PROMYS Europe Connect 2021

Application Problem Set

https://promys-europe.org/

Please attempt each of the following problems. Though they can all be solved with no more than a standard high school mathematics background, most of the problems require considerably more ingenuity than is usually expected in high school. You should keep in mind that we do not expect you to find complete solutions to all of them. Rather, we are looking to see how you approach challenging problems.

We ask that you tackle these problems by yourself. Please see below for further information on citing any sources you use in your explorations.

Here are a few suggestions:

- Think carefully about the meaning of each problem.
- Examine special cases, either through numerical examples or by drawing pictures.
- Be bold in making conjectures.
- Test your conjectures through further experimentation, and try to devise mathematical proofs to support the surviving ones.
- Can you solve special cases of a problem, or state and solve simpler but related problems?

If you think you know the answer to a question, but cannot prove that your answer is correct, tell us what kind of evidence you have found to support your belief. If you use books, articles, or websites in your explorations, be sure to cite your sources.

Be careful if you search online for help. We are interested in your ideas, not in solutions that you have found elsewhere. If you search online for a problem and find a solution (or most of a solution), it will be much harder for you to demonstrate your insight to us.

You may find that most of the problems require some patience. Do not rush through them. It is not unreasonable to spend a month or more thinking about the problems. It might be good strategy to devote most of your time to a small selection of problems which you find especially interesting. Be sure to tell us about progress you have made on problems not yet completely solved. **For each problem you solve, please justify your answer clearly and tell us how you arrived at your solution: this includes your experimentation and any thinking that led you to an argument.**

There are various tools available for typesetting mathematics on a computer. You are welcome to use one of these if you choose, or you are welcome to write your solutions by hand, or you might want to do a bit of both. We are interested in your mathematical ideas, not in your typesetting. What is important is that we can read all of your submitted work, and that you can include all of your ideas (including the ones that didn't fully work).

1. Calculate each of the following:

$$\begin{aligned} 1^3 + 5^3 + 3^3 &= ?? \\ 16^3 + 50^3 + 33^3 &= ?? \\ 166^3 + 500^3 + 333^3 &= ?? \\ 1666^3 + 5000^3 + 3333^3 &= ?? \end{aligned}$$

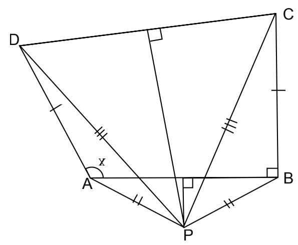
What do you see? Can you state and prove a generalization of your observations?

2. The sequence (x_n) of positive real numbers satisfies the relationship $x_{n-1}x_nx_{n+1} = 1$ for all $n \geqslant 2$. If $x_1 = 1$ and $x_2 = 2$, what are the values of the next few terms? What can you say about the sequence? What happens for other starting values?

The sequence (y_n) satisfies the relationship $y_{n-1}y_{n+1} + y_n = 1$ for all $n \geqslant 2$. If $y_1 = 1$ and $y_2 = 2$, what are the values of the next few terms? What can you say about the sequence? What happens for other starting values?

3. Consider the sequence $t_0 = 3$, $t_1 = 3^3$, $t_2 = 3^{3^3}$, $t_3 = 3^{3^{3^3}}$, ... defined by $t_0 = 3$ and $t_{n+1} = 3^{t_n}$ for $n \geqslant 0$. What are the last two digits in $t_3 = 3^{3^{3^3}}$? Can you say what the last *three* digits are? Show that the last 10 digits of t_k are the same for all $k \geqslant 10$.

4. According to the Journal of Irreproducible Results, any obtuse angle is a right angle!



Here is their argument. Given the obtuse angle x, we make a quadrilateral $ABCD$ with $\angle DAB = x$, and $\angle ABC = 90°$, and $AD = BC$. Say the perpendicular bisector to DC meets the perpendicular bisector to AB at P. Then $PA = PB$ and $PC = PD$. So the triangles PAD and PBC have equal sides and are congruent. Thus $\angle PAD = \angle PBC$. But PAB is isosceles, hence $\angle PAB = \angle PBA$. Subtracting gives $x = \angle PAD - \angle PAB = \angle PBC - \angle PBA = 90°$. This is a preposterous conclusion – just where is the mistake in the "proof" and why does the argument break down there?

