\nabla_i u - \frac{\partial\,\mathrm{tr}(H)}{\partial x_j}[u] - \frac{\partial\,\mathrm{tr}(H)}{\partial z}[u]\,\nabla^j u \\
&\qquad\qquad + \frac{\partial\,\mathrm{tr}(H)}{\partial \xi^a}[u](F^{ja} + H^{ja})\bigg)\nabla_j\tilde{v}. \qquad (4.25)
\end{aligned}
$$

The integral on the last two lines of (4.25) can be bounded from below by $-C\rho^{-1}J_h^{(q+2)}$ in exactly the same way as in the proof of Lemmas 4.4 and 4.5: we integrate by parts using Lemma 3.2, estimate the relevant quantities in terms of $\Delta u + C_1$ and apply Hölder's inequality. The estimate (4.24) then follows. □

### 4.4 Integral estimates for $F[u]^{ij}\Delta^h_{ll}H[u]_{ij}$

In this section we obtain estimates for the quantity $(\mathrm{I}_3)_h = \sum_l \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}F^{ij}\Delta^h_{ll}(H[u])_{ij}$. More precisely, we will prove the following lemma:

**Lemma 4.10.** *Suppose $H \in C^{1,1}_{\rm loc}(\Omega \times \mathbb{R} \times \mathbb{R}^n\,; \mathrm{Sym}_n(\mathbb{R}))$, $R > 0$ is such that $B_{2R} \Subset \Omega$ and $\rho \in (0, \frac{R}{3}]$.*

*a) If $u \in W^{2,q+k}_{\rm loc}(\Omega) \cap W^{1,\infty}_{\rm loc}(\Omega)$ $(q > 1)$, then for $|h|$ sufficiently small, we have*

$$(\mathrm{I}_3)_h \geq \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1} F^{ij} \frac{\partial H_{ij}}{\partial \xi_a}[u] \, \nabla_a \tilde{v} - C J_h^{(q+k)}. \tag{4.26}$$

*b) If $u \in W^{2,q+k-1}_{\rm loc}(\Omega) \cap W^{1,\infty}_{\rm loc}(\Omega)$ $(q > 1)$ and $H(x,z,\xi) = H_1(x,z)|\xi|^2 I$ with $H_1 \geq 0$, then for $|h|$ sufficiently small, we have*

$$(\mathrm{I}_3)_h \geq \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1} F^{ij} \frac{\partial H_{ij}}{\partial \xi_a}[u] \, \nabla_a \tilde{v} - C J_h^{(q+k-1)}. \tag{4.27}$$

**Remark 4.11.** Neither estimate in Lemma 4.10 requires $u$ to be a solution to (4.6).

Before proving Lemma 4.10 we first discuss its consequences, namely the resulting cancellations between $(\mathrm{I}_2)_h$ and $(\mathrm{I}_3)_h$. First consider the case $H = H_1(x,z)|\xi|^2 I$ with $H_1 \geq 0$:

**Corollary 4.12.** *Suppose $f \in C^{1,1}_{\rm loc}(\Omega \times \mathbb{R})$ is positive, $H \in C^{1,1}_{\rm loc}(\Omega \times \mathbb{R} \times \mathbb{R}^n\,; \mathrm{Sym}_n(\mathbb{R}))$ with $H = H_1(x,z)|\xi|^2 I$ and $H_1 \geq 0$, and that $u \in W^{2,q+k-1}_{\rm loc}(\Omega) \cap W^{1,\infty}_{\rm loc}(\Omega)$ $(q > 1)$ is a solution to (4.6). Then for $R > 0$ with $B_{2R} \Subset \Omega$, $\rho \in (0, \frac{R}{3}]$ and $|h|$ sufficiently small, we have*

$$(\mathrm{I}_2)_h + (\mathrm{I}_3)_h \geq -C\rho^{-1} J_h^{(q+k-1)}. \tag{4.28}$$

*In particular,*

$$\frac{q-1}{q^2} \int_{B_{R+\rho}} f^k \frac{\left|\nabla\left((\tilde{v}^+)^{q/2}\right)\right|^2}{\Delta u - \mathrm{tr}(H)} \leq C\rho^{-2} J_h^{(q+k-1)}. \tag{4.29}$$

*Proof.* The estimate (4.28) follows from combining the estimates (4.17) and (4.27). The estimate (4.29) is then obtained by substituting (4.14) and (4.28) into (4.7). □

Similarly, we obtain the following in the case that $H = H_2(x,z,\xi)I$:

**Corollary 4.13.** *Suppose $H \in C^{1,1}_{\rm loc}(\Omega \times \mathbb{R} \times \mathbb{R}^n\,; \mathrm{Sym}_n(\mathbb{R}))$ with $H = H_2(x,z,\xi)I$, and $u \in W^{2,q+k}_{\rm loc}(\Omega) \cap W^{1,\infty}_{\rm loc}(\Omega)$ $(q > 1)$. Then for $R > 0$ with $B_{2R} \Subset \Omega$, $\rho \in (0, \frac{R}{3}]$ and $|h|$ sufficiently small, we have*

$$(\mathrm{I}_2)_h + (\mathrm{I}_3)_h \geq -C\rho^{-1} J_h^{(q+k)}. \tag{4.30}$$

*If, in addition, $u$ solves (4.6) for some positive $f \in C^{1,1}_{\rm loc}(\Omega \times \mathbb{R})$, then*

$$\frac{q-1}{q^2} \int_{B_{R+\rho}} f^k \frac{\left|\nabla\left((\tilde{v}^+)^{q/2}\right)\right|^2}{\Delta u - \mathrm{tr}(H)} \leq C\rho^{-2} J_h^{(q+k)}. \tag{4.31}$$

*Proof.* The estimate (4.30) follows from combining the estimates (4.18) and (4.26). The estimate (4.31) is then obtained by substituting (4.14) and (4.30) into (4.7). ☐

A similar cancellation also holds in the setting of Theorem 1.8, although this requires a little more work:

**Corollary 4.14.** *Suppose $H \in C^{1,1}_{\rm loc}(\Omega \times \mathbb{R} \times \mathbb{R}^n\,; \mathrm{Sym}_n(\mathbb{R}))$, $k = 2$ and $u \in W^{2,q+2}_{\rm loc}(\Omega) \cap W^{1,\infty}_{\rm loc}(\Omega)$ $(q > 1)$. Then for $|h|$ sufficiently small, we have*

$$(\mathrm{I}_2)_h + (\mathrm{I}_3)_h \geq -C\rho^{-2} J_h^{(q+2)}. \tag{4.32}$$

*If, in addition, $u$ solves* (4.6) *for some positive $f \in C^{1,1}_{\rm loc}(\Omega \times \mathbb{R})$, then*

$$\frac{q-1}{q^2}\int_{B_{R+\rho}} f^2 \frac{\big|\nabla\big((\tilde{v}^+)^{q/2}\big)\big|^2}{\Delta u - \mathrm{tr}(H)} \leq C\rho^{-1} J_h^{(q+2)}. \tag{4.33}$$

*Proof.* The estimate (4.33) will immediately follow once (4.32) is established, by substituting (4.14) and (4.32) into (4.7).

Taking $k = 2$ in Lemma 4.10 a) and using $F^{ij} = \mathrm{tr}(A_H)\delta^{ij} - \nabla^i\nabla^j u - H^{ij}$, we see

$$\begin{aligned}(\mathrm{I}_3)_h \geq & \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\,\mathrm{tr}(A_H)\frac{\partial\,\mathrm{tr}(H)}{\partial\xi_a}[u]\,\nabla_a\tilde{v} - \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\nabla^i\nabla^j u\frac{\partial H_{ij}}{\partial \xi_a}[u]\,\nabla_a\tilde{v} \\ & - \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1} H^{ij}\frac{\partial H_{ij}}{\partial\xi_a}[u]\,\nabla_a\tilde{v} - CJ_h^{(q+2)}. \end{aligned} \tag{4.34}$$

Now, the first term on the RHS of (4.34) cancels with the second term on the RHS of (4.24), and the first term on the last line of (4.34) can be estimated by $-C\rho^{-1}J_h^{(q+2)}$, after integrating by parts and applying Hölder's inequality. Therefore, combining (4.24) and (4.34), we obtain

$$(\mathrm{I}_2)_h + (\mathrm{I}_3)_h \geq \frac{1}{q}\int_{B_{R+2\rho}} \eta\frac{\partial H_{ij}}{\partial\xi_a}[u]\Big(\nabla^i\nabla_a u\,\nabla^j(\tilde{v}^+)^q - \nabla^i\nabla^j u\,\nabla_a(\tilde{v}^+)^q\Big) - C\rho^{-1}J_h^{(q+2)}.$$

Now, if $u$ were to have enough regularity, we could integrate by parts here, observe that the third derivatives of $u$ cancel, and obtain (4.32) by estimating the remaining terms in the usual way. To circumvent the lack of regularity, we instead apply the following lemma:

**Lemma 4.15.** *Let $U \subset \mathbb{R}^n$ be a smooth bounded domain and let $B \in L^\infty(U; \mathbb{R}^{n\times n})$ be an antisymmetric matrix with $\mathrm{supp}(B) \Subset U$. For $1 \leq p < \infty$ and $p' := \frac{p}{p-1}$, consider the bilinear form $\mathcal{B} : W^{1,p}(U) \times W^{1,p'}(U) \to \mathbb{R}$ given by*

$$\mathcal{B}(g,h) = \int_U B^a_j\,\nabla_a g\,\nabla^j h. \tag{4.35}$$

*If* $\operatorname{div} B \in L^q(U;\mathbb{R}^n)$ *with* $\frac{1}{p}+\frac{1}{q} = 1-\frac{1}{r}$ *for some* $1\leq q, r\leq\infty$*, then we have the estimate*

$$|\mathcal{B}(g,h)| \leq \int_U |\operatorname{div} B||\nabla g||h| \tag{4.36}$$

*for all* $g\in W^{1,p}(U)$ *and* $h\in W^{1,p'}(U)\cap L^r(U)$.

