PROMYS Europe 2023

Application Problem Set

https://www.promys-europe.org

Please attempt each of the following 10 problems. Though they can all be solved with no more than a standard high school mathematics background, most of the problems require considerably more ingenuity than is usually expected in high school. You should keep in mind that we do not expect you to find complete solutions to all of them. Rather, we are looking to see how you approach challenging problems.

We ask that you tackle these problems by yourself. Please see below for further information on citing any sources you use in your explorations.

Here are a few suggestions:

- Think carefully about the meaning of each problem.
- Examine special cases, either through numerical examples or by drawing pictures.
- Be bold in making conjectures.
- Test your conjectures through further experimentation, and try to devise mathematical proofs to support the surviving ones.
- Can you solve special cases of a problem, or state and solve simpler but related problems?

If you think you know the answer to a question, but cannot prove that your answer is correct, tell us what kind of evidence you have found to support your belief. If you use books, articles, or websites in your explorations, be sure to cite your sources.

Be careful if you search online for help. We are interested in your ideas, not in solutions that you have found elsewhere. If you search online for a problem and find a solution (or most of a solution), it will be much harder for you to demonstrate your insight to us.

You may find that most of the problems require some patience. Do not rush through them. It is not unreasonable to spend a month or more thinking about the problems. It might be good strategy to devote most of your time to a small selection of problems which you find especially interesting. Be sure to tell us about progress you have made on problems not yet completely solved. **For each problem you solve, please justify your answer clearly and tell us how you arrived at your solution: this includes your experimentation and any thinking that led you to an argument.**

There are various tools available for typesetting mathematics on a computer. You are welcome to use one of these if you choose, or you are welcome to write your solutions by hand, or you might want to do a bit of both. We are interested in your mathematical ideas, not in your typesetting. What is important is that we can read all of your submitted work, and that you can include all of your ideas (including the ones that didn't fully work).

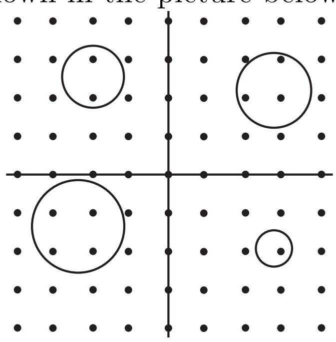
1. Consider the sequence

$$\begin{aligned} a_1 &= 2^1 - 3 = -1, \\ a_2 &= 2^2 - 3 = 1, \\ a_3 &= 2^3 - 3 = 5, \\ a_4 &= 2^4 - 3 = 13, \\ &\vdots \\ a_n &= 2^n - 3, \\ &\vdots \end{aligned}$$

defined for positive integers n. Which elements of this sequence are divisible by 5? What about 13? Are any elements of this sequence divisible by $65 = 5 \cdot 13$? Why or why not?

2. To get the *echo* of a positive integer, we write it twice in a row without a space. For example, the echo of 2023 is 20232023. Is there a positive integer whose echo is a perfect square? If so, how many such positive integers can you find? If not, explain why not.

3. A lattice point is a point (x, y) in the plane, both of whose coordinates are integers. It is easy to see that every lattice point can be surrounded by a small circle which excludes all other lattice points from its interior. It is not much harder to see that it is possible to draw a circle which has exactly two lattice points in its interior, or exactly 3, or exactly 4, as shown in the picture below.



Do you think that for every positive integer n there is a circle in the plane which contains exactly n lattice points in its interior? Justify your answer.

4. Calculate each of the following:

$$\begin{aligned} 1^3 + 5^3 + 3^3 &= ?? \\ 16^3 + 50^3 + 33^3 &= ?? \\ 166^3 + 500^3 + 333^3 &= ?? \\ 1666^3 + 5000^3 + 3333^3 &= ?? \\ \ldots &= ?? \end{aligned}$$

What do you see? Can you state and prove a generalization of your observations?

5. Any positive integer can be written in binary (also called base 2). For example, 37 is 100101 in binary (because $37 = 2^5 + 2^2 + 2^0$), and 45 is 101101 in binary. Let's say that a positive integer is "scattered" if, in its binary expansion, there are never two ones immediately next to each other. For example, 37 is scattered but 45 is not. How many scattered numbers are there less than 4? Less than 8? Less than 2^n?

6. The set S contains some real numbers, according to the following three rules.

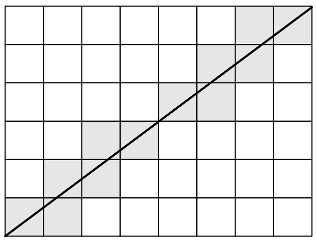
(i) $\frac{1}{1}$ is in S.

(ii) If $\frac{a}{b}$ is in S, where $\frac{a}{b}$ is written in lowest terms (that is, a and b have highest common factor 1), then $\frac{b}{2a}$ is in S.

(iii) If $\frac{a}{b}$ and $\frac{c}{d}$ are in S, where they are written in lowest terms, then $\frac{a+c}{b+d}$ is in S.

These rules are exhaustive: if these rules do not imply that a number is in S, then that number is not in S. Can you describe which numbers are in S?

7. The rectangular floor of a bathroom is covered with square tiles (all of the same size). A spider starts at one corner of the bathroom, and walks to the corner diagonally opposite. For example, the figure below shows a 6×8 bathroom, in which the spider touches 12 tiles on its path. (A spider doesn't touch a tile if it just walks over the grout at the corner of a tile.) For an m by n bathroom, how many tiles does the spider touch on its walk?



8. Let P_0 be an equilateral triangle of area 10. Each side of P_0 is trisected into three segments of equal length, and the corners of P_0 are snipped off, creating a new polygon (in fact, a hexagon) P_1. What is the area of P_1? Now repeat the process to P_1 – i.e. trisect each side and snip off the corners – to obtain a new polygon P_2. What is the area of P_2? Now repeat this process infinitely to create an object P_∞. What can you say about the shape P_∞? What is the area of P_∞?

9. A monkey has filled in a 3×3 grid with the numbers $1, 2, \ldots, 9$. A cat writes down the three numbers obtained by multiplying the numbers in each horizontal row. A dog writes down the three numbers obtained by multiplying the numbers in each vertical column. Can the monkey fill in the grid in such a way that the cat and dog obtain the same lists of three numbers? What if the monkey writes the numbers $1, 2, \ldots, 25$ in a 5×5 grid? Or $1, 2, \ldots, 121$ in a 11×11 grid? Can you find any conditions on n that guarantee that it is possible or any conditions that guarantee that it is impossible for the monkey to write the numbers $1, 2, \ldots, n^2$ in an $n \times n$ grid so that the cat and the dog obtain the same lists of numbers?

10. On a strange railway line, there is just one infinitely long track, so overtaking is impossible. Any time a train catches up to the one in front of it, they link up to form a single train moving at the speed of the slower train. At first, there are three equally spaced trains, each moving at a different speed. You watch, and eventually (after all the linking that will happen has happened), you count the trains. You wonder what would have happened if the trains had started in a different order (but each of the original three trains had kept its same starting speed). On average (averaging over all possible orderings), how many trains will there be after a long time has elapsed? What if at the start there are 4 trains (all moving at different speeds)? Or 5? Or n?