5. A monkey has filled in a 3×3 grid with the numbers $1, 2, \ldots, 9$. A cat writes down the three numbers obtained by multiplying the numbers in each horizontal row. A dog writes down the three numbers obtained by multiplying the numbers in each vertical column. Can the monkey fill in the grid in such a way that the cat and dog obtain the same lists of three numbers? What if the monkey writes the numbers $1, 2, \ldots, 25$ in a 5×5 grid? Or $1, 2, \ldots, 121$ in a 11×11 grid? Can you find any conditions on n that guarantee that it is possible or any conditions that guarantee that it is impossible for the monkey to write the numbers $1, 2, \ldots, n^2$ in an $n \times n$ grid so that the cat and the dog obtain the same lists of numbers?

6. For each positive integer n, consider the pair $(3^n - 1, 5^n - 1)$. For $n = 1$, this is the pair $(2, 4)$ and for $n = 2$, it is $(8, 24)$. Can you find a value for n such that both numbers in the pair $(3^n - 1, 5^n - 1)$ are divisible by $d = 7$? Can you find more than one such n? Do you think there are finitely or infinitely many n such that $3^n - 1$ and $5^n - 1$ are both divisible by $d = 7$? Why? Explain your reasoning as carefully as you can.

Now, what if we replace $d = 7$ with $d = 11$? Or with $d = 13$? What if we take $d = 77$ or $d = 1001$? Describe any patterns you think are interesting.

7. The set S contains some real numbers, according to the following three rules.

(i) $\frac{1}{1}$ is in S.

(ii) If $\frac{a}{b}$ is in S, where $\frac{a}{b}$ is written in lowest terms (that is, a and b have highest common factor 1), then $\frac{b}{2a}$ is in S.

(iii) If $\frac{a}{b}$ and $\frac{c}{d}$ are in S, where they are written in lowest terms, then $\frac{a+c}{b+d}$ is in S.

These rules are exhaustive: if these rules do not imply that a number is in S, then that number is not in S. Can you describe which numbers are in S?

8. Let P_0 be an equilateral triangle of area 10. Each side of P_0 is trisected into three segments of equal length, and the corners of P_0 are snipped off, creating a new polygon (in fact, a hexagon) P_1. What is the area of P_1? Now repeat the process to P_1 – i.e. trisect each side and snip off the corners – to obtain a new polygon P_2. What is the area of P_2? Now repeat this process infinitely to create an object P_∞. What can you say about the shape P_∞? What is the area of P_∞?

9. On a strange railway line, there is just one infinitely long track, so overtaking is impossible. Any time a train catches up to the one in front of it, they link up to form a single train moving at the speed of the slower train. At first, there are three equally spaced trains, each moving at a different speed. You watch, and eventually (after all the linking that will happen has happened), you count the trains. You wonder what would have happened if the trains had started in a different order (but each of the original three trains had kept its same starting speed). On average (averaging over all possible orderings), how many trains will there be after a long time has elapsed? What if at the start there are 4 trains (all moving at different speeds)? Or 5? Or n?

10. A particular two-player game starts with a pile of diamonds and a pile of rubies. On your turn, you can take any number of diamonds, or any number of rubies, or an equal number of each. You must take at least one gem on each of your turns. Whoever takes the last gem wins the game. For example, in a game that starts with 5 diamonds and 10 rubies, a game could look like: you take 2 diamonds, then your opponent takes 7 rubies, then you take 3 diamonds and 3 rubies to win.

You get to choose the starting number of diamonds and rubies, and whether you go first or second. Find all starting configurations (including who goes first) with 8 gems where you are guaranteed to win. If you have to let your opponent go first, what are the starting configurations of gems where you are guaranteed to win? If you can't find all such configurations, describe the ones you do find and any patterns you see.