DEGREE OF MASTER OF SCIENCE

Mathematical Modelling and Scientific Computing

Mathematical Methods II

Trinity TERM 2022

Thursday, 21 April 2022

Opening Time: 9:30am GMT

Mode of completion (format in which you will complete this exam): handwritten

You have 2 hours and 30 minutes of writing time to complete the paper and up to 30 minutes technical time to upload your answer file.

This exam paper contains three sections. You may attempt as many questions as you like but you must answer at least one question in each section. Your best answer in each section will count, along with your next best answer, making a total of four answers.

To upload your work you must submit just one pdf.

Nonlinear Systems

1. Consider the system

$$\begin{aligned}
\dot{z}_1 &= -z_2 + z_1 z_2, \\
\dot{z}_2 &= z_1 - z_1^2 - z_2 z_3, \\
\dot{z}_3 &= -z_3 + z_2^2.
\end{aligned}$$

   (a) [2 marks] Find the fixed points.

   (b) [2 marks] Show that the line $z_1 = 1$ is an invariant set.

   (c) [8 marks] Determine the stable, unstable and/or centre linear subspaces for each fixed point.

   (d) [8 marks] Find a quadratic approximation to the centre manifold in the vicinity of the origin, and use it to show that the local dynamics on the centre manifold are given by

$$\begin{aligned}
\dot{z}_1 &= -z_2 + z_1 z_2, \\
\dot{z}_2 &= z_1 - z_1^2 - \frac{2}{5} z_1^2 z_2 + \frac{2}{5} z_1 z_2^2 - \frac{3}{5} z_2^3.
\end{aligned}$$

   (e) [5 marks] By using an appropriate Lyapunov function, determine whether the origin is Lyapunov stable in the centre manifold.

2. Consider the system

$$\begin{aligned}
\dot{x} &= -\mu x + xy, \\
\dot{y} &= -2y + z, \\
\dot{z} &= -z + x^2,
\end{aligned}$$

   where $\mu \in \mathbb{R}$ is a parameter.

   (a) [3 marks] Find the fixed points, being careful to state for which values of μ each fixed point exists.

   (b) [8 marks] Determine the stable, unstable and/or centre linear subspaces for the fixed point at the origin, being careful to consider all values of μ. For what value of μ is there a bifurcation of this fixed point?

   (c) [10 marks] Find a quadratic approximation to the extended centre manifold in the vicinity of the origin.

   (d) [4 marks] Determine the local dynamics on the extended centre manifold in the vicinity of the origin. Describe the type of bifurcation. Sketch the bifurcation diagram, including the stability of the branches.

Further Mathematical Methods

3. (a) [7 marks] Consider the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (n^2\pi^2 + T\epsilon)y - y^2 = 0,$$

subject to boundary conditions

$$y(0) = y(1) = 0,$$

with $n \in \mathbb{Z}$, $0 < \epsilon \ll 1$ and T a constant.
By expanding

$$y(x) = \epsilon y_1(x) + \epsilon^2 y_2(x) + ...$$

show that

$$y_1(x) = A \sin n\pi x,$$

and determine A for odd n by obtaining a solvability condition at higher order in ϵ.

(b) Consider the integral equation

$$y(x) = f(x) + \lambda \int_{-L}^{L} [g(x)h(t) + g(t)h(x)]y(t) \ \mathrm{d}t, \tag{1}$$

where $g(t)$ and $h(t)$ are continuous functions.

(i) [8 marks] Find the eigenvalues and eigenfunctions for the homogeneous problem, $f(x) = 0$, under the assumption that $g(x)$ and $h(x)$ satisfy

$$\int_{-L}^{L} g(t)^2 \ \mathrm{d}t = \int_{-L}^{L} h(t)^2 \ \mathrm{d}t = 1, \qquad \int_{-L}^{L} g(t)h(t) \ \mathrm{d}t = 0.$$

(ii) [8 marks] Now take $g(x) = \cos x$, $h(x) = x$, $L = \pi$, and $f(x) \neq 0$. Derive conditions on λ and $f(x)$ for which solutions to (1) exist, and state whether these solutions are unique.
[*You may wish to use the results of (i) after scaling g, h and λ appropriately. You do not need to give explicit expressions for the solutions of* (1).]

(iii) [2 marks] Taking $f(x) = x + \beta x^3$, find the value of β that ensures at least one solution of (1) exists for all values of $\lambda \in \mathbb{R}$.

4. (a) Consider the functional

$$J_1[\phi] = \int_0^\infty F_1(s, \phi, \phi') \, \mathrm{d}s,$$

where $F_1(s, \phi, \phi') = \frac{1}{2}(\phi')^2 - \omega^2(1 - \cos\phi)$, $(\cdot)'$ denotes differentiation with respect to s and $\omega \in \mathbb{C}$ is a constant.

