PROMYS Europe 2024

Application Problem Set

https://www.promys-europe.org

Please attempt each of the following 8 problems. Though they can all be solved with no more than a standard high school mathematics background, most of the problems require considerably more ingenuity than is usually expected in high school. You should keep in mind that we do not expect you to find complete solutions to all of them. Rather, we are looking to see how you approach challenging problems.

We ask that you tackle these problems by yourself. Please see below for further information on citing any sources you use in your explorations.

Here are a few suggestions:

- Think carefully about the meaning of each problem.
- Examine special cases, either through numerical examples or by drawing pictures.
- Be bold in making conjectures.
- Test your conjectures through further experimentation, and try to devise mathematical proofs to support the surviving ones.
- Can you solve special cases of a problem, or state and solve simpler but related problems?

If you think you know the answer to a question, but cannot prove that your answer is correct, tell us what kind of evidence you have found to support your belief. If you use books, articles, or websites in your explorations, be sure to cite your sources.

Be careful if you search online for help. We are interested in your ideas, not in solutions that you have found elsewhere. If you search online for a problem and find a solution (or most of a solution), it will be much harder for you to demonstrate your insight to us.

You may find that most of the problems require some patience. Do not rush through them. It is not unreasonable to spend a month or more thinking about the problems. It might be good strategy to devote most of your time to a small selection of problems which you find especially interesting. Be sure to tell us about progress you have made on problems not yet completely solved. **For each problem you solve, please justify your answer clearly and tell us how you arrived at your solution: this includes your experimentation and any thinking that led you to an argument.**

There are various tools available for typesetting mathematics on a computer. You are welcome to use one of these if you choose, or you are welcome to write your solutions by hand, or you might want to do a bit of both. We are interested in your mathematical ideas, not in your typesetting. What is important is that we can read all of your submitted work, and that you can include all of your ideas (including the ones that didn't fully work).

1. A unit fraction is a fraction of the form $\frac{1}{n}$ where n is a positive integer. Note that the unit fraction $\frac{1}{11}$ can be written as the sum of two unit fractions in the following three ways:

$$\frac{1}{11} = \frac{1}{12} + \frac{1}{132} = \frac{1}{22} + \frac{1}{22} = \frac{1}{132} + \frac{1}{12}.$$

Are there any other ways of decomposing $\frac{1}{11}$ into the sum of two unit fractions? In how many ways can we write $\frac{1}{60}$ as the sum of two unit fractions? More generally, in how many ways can the unit fraction $\frac{1}{n}$ be written as the sum of two unit fractions? In other words, how many ordered pairs (a, b) of positive integers a, b are there for which

$$\frac{1}{n} = \frac{1}{a} + \frac{1}{b}\ ?$$

2. Start with a positive integer, then choose a negative integer. We'll use these two numbers to generate a sequence using the following rule: create the next term in the sequence by adding the previous two. For example, if we started with 6 and -5, we would get the sequence

$$\underbrace{6, -5, 1, -4}_{\text{alternating part}}, -3, -7, -10, -17, -27, \ldots$$

which starts with 4 elements that alternate sign before the terms are all negative. If we started with 3 and -2, we would get the sequence

$$\underbrace{3, -2, 1, -1}_{\text{alternating part}}, 0, -1, -1, -2, -3, \ldots$$

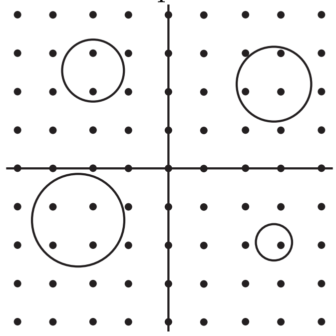
which also starts with 4 elements that alternate sign before the terms are all non-positive (we don't count 0 in the alternating part).

(a) Can you find a sequence of this type that starts with 5 elements that alternate sign? With 10 elements that alternate sign? Can you find a sequence with any number of elements that alternate sign?

(b) Given a particular starting integer, what negative number should you choose to make the alternating part of the sequence as long as possible? For example, if your sequence started with 8, what negative number would give the longest alternating part? What if you started with 10? With n?

3. The tail of a giant hare is attached by a giant rubber band to a stake in the ground. A flea is sitting on top of the stake eyeing the hare (hungrily). Seeing the flea, the hare leaps into the air and lands one kilometer from the stake (with its tail still attached to the stake by the rubber band). The flea does not give up the chase but leaps into the air and lands on the stretched rubber band one centimeter from the stake. The giant hare, seeing this, again leaps into the air and lands another kilometer from the stake (i.e., a total of two kilometers from the stake). The flea is undaunted and leaps into the air again, landing on the rubber band one centimeter further along. Once again the giant hare jumps another kilometer. The flea again leaps bravely into the air and lands another centimeter along the rubber band. If this continues indefinitely, will the flea ever catch the hare? (Assume the earth is flat and continues indefinitely in all directions.)