Before proving Lemma 4.15, we use it to complete the proof of (4.32): for each $i\in\{1,\dots,n\}$, taking $B^a_j = \eta\frac{\partial H_{ij}}{\partial\xi_a}[u] - \eta\frac{\partial H^a_i}{\partial\xi^j}[u]$, $g=\nabla_i u$ and $h=(\tilde{v}^+)^q$ in Lemma 4.15 we obtain

$$\begin{aligned}\int_{B_{R+2\rho}}\eta\frac{\partial H_{ij}}{\partial\xi_a}[u]\left(\nabla^i\nabla_a u\nabla^j(\tilde{v}^+)^q - \nabla^i\nabla^j u\nabla_a(\tilde{v}^+)^q\right) &\overset{(4.36)}{\leq} C\rho^{-1}\int_{B_{R+2\rho}}(\Delta u + C_1)^2(\tilde{v}^+)^q\\ &\leq C\rho^{-1}J_h^{(q+2)}.\end{aligned}$$

It remains to prove Lemma 4.15. By a standard approximation argument, it suffices to prove (4.36) for $g,h\in C^\infty(U)$. We are then justified in integrating by parts in (4.35), giving

$$|\mathcal{B}(g,h)| = \left|\int_U\left(\nabla^j B^a_j\,\nabla_a g + \underbrace{B^a_j\,\nabla_a\nabla^j g}_{=0}\right)h\right| \leq \int_U|\operatorname{div} B||\nabla g||h|,$$

where we have used antisymmetry of $B$ to assert that $B^a_j\,\nabla_a\nabla^j g = 0$. ☐

### 4.4.1 Proof of Lemma 4.10 b)

We now turn our attention back to the proof of Lemma 4.10. Whilst the two estimates (4.26) and (4.27) can be dealt with simultaneously (see the proof of Lemma 4.10 in Section 4.4.2), for illustrative purposes we first provide a more direct proof of (4.27), which includes the $\sigma_k$-Yamabe equation in the positive case. Indeed, when $H = H_1(x,z)|\xi|^2 I$ we are able to calculate $\Delta^h_{ll}(H[u])_{ij}$ explicitly by deriving the following discrete version of the Bochner identity, avoiding the more involved estimates required for the general case. In what follows, we denote

$$u^h_l(x) := u(x+he_l).$$

**Lemma 4.16** (Discrete Bochner identity)**.** *Suppose* $H_1\in C^0(\Omega\times\mathbb{R})$ *and* $l\in\{1,\dots,n\}$*. Then*

$$\begin{aligned}\Delta^h_{ll}\big(H_1[u]|\nabla u|^2\big) = {}& 2H_1\nabla^i u\nabla_i\Delta^h_{ll}u + (H_1[u])^{-h}_l\big|\nabla\nabla^{-h}_l u\big|^2 + (H_1[u])^h_l\big|\nabla\nabla^h_l u\big|^2\\ &+\nabla^{-h}_l\nabla_i u\nabla^i u\nabla^{-h}_l H_1[u] + \nabla^h_l\nabla^i u\nabla_i u\nabla^h_l H_1[u]\\ &+\nabla^h_l\Big(\nabla_i u(\nabla^i u)^{-h}_l\nabla^{-h}_l H_1[u]\Big).\end{aligned}\tag{4.37}$$

Assuming the validity of Lemma 4.16, the proof of (4.27) in Lemma 4.10 b) is then straightforward:

*Proof of Lemma 4.10 b).* Substituting the discrete Bochner identity (4.37) into the definition of $(\mathrm{I}_3)_h$ and dropping the two positive terms, we obtain

$$
\begin{aligned}
(\mathrm{I}_3)_h \geq 2 \int_{B_{R+2\rho}} & \eta(\tilde{v}^+)^{q-1} \operatorname{tr}(F) H_1 \nabla^i u \nabla_i \tilde{v} \\
+ \sum_l \int_{B_{R+2\rho}} & \eta(\tilde{v}^+)^{q-1} \operatorname{tr}(F) \Big( \nabla_l^{-h} \nabla_i u \nabla^i u \nabla_l^{-h} H_1[u] + \nabla_l^h \nabla^i u \nabla_i u \nabla_l^h H_1[u] \\
& + \nabla_l^h \Big( \nabla_i u (\nabla^i u)_l^{-h} \nabla_l^{-h} H_1[u] \Big) \Big). \qquad (4.38)
\end{aligned}
$$

After applying the difference quotient product rule

$$
\nabla_l^h (uv)(x) = u_l^h(x) \nabla_l^h v(x) + v(x) \nabla_l^h u(x) \qquad (4.39)
$$

to the integrand in the last line of (4.38), we may then estimate the last two lines of (4.38) in the usual way. Namely, after applying the bound $\operatorname{tr}(F) \leq C(\Delta u + C_1)^{k-1}$, using Hölder's inequality and appealing to (2.4), we see that the last two lines of (4.38) are collectively bounded from below by $-CJ_h^{(q+k-1)}$. The estimate (4.27) then follows. ◻

*Proof of the discrete Bochner identity (Lemma 4.16).* Using the product rule (4.39) to first calculate $\nabla_l^{-h}(H_1[u]|\nabla u|^2)$, we see

$$
\begin{aligned}
\Delta_{ll}^h \big( H_1[u] |\nabla u|^2 \big) &= \nabla_l^h \Big( \nabla_l^{-h} \big( H_1[u] \nabla^i u \nabla_i u \big) \Big) \\
&= \nabla_l^h \Big( (H_1[u] \nabla^i u)_l^{-h} \nabla_l^{-h} \nabla_i u \Big) + \nabla_l^h \Big( H_1[u] \nabla_i u \nabla_l^{-h} \nabla^i u \Big) + \nabla_l^h \Big( \nabla_i u (\nabla^i u)_l^{-h} \nabla_l^{-h} H_1[u] \Big).
\end{aligned}
$$

On the other hand, noting that $\nabla_l^h u_l^{-h} u(x) = \nabla_l^{-h} u(x)$ and $(\nabla_l^{-h} u)_l^h(x) = \nabla_l^h u(x)$, we also have by (4.39) the identities

$$
\begin{aligned}
\nabla_l^h \Big( (H_1[u] \nabla^i u)_l^{-h} \nabla_l^{-h} \nabla_i u \Big) &= H_1 \nabla^i u \nabla_l^h \nabla_l^{-h} \nabla_i u + \nabla_l^{-h} \nabla_i u \nabla_l^{-h} (H_1[u] \nabla^i u) \\
&= H_1 \nabla^i u \nabla_i \Delta_{ll}^h u + (H_1[u])_l^{-h} \big| \nabla \nabla_l^{-h} u \big|^2 + \nabla_l^{-h} \nabla_i u \nabla^i u \nabla_l^{-h} H_1[u]
\end{aligned}
$$

and

$$
\begin{aligned}
\nabla_l^h \Big( H_1[u] \nabla_i u \nabla_l^{-h} \nabla^i u \Big) &= \nabla_l^h \nabla^i u \nabla_l^h \Big( H_1[u] \nabla_i u \Big) + H_1 \nabla_i u \nabla_l^h \nabla_l^{-h} \nabla^i u \\
&= (H_1[u])_l^h \big| \nabla \nabla_l^h u \big|^2 + \nabla_l^h \nabla^i u \nabla_i u \nabla_l^h H_1[u] + H_1 \nabla_i u \nabla^i \Delta_{ll}^h u.
\end{aligned}
$$