(i) [2 marks] Using the Euler equation, derive a second order differential equation satisfied by the $\phi(s)$ that extremizes $J_1[\phi]$.

(ii) [2 marks] Obtain a first integral of this equation, under the assumption that $\phi \to 0$ as $s \to \infty$.

(iii) [2 marks] Show that any solution of these equations also satisfies

$$\phi'''' = \omega^2 \sin\phi \left[\tfrac{3}{2}(\phi')^2 + \omega^2\right].$$

(b) For this part of the question, consider the functional

$$J_2[\phi, y] = \int_0^\infty F_2(s, \phi, \phi', y, y') \, \mathrm{d}s, \tag{1}$$

which is to be extremized over all functions $\{\phi(s), y(s)\} \in C^2[0, \infty]$.

(i) [3 marks] Starting from the generalization of Euler's equation to two dependent variables, show that the Hamiltonian $H = p_\phi \phi' + p_y y' - F_2$ satisfies

$$\frac{\mathrm{d}H}{\mathrm{d}s} = -\frac{\partial F_2}{\partial s},$$

where $p_\phi = \frac{\partial F_2}{\partial \phi'}$ and $p_y = \frac{\partial F_2}{\partial y'}$ are the generalized momenta.

(ii) [4 marks] Derive Hamilton's equation

$$\frac{\mathrm{d}p_\phi}{\mathrm{d}s} = -\frac{\partial H}{\partial q_\phi}, \tag{2}$$

where $q_\phi = \phi$ is a generalized coordinate.

(c) Consider now the functional $J_2[\phi, y]$ with $F_2 = \frac{1}{2}(\phi')^2 + \frac{1}{2}y^2 \cos\phi$, supplemented by the constraints

$$\int_0^\infty 1 - \cos\phi \, \mathrm{d}s = c \quad \text{and} \quad y' = \sin\phi, \tag{3}$$

for c a constant.

(i) [2 marks] Describe (briefly) why the appropriate F_2 to extremize (1) subject to (3) is

$$F_2(s, y, y', \phi, \phi') = \frac{1}{2}(\phi')^2 + \frac{1}{2}y^2 \cos\phi - P(1 - \cos\phi) - Q(s)\left(\sin\phi - y'\right),$$

highlighting any important differences in the Lagrange multipliers introduced.

(ii) [3 marks] Calculate the associated Hamiltonian, explaining why it is constant in this case, and evaluating this constant under the assumption that $\phi, y \to 0$ as $s \to \infty$.

(iii) [2 marks] Use (2) and your answer to c(ii) to eliminate $Q(s)$.

(iv) [3 marks] Differentiate the resulting equation twice with respect to s and show that

$$\phi'''' + \left[P + \frac{3}{2}(\phi')^2\right]\phi'' + \sin\phi = 0. \tag{4}$$

(v) [2 marks] Deduce that any solution, $\phi_a(s)$, of the equation(s) you derived in part (a) is also a solution of (4) if ω is such that $P = \omega^2 + \omega^{-2}$.

Further Partial Differential Equations

5. Consider a two-dimensional material, which lies in $x, y \in \mathbb{R}$, and is solid for $x \leqslant s(y,t)$ and liquid for $x > s(y,t)$, where t denotes time. The temperature is denoted by T_1 in the solid and T_2 in the liquid and is governed by

$$\frac{\partial^2 T_1}{\partial x^2} + \frac{\partial^2 T_1}{\partial y^2} = 0, \qquad \text{for} \quad x \leqslant s(y,t), \qquad \frac{\partial^2 T_2}{\partial x^2} + \frac{\partial^2 T_2}{\partial y^2} = 0, \qquad \text{for} \quad x > s(y,t).$$

At the solid–liquid interface $x = s(y,t)$ the following conditions hold:

$$T_1 = T_2 = 0, \qquad \frac{\partial T_1}{\partial n} - \frac{\partial T_2}{\partial n} = \frac{\partial s}{\partial t}, \qquad \text{on} \quad x = s(y,t), \qquad \text{(1a,b)}$$

where we have assumed the melting temperature of the solid is $T = 0$ and n denotes the coordinate in the normal direction to the interface, pointing into the liquid. Far from the interface, we impose

$$\frac{\partial T_1}{\partial x} \to F \quad \text{as} \quad x \to -\infty, \qquad T_2 \to 0 \quad \text{as} \quad x \to \infty, \qquad \text{(2a,b)}$$

where $F > 0$ is a constant. We also impose the initial condition $s(y,0) = 0$.

(a) [3 marks] Explain the physical interpretation of the interface and boundary conditions (1b) and (2a) and whether this set-up will give rise to freezing of liquid or melting of solid.