4. A lattice point is a point (x, y) in the plane, both of whose coordinates are integers. It is easy to see that every lattice point can be surrounded by a small circle which excludes all other lattice points from its interior. It is not much harder to see that it is possible to draw a circle which has exactly two lattice points in its interior, or exactly 3, or exactly 4, as shown in the picture below.



Do you think that for every positive integer n there is a circle in the plane which contains exactly n lattice points in its interior? Justify your answer.

5. The set S contains some real numbers, according to the following three rules.

(i) $\frac{1}{1}$ is in S.

(ii) If $\frac{a}{b}$ is in S, where $\frac{a}{b}$ is written in lowest terms (that is, a and b have highest common factor 1), then $\frac{b}{2a}$ is in S.

(iii) If $\frac{a}{b}$ and $\frac{c}{d}$ are in S, where they are written in lowest terms, then $\frac{a+c}{b+d}$ is in S.

These rules are exhaustive: if these rules do not imply that a number is in S, then that number is not in S. Can you describe which numbers are in S? For example, by (i), $\frac{1}{1}$ is in S. By (ii), since $\frac{1}{1}$ is in S, $\frac{1}{2\cdot 1}$ is in S. Since both $\frac{1}{1}$ and $\frac{1}{2}$ are in S, (iii) tells us $\frac{1+1}{1+2}$ is in S.

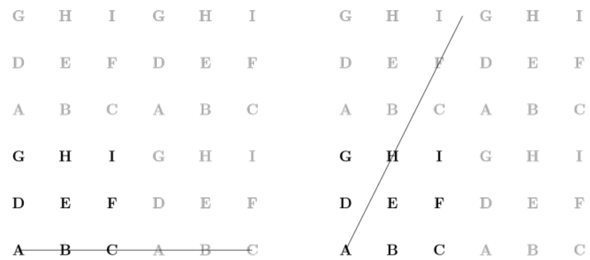
6. Let P_0 be an equilateral triangle of area 10. Each side of P_0 is trisected into three segments of equal length, and the corners of P_0 are snipped off, creating a new polygon (in fact, a hexagon) P_1. What is the area of P_1? Now repeat the process to P_1 – i.e. trisect each side and snip off the corners – to obtain a new polygon P_2. What is the area of P_2? Now repeat this process infinitely to create an object P_∞. What can you say about the shape P_∞? What is the area of P_∞?

7. A monkey has filled in a 3×3 grid with the numbers $1, 2, \ldots, 9$. A cat writes down the three numbers obtained by multiplying the numbers in each horizontal row. A dog writes down the three numbers obtained by multiplying the numbers in each vertical column. Can the monkey fill in the grid in such a way that the cat and dog obtain the same lists of three numbers? What if the monkey writes the numbers $1, 2, \ldots, 25$ in a 5×5 grid? Or $1, 2, \ldots, 121$ in a 11×11 grid? Can you find any conditions on n that guarantee that it is possible or any conditions that guarantee that it is impossible for the monkey to write the numbers $1, 2, \ldots, n^2$ in an $n \times n$ grid so that the cat and the dog obtain the same lists of numbers?

8. Consider the grid of letters that represent points below, which is formed by repeating the bold 3×3 grid. (The points are labeled by the letters A through I.)

G	H	I	G	H	I
D	E	F	D	E	F
A	B	C	A	B	C
G	**H**	**I**	G	H	I
D	**E**	**F**	D	E	F
A	**B**	**C**	A	B	C

We'll draw lines through the bottom left point (the bold A) and at least one other bold letter, then see what set of letters the line hits. We've drawn two example lines for the repeating 3×3 grid below. For the example on the left, the set of letters is {A, B, C}, and for the right, the set of letters is {A, F, H}.



Assuming the 3×3 grid repeats forever in every direction, do any of these lines ever pass through more than 3 different letters? Can you get the same set of letters from two different lines? Find the four different sets of letters that you can get from drawing lines in this grid.

What would happen if you had a repeated 5×5 grid of letters (and still had to draw lines through the bottom left point and at least one other bold point)? Can you predict what would happen with a repeated 7×7 grid? Does your prediction also work for 6×6? Can you justify your predictions?

End of Problem Set