Putting these three identities together, we arrive at (4.37). ◻

### 4.4.2 Proof of Lemma 4.10 in the general case

We now prove Lemma 4.10 in the general case. To simplify our analysis, we will make use of the following semi-convexity property of $H \in C^{1,1}_{\mathrm{loc}}(\Omega \times \mathbb{R} \times \mathbb{R}^n; \mathrm{Sym}_n(\mathbb{R}))$: there exists a constant $C_\Sigma > 0$ such that the mapping $\xi \longmapsto H(x,z,\xi) + C_\Sigma|\xi|^2 I$ is convex for all $(x,z,\xi) \in \Sigma$ (this is an immediate consequence of the $C^{1,1}_{\mathrm{loc}}$ regularity of $H$). We will make use of this property in the form

$$H_{ij}(x,z,\xi) \geq H_{ij}(x,z,\zeta) + \frac{\partial H_{ij}}{\partial \xi_a}(x,z,\zeta)(\xi-\zeta)_a - C_\Sigma \delta_{ij}|\xi-\zeta|^2 \tag{4.40}$$

for all $(x,z,\xi),(x,z,\zeta) \in \Sigma$. Note that in Case 1 of Theorem 1.5, we may take $C_\Sigma = 0$ in (4.40), as $H(x,z,\xi) = H_1(x,z)|\xi|^2 I$ is convex with respect to $\xi$ when $H_1 \geq 0$. The inequality (4.40) will play a role similar to that of the discrete Bochner identity used in the previous subsection (see Lemma 4.16).

*Proof of Lemma 4.10.* We first prove Lemma 4.10 a). It suffices to show that

$$F^{ij}\Delta^h_{ll}(H[u])_{ij} \geq F^{ij}\frac{\partial H_{ij}}{\partial \xi_a}[u]\,\nabla_a \Delta^h_{ll} u + \text{error terms} \quad \forall l \in \{1,\dots,n\}, \tag{4.41}$$

where the error terms satisfy

$$\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}|\text{error terms}| \leq C J_h^{(q+k)}. \tag{4.42}$$

To keep notation succinct, we denote $x^\pm = x \pm he_l$ in what follows.

***Step 1:*** We first prove a lower bound for $F^{ij}(x)\Delta^h_{ll}(H[u](x))_{ij}$, identifying the error terms in (4.41). Observe that by (4.40) and the fact that $F^{ij}$ is positive definite in $\Gamma_k^+$, we have

$$\begin{aligned}
&\frac{F^{ij}(x)}{h^2}\Big[(H[u](x^\pm))_{ij} - H(x^\pm, u(x^\pm), \nabla u(x))_{ij}\Big] \\
&\geq \frac{F^{ij}(x)}{h^2}\frac{\partial H_{ij}}{\partial \xi_a}\big(x^\pm, u(x^\pm), \nabla u(x)\big)\big(\nabla_a u(x^\pm) - \nabla_a u(x)\big) - \frac{C_\Sigma|F|}{h^2}\big|\nabla u(x^\pm) - \nabla u(x)\big|^2 \\
&\geq \frac{F^{ij}(x)}{h^2}\frac{\partial H_{ij}}{\partial \xi_a}[u](x)\big(\nabla_a u(x^\pm) - \nabla_a u(x)\big) - \frac{C_\Sigma|F|}{h^2}\big|\nabla u(x^\pm) - \nabla u(x)\big|^2 \\
&\quad - \frac{C|F|}{|h|}\big|\nabla u(x^\pm) - \nabla u(x)\big| \quad \text{for a.e. } x \in B_{R+2\rho},
\end{aligned}$$

where to obtain the second inequality we have estimated

$$\left|\frac{\partial H_{ij}}{\partial \xi_a}\big(x^\pm, u(x^\pm), \nabla u(x)\big) - \frac{\partial H_{ij}}{\partial \xi_a}[u](x)\right| \leq \|H\|_{C^{1,1}(\Sigma)}\big(|x^\pm - x| + |u(x^\pm) - u(x)|\big) \leq C|h|.$$

Recalling the definition of $\Delta^h_{ll}(H[u](x))_{ij}$, we therefore see that for a.e. $x \in B_{R+2\rho}$,

$$
\begin{aligned}
&F^{ij}(x)\Delta^h_{ll}(H[u](x))_{ij}\\
&\qquad \geq F^{ij}(x)\frac{\partial H_{ij}}{\partial \xi_a}[u](x)\nabla_a\Delta^h_{ll}u(x)\\
&\qquad\quad + \frac{F^{ij}(x)}{h^2}\bigg(H(x^+,u(x^+),\nabla u(x))_{ij} - 2(H[u](x))_{ij} + H(x^-,u(x^-),\nabla u(x))_{ij}\bigg)\\
&\qquad\quad - C_\Sigma|F||\nabla^h_l\nabla u|^2 - C_\Sigma|F||\nabla^{-h}_l\nabla u|^2 - C|F||\nabla^h_l\nabla u| - C|F||\nabla^{-h}_l\nabla u|. \qquad (4.43)
\end{aligned}
$$

***Step 2:*** To prove (4.26), we need to show that the error terms in last two lines of (4.43) satisfy (4.42). Formally, these terms behave like $|F|(|\nabla^2 u|^2 + |\nabla^2 u|)$, and so by the estimate $|F| \leq C(\Delta u + C_1)^{k-1}$, the bound (4.42) is then conceivable. We now give the details.

Denote the terms on the penultimate line of (4.43) collectively by $E_1$, and the terms on the last line of (4.43) collectively by $E_2$. The error terms in $E_2$ are easier to deal with. Indeed, by the bound $|F| \leq C(\Delta u + C_1)^{k-1}$, Hölder's inequality and (2.4), we have

$$
\int_{B_{R+2\rho}}\eta(\tilde{v}^+)^{q-1}|F||\nabla^{\pm h}_l\nabla u|^2 \leq C(J^{(q+k)}_h)^{\frac{q+k-2}{q+k}}\left(\int_{B_{R+2\rho}}|\nabla^{\pm h}_l\nabla u|^{q+k}\right)^{\frac{2}{q+k}} \leq CJ^{(q+k)}_h. \quad (4.44)
$$

In exactly the same way, one can show $\int_{B_{R+2\rho}}\eta(\tilde{v}^+)^{q-1}|F||\nabla^{\pm h}_l\nabla u| \leq CJ^{(q+k-1)}_h$, and combining these estimates we obtain $\int_{B_{R+2\rho}}\eta(\tilde{v}^+)^{q-1}|E_2| \leq CJ^{(q+k)}_h$.

We now treat the error terms in $E_1$. We first observe that by the fundamental theorem of calculus followed by the chain rule, we have the identities

$$
\begin{aligned}
&H(x^\pm,u(x^\pm),\xi)_{ij} - H(x,u(x),\xi)_{ij}\\
&= \int_0^1 \frac{d}{dt}H(x\pm the_l,u(x^\pm),\xi)_{ij}\,dt + \int_0^1\frac{d}{dt}H(x,u(x\pm the_l),\xi)_{ij}\,dt\\
&= \pm h\int_0^1\frac{\partial H_{ij}}{\partial x^l}(x\pm the_l,u(x^\pm),\xi)\,dt \pm h\int_0^1\frac{\partial H_{ij}}{\partial z}(x,u(x\pm the_l),\xi)\nabla_l u(x\pm the_l)\,dt,
\end{aligned}
$$

and therefore

$$
\begin{aligned}
&H(x^+,u(x^+),\xi)_{ij} - 2H(x,u(x),\xi)_{ij} + H(x^-,u(x^-),\xi)_{ij}\\
&= h\int_0^1\left(\frac{\partial H_{ij}}{\partial z}(x,u(x+the_l),\xi)\nabla_l u(x+the_l) - \frac{\partial H_{ij}}{\partial z}(x,u(x-the_l),\xi)\nabla_l u(x-the_l)\right)dt\\
&\quad + h\int_0^1\left(\frac{\partial H_{ij}}{\partial x^l}(x+the_l,u(x^+),\xi) - \frac{\partial H_{ij}}{\partial x^l}(x-the_l,u(x^-),\xi)\right)dt. \qquad (4.45)
\end{aligned}
$$

Now, by the $C^{1,1}_{\rm loc}$ regularity of $H$ and the Lipschitz regularity of the mapping $(x,z,p) \mapsto \frac{\partial H_{ij}}{\partial z}(x,z,\xi)p_l$ for fixed $\xi$ and each $l \in \{1,\dots,n\}$, we can estimate the last line of (4.45) from above by $Ch^2$ and the middle line of (4.45) from above by

$$Ch^2 + Ch^2 \int_0^1 \frac{1}{t|h|}\Big|\nabla_l u(x+the_l) - \nabla_l u(x-the_l)\Big|\,dt.$$

Applying these estimates in (4.45) and taking $\xi = \nabla u(x)$, we therefore see that

$$|E_1| \leq C|F| + C|F|\int_0^1 |\nabla_l^{th}\nabla_l u(x)|\,dt + C|F|\int_0^1 |\nabla_l^{-th}\nabla_l u(x)|\,dt. \tag{4.46}$$

Using (4.46), one readily obtains the estimate $\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}|E_1| \leq CJ_h^{(q+k-1)}$, applying the same line of argument as seen above for $E_2$. For example, by Fubini's theorem and Young's inequality, we have

$$\begin{aligned}\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}|F|\bigg(\int_0^1 |\nabla_l^{\pm th}\nabla_l u(x)|\,dt\bigg)\,dx &= \int_0^1\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}|F||\nabla_l^{\pm th}\nabla_l u(x)|\,dx\,dt \\ &\leq CJ_h^{(q+k-1)} + C\int_0^1\int_{B_{R+2\rho}} |\nabla_l^{\pm th}\nabla_l u|^{q+k-1}\,dx\,dt \overset{(2.4)}{\leq} CJ_h^{(q+k-1)}.\end{aligned}$$

This completes the proof of Lemma 4.10 a).

