A11473W1

DEGREE OF MASTER OF SCIENCE

Mathematical Modelling and Scientific Computing

Mathematical Methods I

Hilary TERM 2022

Thursday, 13 January 2022

Opening Time: 9:30am GMT

Mode of completion (format in which you will complete this exam): handwritten

You have 2 hours and 30 minutes of writing time to complete the paper and up to 30 minutes technical time to upload your answer file.

This exam paper contains two sections. You may attempt as many questions as you like but you must answer at least one question in each section. Your best answer in each section will count, along with your next best two answers, making a total of four answers.

To upload your work you must submit just one pdf.

Applied Partial Differential Equations

1. Consider solutions $h(x,t)$ of the partial differential equation

$$h_t + (h^n h_{xxx})_x = 0 \qquad \text{on } -s(t) < x < s(t) \tag{1a}$$

that satisfy the following boundary conditions at $x = \pm s(t)$:

$$h(\pm s(t), t) = 0, \tag{1b}$$
$$h_x(\pm s(t), t) = 0, \tag{1c}$$
$$h^n h_{xxx} \to 0 \qquad \text{for } x \to \pm s(t), \tag{1d}$$

for $t > 0$, where $n > 0$ is a constant.

(a) [6 marks] Show that solutions $h(x,t)$ and $s(t)$ of (1a)-(1d) conserve

$$I = \int_{-s(t)}^{s(t)} h(x,t)\,\mathrm{d}x.$$

(b) [13 marks] Show that for a suitable choice of a, b and c, the combination of the equations (1a)-(1d) with $I = 1$ is invariant under the scalings

$$t = \varepsilon^a \bar{t}, \quad x = \varepsilon^b \bar{x}, \quad s(t) = \varepsilon^b \bar{s}(\bar{t}), \quad h(x,t) = \varepsilon^c \bar{h}(\bar{x}, \bar{t}).$$

for all $\varepsilon > 0$. Use this result to determine constants α and β so that

$$h(x,t) = t^\alpha H(\xi) \quad \text{with } \xi = x/t^\beta \text{ and } s(t) = \sigma t^\beta$$

is a self-similar solution of (1a)-(1d) for which $I = 1$. In particular, show that $\alpha = -\beta$. State the resulting boundary value problem for a third order ordinary differential equation for H and σ. Which initial condition does the self similar solution satisfy, if we extend $H(\xi)$ so that $H(\xi) = 0$ for $|\xi| > \sigma$? Give reasons for your answer.

(c) [6 marks] Determine the self-similar solution explicitly in the case $n = 1$.

2. Consider the first order quasilinear partial differential equation in conservation form

$$u_t + \left(u^3\right)_x = 0, \tag{1}$$

on the domain $t > 0$, with initial condition

$$u(x,0) = \begin{cases} 1 & \text{for } x \leqslant -1, \\ -x & \text{for } x > -1. \end{cases} \tag{2}$$

(a) [6 marks] State the characteristic equations for (1) with this initial condition, and obtain the solution in parametric form.

(b) [9 marks] Determine the envelope of the characteristic projections. Hence, or otherwise, find the domain of definition of the classical solution $u(x,t)$, that is, the points (x,t) that are visited by exactly one characteristic projection. Sketch it, including a couple of characteristic projections.
[*You are not required to determine* $u(x,t)$.]

(c) [10 marks] A shock develops at $x = st$ with

$$u(x,t) = \begin{cases} 1 & \text{for } x \leqslant st, \\ u_+ & \text{for } x > st, \end{cases}$$

Derive a condition for the speed s of the shock and specify for which range of $u_+ < 1$ the shock is causal.
[*It is sufficient for causality if the characteristic speed in the left (and in the right) of the shock is not slower (and not faster) than the shock speed, respectively, i.e. equality of shock speed and characteristic speed is permitted.*]
Now find a similarity solution of (1) of the form $u(x,t) = v(\eta)$, $\eta = x/t$, $t > 0$, where

$$v(\eta) = \begin{cases} 1 & \text{for } \eta \leqslant \eta_1 \\ g(\eta) & \text{for } \eta_1 < \eta \leqslant \eta_2 \\ -1 & \text{for } \eta > \eta_2, \end{cases}$$

with a differentiable function $g(\eta)$ with $-1 \leqslant g(\eta) \leqslant 1$ and $g(\eta_2) = -1$, and values η_1 and η_2 which you are to determine.

3. Consider the following system of partial differential equations for $u(x, y)$ and $v(x, y)$:

$$u_x - v_y = 0, \qquad v_x - u_y = f(x, y), \tag{1}$$

where $f(x, y)$ is a given smooth function.

(a) [11 marks] Determine the characteristics and the corresponding Riemann invariants for (1), and use these results to find the solution for initial data $u = 0$, $v = 0$, specified on the axis $y = 0$.

(b) [7 marks] Now consider (1) with data $u = 0$, $v = 0$ on $\Gamma_1 = \{(x, 0) : x \geqslant 0\}$, and $\alpha u + v = 0$ on $\Gamma_2 = \{(0, y) : y \geqslant 0\}$, where α is a given real number.