(b) [7 marks] Suppose that the solutions for T_1, T_2 and s are all independent of y and given by $T_1 = T_1^*(x,t)$, $T_2 = T_2^*(x,t)$ and $s = s^*(t)$. Find expressions for T_1^*, T_2^* and s^* and sketch the graph of the temperature profile as a function of x at two different times, $t > 0$.

(c) [6 marks] Now suppose that the interface is perturbed about the base state via $s(t) = s^*(t) + \epsilon\eta(y,t)$, where $\epsilon \ll 1$. By assuming that the solutions for the temperature in this perturbed state take the form $T_j = T_j^*(x,t) + \epsilon u_j(x,y,t)$ for $j = 1, 2$, show that the boundary condition (1b) gives, at $O(\epsilon)$,

$$\frac{\partial u_1}{\partial x} - \frac{\partial u_2}{\partial x} = \frac{\partial \eta}{\partial t} \qquad \text{on} \quad x = s^*(t).$$

(d) [7 marks] By posing the ansätze

$$u_j = A_j \exp\left(\sigma t + \mathrm{i}ky + k(x - s^*(t))\right) + B_j \exp\left(\sigma t + \mathrm{i}ky - k(x - s^*(t))\right), \quad j = 1, 2,$$
$$\eta = \exp\left(\sigma t + \mathrm{i}ky\right),$$

where $k > 0$, determine the solutions u_1 and u_2 and an expression for σ. Hence deduce the stability of the interface and give your reason for this conclusion.

(e) [2 marks] Use your results and those in the lecture notes to describe a physical experiment that should lead to the freezing of a liquid to a solid in either (i) a stable or (ii) an unstable manner.

6. Consider a one-dimensional salty material, which is mush for $0 \leqslant x \leqslant h(t)$ and liquid for $x > h(t)$, where t denotes time and $h(0) = 0$. The salt concentration S is governed by

$$\frac{\partial S}{\partial t} = \frac{\partial^2 S}{\partial x^2}, \qquad \text{for} \quad x \geqslant 0. \tag{1}$$

The temperature is denoted by T_1 in the mush and T_2 in the liquid, and is governed by

$$T_1 = F(S), \qquad \text{for} \quad 0 \leqslant x \leqslant h(t), \qquad \frac{\partial T_2}{\partial t} = k\frac{\partial^2 T_2}{\partial x^2}, \qquad \text{for} \quad x > h(t), \tag{2a,b}$$

for some constant k and prescribed function F. At the mush–liquid interface $x = h(t)$ the following conditions hold:

$$T_1 = T_2, \qquad \frac{\partial T_1}{\partial x} = \frac{\partial T_2}{\partial x}. \tag{3a,b}$$

We also apply the conditions

$$S(0,t) = S_0, \qquad S \to 0 \quad \text{as} \quad x \to \infty, \qquad T_2 \to 0 \quad \text{as} \quad x \to \infty, \tag{4a–c}$$

for some constant S_0.

(a) [7 marks] Show that a similarity solution exists of the form $S = f(\eta)$ and $T_2 = g(\eta)$ for equations (1) and (2), where $\eta = x/t^{\omega}$, for some ω whose value you should specify, and find the differential equations satisfied by f and g.

(b) [2 marks] Explain why the position of the interface is of the form $h = \beta t^{\gamma}$ and state the required value of γ along with any conditions on β that are required for the mushy layer to grow with time.

(c) [7 marks] Use equations (1) and (2) and boundary conditions (3a) and (4) to find the similarity solutions $f(\eta)$ and $g(\eta)$ in terms of the parameters k, S_0 and β.

[You may find it helpful to recall the definitions of the error function $\mathrm{erf}(z)$ *and the complementary error function* $\mathrm{erfc}(z)$*:*

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z \mathrm{e}^{-y^2}\,\mathrm{d}y, \qquad \mathrm{erfc}(z) = 1 - \mathrm{erf}(z). \qquad]$$

(d) [4 marks] Suppose that $F(S) = T_0 - \alpha S$ for some constants T_0 and α. Use boundary condition (3b) to show that

$$\left[T_0 - \alpha S_0\,\mathrm{erfc}\left(\frac{\beta}{2}\right)\right]\exp\left(\frac{\beta^2}{4}\left(1 - \frac{1}{k}\right)\right) = -\alpha S_0\sqrt{k}\,\mathrm{erfc}\left(\frac{\beta}{2\sqrt{k}}\right). \tag{5}$$

(e) [5 marks] If $k > 1$, use (5) to show that a critical concentration of salt, $S_0 = S_0^*$, is required for a mushy layer to exist, where

$$S_0^* = -\frac{T_0}{\alpha\left(\sqrt{k} - 1\right)}.$$

[You may use without proof the fact that β is a monotonically increasing function of S_0.]

Show that $F(S_0^*) < 0$ if $k > 1$ provided $T_0 < 0$ and explain why we would expect $F < 0$ for this problem to make physical sense.