***Step 3:*** It remains to prove Lemma 4.10 b) (see Section 4.4.1 for an alternative proof which is independent of calculations in Steps 1 and 2 above). Note that in this case, we may take $C_\Sigma = 0$ in (4.43) and so the error terms on the last two lines of (4.43) formally behave like $|F||\nabla^2 u|$. By the same argument as in Step 2, the error terms $E_1$ and $E_2$ considered in Step 2 therefore satisfy $\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}|E_i| \leq CJ_h^{(q+k-1)}$, and the conclusion follows. □

# 5 Proof of main results

In this section we use Corollaries 4.12, 4.13 and 4.14 to prove Theorems 1.5 and 1.8, as outlined at the end of Section 2. We will first give a detailed proof of Case 1 of Theorem 1.5 when $f = f(x,z)$ in Section 5.1, and then indicate the necessary adjustments for remaining cases, still when $f = f(x,z)$, in Section 5.2. In Section 5.3, we extend these results to the case $f = f(x,z,\xi)$, completing the proofs of Theorems 1.5 and 1.8.

## 5.1 Proof of Case 1 of Theorem 1.5 when $f = f(x,z)$

In this case, we recall that by Corollary 4.12 we have the estimate

$$\int_{B_{R+\rho}} f^k \frac{\left|\nabla\left((\tilde{v}^+)^{q/2}\right)\right|^2}{\Delta u - \mathrm{tr}(H)} \leq \frac{Cq}{\rho^2} J_h^{(q+k-1)}, \tag{5.1}$$

where $u \in W^{2,q+k-1}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ $(q > 1)$. Let $\theta \in (0,1)$ be such that $\frac{2-\theta}{\theta} \leq q+k-1$ (we will eventually take $\theta = \frac{4}{kn+2}$). Also denote by $(2-\theta)^* := n(2-\theta)/(n-2+\theta)$ the Sobolev conjugate of $2-\theta$. We first obtain from (5.1) the following:

**Lemma 5.1.** *Suppose $f \in C^{1,1}_{\mathrm{loc}}(\Omega \times \mathbb{R})$ is positive, $H = H_1(x,z)|\xi|^2 I$ with $H_1 \in C^{1,1}_{\mathrm{loc}}(\Omega \times \mathbb{R})$ and $H_1 \geq 0$, and that $u \in W^{2,q+k-1}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ $(q > 1)$ is a solution to (4.6). Then*

$$\left(\int_{B_{R+\rho}} (\Delta u + C_1)^{\frac{q(2-\theta)^*}{2}}\right)^{\frac{2}{(2-\theta)^*}} \leq \frac{Cq}{\rho^2}\left(\int_{B_{R+3\rho}} (\Delta u + C_1)^{\frac{2-\theta}{\theta}}\right)^{\frac{\theta}{2-\theta}} \int_{B_{R+3\rho}} (\Delta u + C_1)^{q+k-1}. \tag{5.2}$$

*Proof.* The estimate (5.2) will follow immediately once we establish the estimate

$$\left(\int_{B_{R+\rho}} (\tilde{v}^+)^{\frac{q(2-\theta)^*}{2}}\right)^{\frac{2}{(2-\theta)^*}} \leq \frac{Cq}{\rho^2}\left|J_h^{(\frac{2-\theta}{\theta})}\right|^{\frac{\theta}{2-\theta}} J_h^{(q+k-1)}, \tag{5.3}$$

since we can then apply Fatou's lemma and the fact that $\tilde{v}^+ \to \Delta u + C_1$ a.e. as $h \to 0$ to the term on the LHS of (5.3), and Lemma 2.1 to the terms on the RHS of (5.3).

Keeping in mind the lower bound $\inf_{B_{2R}} f > \frac{1}{C} > 0$, we first observe that by Hölder's inequality and (5.1), we have

$$\begin{aligned}\left(\int_{B_{R+\rho}} \left|\nabla\left((\tilde{v}^+)^{q/2}\right)\right|^{2-\theta}\right)^{\frac{2}{2-\theta}} &\leq C\left(\int_{B_{R+\rho}} (\Delta u - \mathrm{tr}(H))^{\frac{2-\theta}{\theta}}\right)^{\frac{\theta}{2-\theta}} \int_{B_{R+\rho}} f^k \frac{\left|\nabla\left((\tilde{v}^+)^{q/2}\right)\right|^2}{\Delta u - \mathrm{tr}(H)} \\ &\overset{(5.1)}{\leq} \frac{Cq}{\rho^2}\left|J_h^{(\frac{2-\theta}{\theta})}\right|^{\frac{\theta}{2-\theta}} J_h^{(q+k-1)}. \end{aligned} \tag{5.4}$$

On the other hand, since $\frac{q(2-\theta)}{2} \leq q+k-1$, Hölder's inequality gives

$$\left(\int_{B_{R+\rho}} (\tilde{v}^+)^{\frac{q(2-\theta)}{2}}\right)^{\frac{2}{2-\theta}} \leq \left(\int_{B_{R+\rho}} (\tilde{v}^+)^{\frac{2-\theta}{\theta}}\right)^{\frac{\theta}{2-\theta}} \int_{B_{R+\rho}} (\tilde{v}^+)^{q-1} \leq \left|J_h^{(\frac{2-\theta}{\theta})}\right|^{\frac{\theta}{2-\theta}} J_h^{(q+k-1)}. \tag{5.5}$$

Applying the Sobolev inequality to $(\tilde{v}^+)^{q/2} \in W^{1,2-\theta}$, and appealing to (5.4) and (5.5), we arrive at (5.3). $\square$

The inequality (5.2) is of reverse Hölder-type if $\theta$ satisfies

$$\frac{2-\theta}{\theta} < q + k - 1 < \frac{q(2-\theta)^*}{2}.$$

For example, if we fix $\theta = \frac{4}{kn+2}$ and finally impose the assumption $q + k - 1 > \frac{kn}{2}$, we see that $(2-\theta)/\theta = kn/2 < q + k - 1$ and

$$\frac{q(2-\theta)^*}{2} - (q+k-1) > \left(\frac{kn}{2} - k + 1\right)\left(\frac{kn}{2+kn-2k} - 1\right) - k + 1 = 0.$$

In what follows, we denote

$$\beta := \frac{(2-\theta)^*}{2} = \frac{kn}{kn+2-2k} > 1.$$

*Proof of Case 1 of Theorem 1.5 when $f = f(x,z)$.* With $\theta = \frac{4}{kn+2}$, we obtain from (5.2) the estimate

$$\left(\int_{B_{R+\rho}} (\Delta u + C_1)^{\beta q}\right)^{1/\beta} \leq \frac{Cq}{\rho^2} \int_{B_{R+3\rho}} (\Delta u + C_1)^{q+k-1} \tag{5.6}$$

for all $q > \frac{kn}{2} - k + 1$ and $\rho \in (0, \frac{R}{3}]$. The constant $C$ in (5.6) and below now depends on $\int_{B_{R+3\rho}} (\Delta u + C_1)^{kn/2}$, which is finite due to our hypotheses.

We now carry out the Moser iteration argument. Let $p > \frac{kn}{2}$ be as in the statement of Theorem 1.5, and define a sequence $q_j$ inductively by

$$q_0 = p - k + 1, \quad q_j = \beta q_{j-1} - k + 1 \text{ for } j \geq 1.$$

Then $q_j = \beta q_{j-1} - (k-1) = \beta^j q_0 - (k-1)(\beta^{j-1} + \cdots + \beta + 1)$, which implies

$$\frac{q_j}{\beta^j} = q_0 - (k-1)\left(\frac{1-\beta^{-j}}{\beta - 1}\right) \overset{j\to\infty}{\longrightarrow} q_0 - \frac{k-1}{\beta-1} > 0. \tag{5.7}$$

Note that the limit in (5.7) is positive by definition of $\beta$ and the fact that $q_0 > \frac{kn}{2} - k + 1$. In particular, $q_j \to \infty$ as $j \to \infty$.