(i) Why is only one condition specified on Γ_2? Using the Riemann invariants obtained in part (a), state for which α and β solutions for u and v can be determined on Γ_2 for general f. Determine u and v on Γ_2.

(ii) Explain briefly why u and v are continuous along $\Gamma = \Gamma_1 \cup \Gamma_2$, including the origin. Deduce that the solutions u and v on $\Omega = \{(x, y) : x \geqslant 0,\ y \geqslant 0\}$ are continuous.

(c) [7 marks] Solve (1) with $f(x, y) = \exp(x)$ on $\Omega = \{(x, y) : x \geqslant 0,\ y \geqslant 0\}$, with data

$$u = 0, \quad v = 0 \ \text{ on } \ \Gamma_1 = \{(x, 0) : x \geqslant 0\},$$

and

$$u - v = 0 \text{ on } \ \Gamma_2 = \{(0, y) : y \geqslant 0\}.$$

[*Hint: In part (c) you may use, after verification, that the general solution of the system of PDEs (4) for the specified f is given by*

$$u = h(x + y) + k(y - x), \qquad v = h(x + y) - k(y - x) + \mathrm{e}^x,$$

where h and k are arbitrary differentiable functions.]

4. Consider the following partial differential equation:

$$(u_y)^3 + (x-1)u_x + u = 0, \qquad (1)$$

for $u(x,t)$.

(a) [10 marks] Formulate Charpit's equations for (1) and find their solution for initial data given by the smooth functions $x_0(s)$, $y_0(s)$, $u_0(s)$, $p_0(s)$ and $q_0(s)$.

(b) [8 marks] For the initial condition of (1) given by

$$u(x,1) = 2(x-1)^3 \quad \text{on} \quad 1 \leqslant x \leqslant 2, \qquad (2)$$

determine the appropriate initial data for Charpit's equations. Hence obtain the solution of (1) and (2) in parametric form.

(c) [7 marks] Indicate on a sketch, and describe carefully, where the solution of (1) is uniquely determined by the initial data specified in (2).

[*You are not required to determine* $u(x,y)$.]

Supplementary Applied Mathematics

5. The differential operator L is defined as follows:

$$Ly \equiv \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\beta\frac{\mathrm{d}y}{\mathrm{d}x} + \beta^2 y, \quad \text{for } 0 \leqslant x \leqslant \frac{\pi}{2},$$

where β is a positive constant.

(a) [10 marks] Find the eigenvalues λ and corresponding eigenfunctions y for

$$Ly = \lambda y, \tag{1}$$

with boundary conditions

$$\frac{\mathrm{d}y}{\mathrm{d}x}(0) + \beta y(0) = 0, \quad y(\pi/2) = 0.$$

[You may assume, without proof, that all eigenvalues are strictly negative.]

(b) [6 marks] Use an appropriate weighting function $\rho(x)$ to rewrite equation (1) in Sturm-Liouville form as $\hat{L}\hat{y} = \hat{\lambda}\rho\hat{y}$. State clearly the eigenvalues $\hat{\lambda}$ and eigenfuctions $\hat{y}$ with boundary conditions

$$\frac{\mathrm{d}\hat{y}}{\mathrm{d}x}(0) + \beta\hat{y}(0) = 0, \quad \hat{y}(\pi/2) = 0.$$

(c) [9 marks] Consider, now, the boundary value problem

$$Ly = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\beta\frac{\mathrm{d}y}{\mathrm{d}x} + \beta^2 y = f(x),$$

with boundary conditions

$$\frac{\mathrm{d}y}{\mathrm{d}x}(0) + \beta y(0) = \theta, \quad y(\pi/2) = 0,$$

where $f(x)$ is a smooth, differentiable function and θ is a real constant. Suppose that this boundary value problem can be solved using an eigenfunction expansion of the form

$$y = \sum_{k=0}^{\infty} c_k \hat{y}_k(x).$$

Using the results from part (b), or otherwise, derive expressions for the coefficients c_k.

[You do not need to evaluate the integrals in your expression for c_k.]

6. Let the differential operator L be defined by

$$Ly \equiv (2x+1)^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 8(2x+1)\frac{\mathrm{d}y}{\mathrm{d}x} + 24y.$$

(a) [8 marks] Find the general solution of the homogeneous differential equation

$$Ly(x) = 0.$$

[Hint. Show that $y_1(x) = (2x+1)^2$ solves $Ly = 0$, and then seek a solution of the form $y(x) = u(x)y_1(x)$.]

(b) [9 marks] Determine the Green's function for the boundary value problem

$$Ly(x) = f(x),$$
$$y(0) = \alpha, \quad y(1) = \beta.$$

(c) [8 marks] Use the Green's function obtained in (b) to write down the solution for general data $\{f(x), \alpha, \beta\}$, where $f(x)$ is a given function, and α and β are given real constants.