Applying (5.6) iteratively with $q = q_j$ and $\rho = 3^{-j-1}R$, we have for each $j \geq 0$

$$\begin{aligned}\left(\int_{B_{(1+3^{-j-1})R}} (\Delta u + C_1)^{\beta q_j}\right)^{\beta^{-j-1}} &\leq \left(9^j C q_j \int_{B_{(1+3^{-j})R}} (\Delta u + C_1)^{\beta q_{j-1}}\right)^{\beta^{-j}} \\ &\overset{(5.7)}{\leq} \prod_{i=0}^{j} \left((9\beta)^i C\right)^{\beta^{-i}} \int_{B_{2R}} (\Delta u + C_1)^p \\ &\leq (9\beta)^{\sum_{i=0}^\infty i\beta^{-i}} C^{\sum_{i=0}^\infty \beta^{-i}} \int_{B_{2R}} (\Delta u + C_1)^p.\end{aligned}$$

Letting $j \to \infty$ and appealing once again to (5.7), we arrive at

$$\|\Delta u + C_1\|_{L^\infty(B_R)} \le C\left(\int_{B_{2R}} (\Delta u + C_1)^p\right)^{\left(q_0 - \frac{k-1}{\beta-1}\right)^{-1}},$$

which implies the desired bound on $\|\nabla^2 u\|_{L^\infty(B_R)}$ by the choice of $C_1$. ☐

## 5.2 Proof of Case 2 of Theorem 1.5 and Theorem 1.8 when $f = f(x,z)$

In these cases, we recall that by Corollaries 4.13 and 4.14 we have the estimate

$$\int_{B_{R+\rho}} f^k \frac{\left|\nabla\left((\tilde{v}^+)^{q/2}\right)\right|^2}{\Delta u - \mathrm{tr}(H)} \le \frac{Cq}{\rho^2} J_h^{(q+k)}, \tag{5.8}$$

where $u \in W^{2,q+k}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ $(q > 1)$.

*Proof of Case 2 of Theorem 1.5 and Theorem 1.8 when $f = f(x,z)$.* We let $\theta \in (0,1)$ be such that $\frac{2-\theta}{\theta} \le q + k$. Following the same arguments as in Section 5.1, one readily obtains the following counterpart to the estimate (5.2):

$$\left(\int_{B_{R+\rho}} (\Delta u + C_1)^{\frac{q(2-\theta)^*}{2}}\right)^{\frac{2}{(2-\theta)^*}} \le \frac{Cq}{\rho^2}\left(\int_{B_{R+3\rho}} (\Delta u + C_1)^{\frac{2-\theta}{\theta}}\right)^{\frac{\theta}{2-\theta}} \int_{B_{R+3\rho}} (\Delta u + C_1)^{q+k}. \tag{5.9}$$

Taking $\theta = \frac{4}{(k+1)n+2}$ and imposing $q + k > \frac{(k+1)n}{2}$, we see

$$\frac{2-\theta}{\theta} = \frac{(k+1)n}{2} < q + k < \frac{q(2-\theta)^*}{2}.$$

We thus obtain from (5.9) the estimate

$$\left(\int_{B_{R+\rho}} (\Delta u + C_1)^{\beta q}\right)^{1/\beta} \le \frac{Cq}{\rho^2}\int_{B_{R+3\rho}} (\Delta u + C_1)^{q+k},$$

where

$$\beta := \frac{(k+1)n}{(k+1)n + 2 - 2(k+1)} > 1$$

and $C$ now depends on $\int_{B_{R+3\rho}} (\Delta u + C_1)^{(k+1)n/2}$. The Moser iteration argument then follows through as before, using $p > \frac{(k+1)n}{2}$ and defining $q_j$ inductively by $q_0 = p - k$ and $q_j = \beta q_{j-1} - k$ for $j \ge 1$. ☐

## 5.3 Proof of Theorems 1.5 and 1.8 for $f = f(x, z, \xi)$

In this section we explain how the preceding arguments may be adjusted to treat the general case $f = f(x, z, \xi)$, thus completing the proofs of Theorems 1.5 and 1.8:

*Proof of Theorems 1.5 and 1.8.* The arguments up until (4.12) remain valid for $f = f(x, z, \xi)$, but the last term in (4.10) can no longer be estimated as in (4.12). Consequently, under otherwise the same hypotheses, the conclusion of Lemma 4.2 now reads

$$(\mathrm{I}_1)_h + (\mathrm{I}_2)_h + (\mathrm{I}_3)_h + (\mathrm{I}_4)_h \leq C\rho^{-2} J_h^{(q+k-1)},$$

where $(\mathrm{I}_1)_h$, $(\mathrm{I}_2)_h$ and $(\mathrm{I}_3)_h$ are as before and

$$(\mathrm{I}_4)_h := \sum_l \int_{B_{R+2\rho}} k\eta(\tilde{v}^+)^{q-1} f^{k-1} \Delta_{ll}^h f[u].$$

The estimates for $(\mathrm{I}_1)_h$, $(\mathrm{I}_2)_h$ and $(\mathrm{I}_3)_h$ are unchanged (see Lemmas 4.3, 4.4, 4.5, 4.8 and 4.10), since they do not involve differentiating $f$. The integrand of $(\mathrm{I}_4)_h$ was previously a lower order term, but is now formally of third order in $u$. However, this can be treated using some of the ideas already seen in the proof of Lemma 4.10. Indeed, by the same argument leading to (4.43), we have for each $l \in \{1, \ldots, n\}$ and a.e. $x \in B_{R+2\rho}$ the estimate

$$\begin{aligned} \Delta_{ll}^h f[u](x) \geq{} & \frac{\partial f}{\partial \xi_a}[u](x)\nabla_a \Delta_{ll}^h u(x) - C_\Sigma|\nabla_l^h \nabla u|^2 - C_\Sigma |\nabla_l^{-h}\nabla u|^2 - C|\nabla_l^h \nabla u| - C|\nabla_l^{-h}\nabla u| \\ & + \frac{1}{h^2}\Big(f(x^+, u(x^+), \nabla u(x)) - 2f[u](x) + f(x^-, u(x^-), \nabla u(x))\Big). \end{aligned} \tag{5.10}$$

As before, the constant $C_\Sigma > 0$ is such that the mapping $\xi \longmapsto f(x, z, \xi) + C_\Sigma|\xi|^2$ is convex for all $(x, z, \xi) \in \Sigma$. Denoting all but the first term on the RHS of (5.10) as error terms, it follows from (5.10) that

$$(\mathrm{I}_4)_h \geq \int_{B_{R+2\rho}} k\eta(\tilde{v}^+)^{q-1} f^{k-1} \frac{\partial f}{\partial \xi_a}[u]\nabla_a \tilde{v} - \int_{B_{R+2\rho}} k\eta(\tilde{v}^+)^{q-1} f^{k-1}|\text{error terms}|. \tag{5.11}$$

Now, in the same way that we dealt with the error terms in Step 2 of the proof of Lemma 4.10, one readily obtains $\int_{B_{R+2\rho}} k\eta(\tilde{v}^+)^{q-1} f^{k-1}|\text{error terms}| \leq CJ_h^{(q+1)}$. For the first integral on the RHS of (5.11), we integrate by parts and apply Hölder's inequality to obtain

$$\bigg|\int_{B_{R+2\rho}} k\eta(\tilde{v}^+)^{q-1} f^{k-1}\frac{\partial f}{\partial \xi_a}[u]\nabla_a \tilde{v}\bigg| \leq C\rho^{-1} J_h^{(q+1)}.$$

Returning to (5.11), we therefore obtain $(\mathrm{I}_4)_h \geq -C\rho^{-1} J_h^{(q+1)}$. As a consequence, the estimates (5.1) and (5.8) hold, and the arguments of Section 5 therefore apply without any changes. ∎

# 6 The case $k \geq 3$ for general $H$

In this final section we consider a minor extension of Theorem 1.8. Recall that our proof of Theorems 1.5 and 1.8 exploited a cancellation phenomenon between higher order terms arising from $(\mathrm{I}_2)_h$ and $(\mathrm{I}_3)_h$, where the divergence structure of $F^{ij}$ played a role in estimating $(\mathrm{I}_2)_h$. When $3 \leq k \leq n$ and $H$ is not necessarily a multiple of the identity, the divergence structure given in (3.4) is more involved and the resulting arguments fall outside the scope of the present paper. That said, if one assumes higher integrability on $\nabla^2 u$ from the outset, the terms $(\mathrm{I}_2)_h$ and $(\mathrm{I}_3)_h$ may be estimated by using Cauchy's inequality and absorbing the resulting negative higher order terms into the positive term $(\mathrm{I}_1)_h$. This avoids the need to prove any cancellation between $(\mathrm{I}_2)_h$ and $(\mathrm{I}_3)_h$. We establish:

**Theorem 6.1.** *Let $\Omega$ be a domain in $\mathbb{R}^n$ $(n \geq 3)$, $f = f(x,z,\xi) \in C^{1,1}_{\mathrm{loc}}(\Omega\times\mathbb{R}\times\mathbb{R}^n)$ a positive function and $H \in C^{1,1}_{\mathrm{loc}}(\Omega\times\mathbb{R}\times\mathbb{R}^n\,;\mathrm{Sym}_n(\mathbb{R}))$. Suppose $3 \leq k \leq n$, $p > kn$ and $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ is a solution to (1.4). Then $u \in C^{1,1}_{\mathrm{loc}}(\Omega)$, and for any concentric balls $B_R \subset B_{2R} \Subset \Omega$ we have*

$$\|\nabla^2 u\|_{L^\infty(B_R)} \leq C,$$

*where $C$ is a constant depending only on $n, p, R, f, H$ and an upper bound for $\|u\|_{W^{2,p}(B_{2R})}$.*

*Proof.* Following the proof of Theorem 1.8 in Section 5.3 but leaving the terms $(\mathrm{I}_2)_h$ and $(\mathrm{I}_3)_h$ untreated, we have for $u \in W^{2,q+k-1}_{\mathrm{loc}}(\Omega)\cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ $(q>1)$

$$\frac{q-1}{Cq^2}\int_{B_{R+2\rho}} \eta\frac{\left|\nabla\left((\tilde{v}^+)^{q/2}\right)\right|^2}{\mathrm{tr}(A_H)} + (\mathrm{I}_2)_h + (\mathrm{I}_3)_h \leq C\rho^{-2}J_h^{(q+k-1)}. \tag{6.1}$$

We now suppose further that $\nabla^2 u \in L^{q+2k-1}_{\mathrm{loc}}(\Omega)$ $(q>1)$. By Cauchy's inequality and the bound $|\operatorname{div} F[u]| \leq C(\Delta u + C_1)^{k-1}$, we see that for all $\delta > 0$

$$\begin{aligned}(\mathrm{I}_2)_h &= \frac{2}{q}\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q/2}\nabla_i F[u]^{ij}\nabla_j(\tilde{v}^+)^{q/2}\\ &\geq -\frac{\delta(q-1)}{q^2}\int_{B_{R+2\rho}}\eta\frac{\left|\nabla\left((\tilde{v}^+)^{q/2}\right)\right|^2}{\mathrm{tr}(A_H)} - \frac{1}{\delta(q-1)}\int_{B_{R+2\rho}}\eta(\tilde{v}^+)^q\,\mathrm{tr}(A_H)\left|\operatorname{div}F[u]\right|^2\\ &\geq -\frac{\delta(q-1)}{q^2}\int_{B_{R+2\rho}}\eta\frac{\left|\nabla\left((\tilde{v}^+)^{q/2}\right)\right|^2}{\mathrm{tr}(A_H)} - \frac{C}{\delta(q-1)}J_h^{(q+2k-1)}.\end{aligned} \tag{6.2}$$

By similar reasoning, it also holds that

$$\begin{aligned}(\mathrm{I}_3)_h &\overset{(4.26)}{\geq} \int_{B_{R+2\rho}}\eta(\tilde{v}^+)^{q-1}F^{ij}\frac{\partial H_{ij}}{\partial\xi_a}[u]\,\nabla_a\tilde{v} - CJ_h^{(q+k)}\\ &\geq -\frac{\delta(q-1)}{q^2}\int_{B_{R+2\rho}}\eta\frac{\left|\nabla\left((\tilde{v}^+)^{q/2}\right)\right|^2}{\mathrm{tr}(A_H)} - \frac{C}{\delta(q-1)}J_h^{(q+2k-1)}.\end{aligned} \tag{6.3}$$

Taking $\delta$ sufficiently small in (6.2) and (6.3), and then substituting these estimates into (6.1), we obtain

$$\frac{q-1}{q^2}\int_{B_{R+2\rho}} \eta \frac{\left|\nabla\left((\tilde{v}^+)^{q/2}\right)\right|^2}{\mathrm{tr}(A_H)} \le C\rho^{-2} J_h^{(q+2k-1)}. \tag{6.4}$$

The argument then proceeds as in Section 5.1: we let $\theta \in (0,1)$ be such that $\frac{2-\theta}{\theta} \le q+2k-1$ and obtain from (6.4) the estimate

$$\left(\int_{B_{R+\rho}} (\Delta u + C_1)^{\frac{q(2-\theta)^*}{2}}\right)^{\frac{2}{(2-\theta)^*}} \le \frac{Cq}{\rho^2}\left(\int_{B_{R+3\rho}} (\Delta u + C_1)^{\frac{2-\theta}{\theta}}\right)^{\frac{\theta}{2-\theta}} \int_{B_{R+3\rho}} (\Delta u + C_1)^{q+2k-1}. \tag{6.5}$$

Taking $\theta = \frac{2}{kn+1}$ and imposing $q+2k-1 > kn$, we see that $\frac{2-\theta}{\theta} = kn < q+2k-1 < \frac{q(2-\theta)^*}{2}$, and we therefore obtain from (6.5) the estimate

$$\left(\int_{B_{R+\rho}} (\Delta u + C_1)^{\beta q}\right)^{1/\beta} \le \frac{Cq}{\rho^2}\int_{B_{R+3\rho}} (\Delta u + C_1)^{q+2k-1},$$

where $\beta := kn/(kn+1-2k) > 1$ and $C$ now depends on $\int_{B_{R+3\rho}} (\Delta u + C_1)^{kn}$. The Moser iteration argument then goes through as before, giving the desired conclusion. $\square$

# A A remark on the regularity of solutions to the $\sigma_2$-Yamabe equation obtained by vanishing viscosity

Let $(M^4, g_0)$ be a 4-manifold with scalar curvature $R_0 > 0$ and Schouten tensor $A_0$ satisfying $\int_{M^4} \sigma_2(A_0)\, dv_0 > 0$. In [**12**], the existence of smooth solutions $g_{w_\delta} = e^{2w_\delta} g_0$ with positive scalar curvature to the fourth order equation

$$\sigma_2(A_{g_{w_\delta}}) = \frac{\delta}{4}\Delta_{g_{w_\delta}} R_{g_{w_\delta}} - 2\gamma_1 |\eta|^2_{g_{w_\delta}} \tag{A.1}$$

is established for each $\delta \in (0,1]$, where $\gamma_1 < 0$ is a carefully chosen conformal invariant and $\eta$ is any fixed non-vanishing $(0,2)$-tensor. Moreover, solutions are shown to satisfy the uniform estimates

$$\|w_\delta\|_{W^{2,s}(M^4,g_0)} \le C \quad \text{for all } \delta \in (0,1], \quad 1 \le s < 5, \tag{A.2}$$

where the constant $C = C(s)$ is independent of $\delta$. A heat flow argument is then applied to obtain a conformal metric $g$ with $\lambda(A_g) \in \Gamma_2^+$. In this appendix, we show that in the case that $(M^4, g_0)$ is locally conformally flat, we may take the limit $\delta \to 0$ more directly in (A.1) to obtain the desired conformal metric with $\lambda(A_g) \in \Gamma_2^+$. More precisely, using Theorem 1.1

and a result of [**46**], we show that, along a subsequence, the solutions $w_\delta$ converge weakly to a smooth solution of the equation $\sigma_2(A_{g_{w_\delta}}) = -2\gamma_1|\eta|^2_{g_{w_\delta}} > 0$.

To this end, fix $4 < s < 5$. We first observe that by (A.2), we can find a sequence $\delta_i \to 0$ for which $w_i := w_{\delta_i}$ converges weakly in $W^{2,s}(M^4, g_0)$, say to $w \in W^{2,s}(M^4, g_0)$. By the Morrey embedding $W^{2,s}(M^4, g_0) \hookrightarrow C^{1,1-\frac{4}{s}}(M^4, g_0)$, we may assume $w_i \to w$ in $C^{1,\alpha}(M^4, g_0)$ for some $\alpha > 0$. It then follows from [**46, Proposition 5.3**] and the estimate (A.2) that for all $\varphi \in C^0(M^4)$, we have

$$\lim_{i\to\infty} \int_{M^4} \sigma_2(A_{g_{w_i}})\varphi \, dv_0 = \int_{M^4} \sigma_2(A_{g_w})\varphi \, dv_0. \tag{A.3}$$

Substituting the equation (A.1) into (A.3) and integrating by parts, we therefore see that

$$\int_{M^4} \sigma_2(A_{g_w})\varphi \, dv_0 = \lim_{i\to\infty} \int_{M^4} \left( \frac{\delta_i}{4} R_{g_{w_i}} \Delta_{g_{w_i}} \varphi - 2\gamma_1 |\eta|^2_{g_{w_i}} \varphi \right) dv_0 = -\int_{M^4} 2\gamma_1 |\eta|^2_{g_w} \varphi \, dv_0$$

for all $\varphi \in C^2(M^4, g_0)$. It follows that $w \in W^{2,s}(M^4, g_0)$ solves

$$\sigma_2(A_{g_w}) = -2\gamma_1 |\eta|^2_{g_w} > 0 \quad \text{a.e. in } M^4. \tag{A.4}$$

Moreover, as $R_{g_{w_i}} > 0$ for each $i$, it follows that $R_{g_w} \geq 0$, and by (A.4) we therefore have $R_{g_w} > 0$ a.e. If $(M^4, g_0)$ is locally conformally flat, we therefore obtain from Theorem 1.1 that $u := e^{-w} \in C^{1,1}(M^4, g_0)$, and consequently (A.4) is uniformly elliptic at $w$.

At this point, we apply the Evans-Krylov theorem to obtain $u \in C^{2,\alpha}(M^4, g_0)$. Indeed, by the proof of [**7, Theorem 6.6**], it suffices to observe that, by Lemmas 4.1 and 4.16, $v = \sum_l \Delta^h_{ll} u$ is a subsolution to a uniformly elliptic linear equation, namely

$$F^{ij} \nabla_i \nabla_j v + B^i D_i v \geq C,$$

where $F^{ij}$ is uniformly elliptic and $F^{ij}$, $B^i$ and $C$ are bounded. Furthermore, since $f(x,u) := -2\gamma_1 |\eta(x)|^2_{u^{-2}g_0} = -2\gamma_1 u^4 |\eta(x)|^2_{g_0}$ is smooth, standard elliptic regularity ensures that $u$ (and hence $w$) belongs to $C^\infty(M^4, g_0)$.

# B Proof of Lemma 2.1

The proof is a standard argument using Taylor's theorem. We claim that

$$\begin{aligned} \|v_h - \Delta u\|_{L^s(\Omega')} \leq & \sum_{l=1}^n \int_0^1 \left\| \nabla_l \nabla_l u(x + the_l) - \nabla_l \nabla_l u(x) \right\|_{L^s(\Omega')} dt \\ & + \sum_{l=1}^n \int_0^1 \left\| \nabla_l \nabla_l u(x - the_l) - \nabla_l \nabla_l u(x) \right\|_{L^s(\Omega')} dt \end{aligned} \tag{B.1}$$

for all $u \in W^{2,s}(\Omega)$ and $\Omega' \Subset \Omega$ satisfying $|h| < \operatorname{dist}(\Omega', \partial\Omega)$, from which the conclusion follows by the continuity of the translation operator in $L^s(\Omega)$. By density it suffices to prove (B.1) for $u \in C^2(\Omega)$. Let $\Omega'$ be as above. Then for each $x \in \Omega'$ and $l \in \{1, \ldots, n\}$, we have by Taylor's theorem

$$u(x \pm he_l) = u(x) \pm h\nabla_l u(x) + h^2 \int_0^1 (1-t)\nabla_l\nabla_l u(x \pm the_l)\,dt,$$

and thus

$$\begin{aligned} v_h(x) - \Delta u(x) &= \sum_{l=1}^n \int_0^1 (1-t)\Big(\nabla_l\nabla_l u(x+the_l) - \nabla_l\nabla_l u(x)\Big)\,dt \\ &\quad + \sum_{l=1}^n \int_0^1 (1-t)\Big(\nabla_l\nabla_l u(x-the_l) - \nabla_l\nabla_l u(x)\Big)\,dt. \end{aligned} \tag{B.2}$$

Let $s'$ be such that $\frac{1}{s} + \frac{1}{s'} = 1$. It follows from (B.2) and Hölder's inequality that for all $g \in L^{s'}(\Omega')$ satisfying $\|g\|_{L^{s'}(\Omega')} \leq 1$, we have

$$\begin{aligned} \int_{\Omega'} \big(v_h(x) - \Delta u(x)\big)g(x)\,dx &= \sum_{l=1}^n \int_0^1 (1-t)\int_{\Omega'} \Big(\nabla_l\nabla_l u(x+the_l) - \nabla_l\nabla_l u(x)\Big)g(x)\,dx\,dt \\ &\quad + \sum_{l=1}^n \int_0^1 (1-t)\int_{\Omega'} \Big(\nabla_l\nabla_l u(x-the_l) - \nabla_l\nabla_l u(x)\Big)g(x)\,dx\,dt \\ &\leq \sum_{l=1}^n \int_0^1 \Big\|\nabla_l\nabla_l u(x+the_l) - \nabla_l\nabla_l u(x)\Big\|_{L^s(\Omega')}\,dt \\ &\quad + \sum_{l=1}^n \int_0^1 \Big\|\nabla_l\nabla_l u(x-the_l) - \nabla_l\nabla_l u(x)\Big\|_{L^s(\Omega')}\,dt. \end{aligned} \tag{B.3}$$

Taking the supremum over such $g$ in (B.3), we obtain (B.1).

# References


[**1**] D. P. Abanto and J. M. Espinar, *Escobar type theorems for elliptic fully nonlinear degenerate equations*, Amer. J. Math., 141 (2019), pp. 1179–1216.

[**2**] E. Barbosa, M. P. Cavalcante, and J. M. Espinar, *Min-Oo conjecture for fully nonlinear conformally invariant equations*, Comm. Pure Appl. Math., 72 (2019), pp. 2259–2281.

[**3**] L. Bo and W. Sheng, *Some rigidity properties for manifolds with constant k-curvature of modified Schouten tensor*, J. Geom. Anal., 29 (2019), pp. 2862–2887.

[**4**] T. P. Branson and A. R. Gover, *Variational status of a class of fully nonlinear curvature prescription problems*, Calc. Var. PDE, 32 (2008), pp. 253–262.

[**5**] S. Brendle and J. A. Viaclovsky, *A variational characterization for* $\sigma_{n/2}$, Calc. Var. PDE, 20 (2004), pp. 399–402.

[**6**] L. Caffarelli, L. Nirenberg, and J. Spruck, *The Dirichlet problem for nonlinear second-order elliptic equations. III. Functions of the eigenvalues of the Hessian*, Acta Math., 155 (1985), pp. 261–301.

[**7**] L. A. Caffarelli and X. Cabré, *Fully nonlinear elliptic equations*, vol. 43 of American Mathematical Society Colloquium Publications, American Mathematical Society, Providence, RI, 1995.

[**8**] J. S. Case, *The weighted* $\sigma_k$*-curvature of a smooth metric measure space*, Pacific J. Math., 299 (2019), pp. 339–399.

[**9**] J. S. Case and Y. Wang, *Boundary operators associated to the* $\sigma_k$*-curvature*, Adv. Math., 337 (2018), pp. 83–106.

[**10**] ———, *On a fully nonlinear sharp Sobolev trace inequality*, https://arxiv.org/abs/1910.14232, (preprint 2019).

[**11**] G. Catino and Z. Djadli, *Conformal deformations of integral pinched 3-manifolds*, Adv. Math., 223 (2010), pp. 393–404.

[**12**] S.-Y. A. Chang, M. J. Gursky, and P. C. Yang, *An equation of Monge-Ampère type in conformal geometry, and four-manifolds of positive Ricci curvature*, Ann. of Math. (2), 155 (2002), pp. 709–787.

[**13**] S.-Y. A. Chang, Z.-C. Han, and P. C. Yang, *Classification of singular radial solutions to the* $\sigma_k$ *Yamabe equation on annular domains*, J. Differential Equations, 216 (2005), pp. 482–501.

[**14**] S.-y. S. Chen, *Local estimates for some fully nonlinear elliptic equations*, Int. Math. Res. Not., (2005), pp. 3403–3425.

[**15**] T. C. Collins and C. Mooney, *Dimension of the minimum set for the real and complex Monge-Ampère equations in critical Sobolev spaces*, Anal. PDE, 10 (2017), pp. 2031–2041.

[**16**] H. Fang and W. Wei, $\sigma_2$ *Yamabe problem on conic 4-spheres*, Calc. Var. PDE, 58 (2019), pp. Paper No. 119, 19.

[**17**] ——, *A $\sigma_2$ Penrose inequality for conformal asymptotically hyperbolic 4-discs*, https://arxiv.org/abs/2003.02875, (preprint 2020).

[**18**] Y. Ge, C.-S. Lin, and G. Wang, *On the $\sigma_2$-scalar curvature*, J. Differential Geom., 84 (2010), pp. 45–86.

[**19**] Y. Ge and G. Wang, *On a fully nonlinear Yamabe problem*, Ann. Sci. École Norm. Sup. (4), 39 (2006), pp. 569–598.

[**20**] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001.

[**21**] M. d. M. González, *Singular sets of a class of locally conformally flat manifolds*, Duke Math. J., 129 (2005), pp. 551–572.

[**22**] ——, *Removability of singularities for a class of fully non-linear elliptic equations*, Calc. Var. PDE, 27 (2006), pp. 439–466.

[**23**] M. d. M. González, Y. Y. Li, and L. Nguyen, *Existence and uniqueness to a fully nonlinear version of the Loewner-Nirenberg problem*, Commun. Math. Stat., 6 (2018), pp. 269–288.

[**24**] B. Guan, *Conformal metrics with prescribed curvature functions on manifolds with boundary*, Amer. J. Math., 129 (2007), pp. 915–942.

[**25**] ——, *Complete conformal metrics of negative Ricci curvature on compact manifolds with boundary*, Int. Math. Res. Not. IMRN, (2008). Art. ID rnn 105, 25pp.

[**26**] P. Guan and G. Wang, *A fully nonlinear conformal flow on locally conformally flat manifolds*, J. Reine Angew. Math., 557 (2003), pp. 219–238.

[**27**] ——, *Local estimates for a class of fully nonlinear equations arising from conformal geometry*, Int. Math. Res. Not., (2003), pp. 1413–1432.

[**28**] M. Gursky and J. Streets, *A formal Riemannian structure on conformal classes and uniqueness for the $\sigma_2$-Yamabe problem*, Geom. Topol., 22 (2018), pp. 3501–3573.

[**29**] M. J. Gursky, J. Streets, and M. Warren, *Existence of complete conformal metrics of negative Ricci curvature on manifolds with boundary*, Calc. Var. PDE, 41 (2011), pp. 21–43.

[**30**] M. J. Gursky and J. A. Viaclovsky, *Fully nonlinear equations on Riemannian manifolds with negative curvature*, Indiana Univ. Math. J., 52 (2003), pp. 399–419.

[**31**] ——, *Prescribing symmetric functions of the eigenvalues of the Ricci tensor*, Ann. of Math. (2), 166 (2007), pp. 475–531.

[**32**] Q. Han, X. Li, and Y. Li, *Asymptotic expansions of solutions of the Yamabe equation and the $\sigma_k$-Yamabe equation near isolated singular points*, https://arxiv.org/abs/1909.07466, (preprint 2019).

[**33**] Z.-C. Han, *Local pointwise estimates for solutions of the $\sigma_2$ curvature equation on 4-manifolds*, Int. Math. Res. Not., (2004), pp. 4269–4292.

[**34**] ——, *A Kazdan-Warner type identity for the $\sigma_k$ curvature*, C. R. Math. Acad. Sci. Paris, 342 (2006), pp. 475–478.

[**35**] Z.-C. Han, Y. Y. Li, and E. V. Teixeira, *Asymptotic behavior of solutions to the $\sigma_k$-Yamabe equation near isolated singularities*, Invent. Math., 182 (2010), pp. 635–684.

[**36**] F. Jiang and N. S. Trudinger, *Oblique boundary value problems for augmented Hessian equations II*, Nonlinear Anal., 154 (2017), pp. 148–173.

[**37**] ——, *Oblique boundary value problems for augmented Hessian equations I*, Bull. Math. Sci., 8 (2018), pp. 353–411.

[**38**] ——, *Oblique boundary value problems for augmented Hessian equations III*, Comm. Partial Differential Equations, 44 (2019), pp. 708–748.

[**39**] Q. Jin, A. Li, and Y. Y. Li, *Estimates and existence results for a fully nonlinear Yamabe problem on manifolds with boundary*, Calc. Var. PDE, 28 (2007), pp. 509–543.

[**40**] A. Li and Y. Y. Li, *On some conformally invariant fully nonlinear equations*, Comm. Pure Appl. Math., 56 (2003), pp. 1416–1464.

[**41**] ——, *On some conformally invariant fully nonlinear equations. II. Liouville, Harnack and Yamabe*, Acta Math., 195 (2005), pp. 117–154.

[**42**] J. Li and W. Sheng, *Deforming metrics with negative curvature by a fully nonlinear flow*, Calc. Var. PDE, 23 (2005), pp. 33–50.

[**43**] Y. Y. Li, *Local gradient estimates of solutions to some conformally invariant fully nonlinear equations*, Comm. Pure Appl. Math., 62 (2009), pp. 1293–1326.

[**44**] Y. Y. Li and L. Nguyen, *A compactness theorem for a fully nonlinear Yamabe problem under a lower Ricci curvature bound*, J. Funct. Anal., 266 (2014), pp. 3741–3771.

[**45**] ——, *Solutions to the $\sigma_k$-Loewner-Nirenberg problem on annuli are locally Lipschitz and not differentiable*, To appear in J. Math. Study, (2020). https://arxiv.org/abs/2001.04257.

[**46**] ——, *Existence and uniqueness of Green's function to a nonlinear Yamabe problem*, https://arxiv.org/abs/2001.00993, (preprint 2020).

[**47**] Y. Y. Li, L. Nguyen, and B. Wang, *On the $\sigma_k$-Nirenberg problem*, https://arxiv.org/abs/2008.08437, (preprint 2020).

[**48**] M. Lin and N. S. Trudinger, *The Dirichlet problem for the prescribed curvature quotient equations*, Topol. Methods Nonlinear Anal., 3 (1994), pp. 307–323.

[**49**] S. Liu and J. Bao, *The local regularity for strong solutions of the Hessian quotient equation*, J. Math. Anal. Appl., 303 (2005), pp. 462–476.

[**50**] G. Loeper, *On the regularity of solutions of optimal transportation problems*, Acta Math., 202 (2009), pp. 241–283.

[**51**] X.-N. Ma, N. S. Trudinger, and X.-J. Wang, *Regularity of potential functions of the optimal transportation problem*, Arch. Ration. Mech. Anal., 177 (2005), pp. 151–183.

[**52**] R. C. Reilly, *On the Hessian of a function and the curvatures of its graph*, Michigan Math. J., 20 (1973), pp. 373–383.

[**53**] W.-M. Sheng, N. S. Trudinger, and X.-J. Wang, *The Yamabe problem for higher order curvatures*, J. Differential Geom., 77 (2007), pp. 515–553.

[**54**] A. Silva Santos, *Solutions to the singular $\sigma_2$-Yamabe problem with isolated singularities*, Indiana Univ. Math. J., 66 (2017), pp. 741–790.

[**55**] Z. Sui, *Complete conformal metrics of negative Ricci curvature on Euclidean spaces*, J. Geom. Anal., 27 (2017), pp. 893–907.

[**56**] N. S. Trudinger and X.-J. Wang, *On Harnack inequalities and singularities of admissible metrics in the Yamabe problem*, Calc. Var. PDE, 35 (2009), pp. 317–338.

[**57**] ——, *On the second boundary value problem for Monge-Ampère type equations and optimal transportation*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 8 (2009), pp. 143–174.

[**58**] J. Urbas, *Regularity of generalized solutions of Monge-Ampère equations*, Math. Z., 197 (1988), pp. 365–393.

[**59**] ——, *On the existence of nonclassical solutions for two classes of fully nonlinear elliptic equations*, Indiana Univ. Math. J., 39 (1990), pp. 355–382.

[**60**] ——, *Some interior regularity results for solutions of Hessian equations*, Calc. Var. PDE, 11 (2000), pp. 1–31.

[**61**] ——, *An interior second derivative bound for solutions of Hessian equations*, Calc. Var. PDE, 12 (2001), pp. 417–431.

[**62**] ——, *An approximation result for solutions of Hessian equations*, Calc. Var. PDE, 29 (2007), pp. 219–230.

[**63**] J. A. Viaclovsky, *Conformal geometry, contact geometry, and the calculus of variations*, Duke Math. J., 101 (2000), pp. 283–316.

[**64**] ——, *Estimates and existence results for some fully nonlinear elliptic equations on Riemannian manifolds*, Comm. Anal. Geom., 10 (2002), pp. 815–846.

[**65**] X.-J. Wang, *A priori estimates and existence for a class of fully nonlinear elliptic equations in conformal geometry*, Chinese Ann. Math. Ser. B, 27 (2006), pp. 169–178.