A11474W1

DEGREE OF MASTER OF SCIENCE

Mathematical Modelling and Scientific Computing

Numerical Linear Algebra and Numerical Solution of Differential Equations

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Hilary TERM 2022

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Opening Time: 9:30am GMT

Mode of completion (format in which you will complete this exam): handwritten

You have 2 hours and 30 minutes of writing time to complete the paper and up to 30 minutes technical time to upload your answer file.

This exam paper contains two sections. You may attempt as many questions as you like but you must answer at least one question in each section. Your best answer in each section will count, along with your next best two answers, making a total of four answers.

To upload your work you must submit just one pdf.

Numerical Linear Algebra

1. Let $A \in \mathbb{R}^{m\times n}, m \geqslant n$, and let $\sigma_1(A) \geqslant \sigma_2(A) \geqslant \cdots \geqslant \sigma_n(A) \geqslant 0$ denote its singular values. An SVD is a decomposition $A = U\Sigma V^T$ where $U \in \mathbb{R}^{m\times n}$ is orthonormal (that is, $U^TU = I_n$), $\Sigma = \mathrm{diag}(\sigma_1(A), \ldots, \sigma_n(A))$, and V is orthogonal. Let $\|\cdot\|_2$ denote the spectral norm, so $\|A\|_2 = \sigma_1(A)$.

   (a) [4 marks] Suppose A is orthonormal, that is, $A^TA = I_n$. Find the singular values and an SVD of A.

   (b) [4 marks] Suppose $A = \begin{bmatrix} A_1 \\ 0 \end{bmatrix}$ where A_1 is $n \times n$ symmetric with eigenvalue decomposition $A_1 = V_1\Lambda_1V_1^T$, where $\Lambda_1 = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is a diagonal matrix and V_1 is orthogonal. Find an SVD of A.

   (c) [4 marks] Suppose $A = \sum_{i=1}^{r} x_iy_i^T$, where $x_i, y_i \in \mathbb{R}^n$ are vectors and $r < n$. Show that $\sigma_{r+1}(A) = 0$.

   (d) [6 marks] Suppose that the entries of A are all 1, except the diagonal entries, which are all $1 + 10^{-10}$.

      (i) Show that the second singular value $\sigma_2(A) \leqslant 10^{-10}$.

      (ii) Find a rank-1 matrix B such that $\|A - B\|_2 \leqslant 10^{-10}$.

   (e) [7 marks] Suppose $A = [A_1 \ A_2]$, where A_1 is $m \times n_1$, and A_2 is $m \times n_2$, with $n_1 + n_2 = n$.

      (i) Show that $\sigma_k(A) \geqslant \sigma_k(A_1)$ for $k = 1, 2, \ldots, n_1$.

      (ii) Give an example where $\sigma_{n_2}(A_2) > 0$ but $\sigma_k(A) = \sigma_k(A_1)$ for all $k \leqslant n_1$.

2. Let $A \in \mathbb{R}^{m\times n}, m \geqslant n$.

(a) [4 marks] Let $Q \in \mathbb{R}^{m\times n}$ be orthonormal, that is, $Q^TQ = I_n$. Prove constructively that there exists a matrix $Q_\perp \in \mathbb{R}^{m\times(m-n)}$ such that $[Q\ Q_\perp]$ is square orthogonal.

(b) [4 marks] Suppose that $m > n$. Show that A^T has a nonzero null vector v such that $A^Tv = 0$.

(c) [4 marks] Show that there exists a decomposition of the form $A = QM$, where M is 'antitriangular', that is, $M_{ij} = 0$ if $i + j \leqslant n$.

[*Here and below, you may use the fact that every matrix has a QR factorisation.*]

(d) [4 marks] Show that there exists a decomposition of the form $A = QL$, where L is *lower* (not upper) triangular.

(e) [9 marks] Let A be orthogonal (so $m = n$). Consider the linear system $Ax = b$.

(i) Find x, given A and b.

(ii) Let ΔA be a matrix such that $\|\Delta A\|_2 \leqslant 10^{-10}$. Consider the perturbed linear system $(A+\Delta A)(x+\Delta x) = b$. Is it possible that $\|\Delta x\|/\|x\| > 10^{-9}$? Find an example or disprove.

Numerical Solution of Differential Equations

3. Consider the two-point boundary-value problem

$$-(p(x)u'(x))' = f(x), \quad x \in (0,1),$$

$$u(0) = A, \quad u(1) = B,$$

where $A, B \in \mathbb{R}$, $p \in C^1([0,1])$, and there exists a positive constant c_0 such that $p(x) \geqslant c_0$ for all $x \in [0,1]$. Suppose further that $f \in L^2((0,1))$. On a uniform finite difference mesh $\overline{\Omega}_h := \{x_i := ih : i = 0, \ldots, N\}$ of spacing $h := 1/N$, where $N \geqslant 2$, the boundary-value problem is approximated by the following finite difference scheme

$$-\frac{1}{h}\left(p(x_{i+1/2})\frac{U_{i+1}-U_i}{h} - p(x_{i-1/2})\frac{U_i - U_{i-1}}{h}\right) = T_h f(x_i), \quad i = 1, \ldots, N-1,$$
$$U_0 = A, \quad U_N = B, \tag{1}$$

where $x_{i\pm 1/2} := x_i \pm \frac{1}{2}h$ and

$$T_h f(x_i) := \frac{1}{h}\int_{x_{i-1/2}}^{x_{i+1/2}} f(x)\,\mathrm{d}x.$$

(a) [9 marks] Show the existence of a unique solution $\{U_i\}_{i=0}^N$ to the difference scheme (1).

(b) [9 marks] Show that the global error $e := u - U$ satisfies the following equalities:

$$-D_x^+(p(x_{i-1/2})D_x^- e_i) = D_x^+ \varphi_i, \quad i = 1, \ldots, N-1,$$

$$e_0 = 0, \quad e_N = 0,$$

where D_x^+ and D_x^- are the first-order forward and backward finite difference operator, respectively, and

$$\varphi_i := p(x_{i-1/2})\left(u'(x_{i-1/2}) - D_x^- u(x_i)\right), \quad i = 1, \ldots, N.$$

Hence deduce that, if u''' is an integrable function on the interval $[0,1]$, then

$$\varphi_i = p(x_{i-1/2})\left(\frac{1}{h}\int_{x_{i-1}}^{x_i}\int_{x}^{x_{i-1/2}}\int_{x_{i-1/2}}^{s} u'''(t)\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}x\right), \quad i = 1, \ldots, N.$$

(c) [7 marks] Suppose that $u''' \in L^2((0,1))$. Show that

$$|\varphi_i| \leqslant h^{3/2}\|p\|_{C([0,1])}\left(\int_{x_{i-1}}^{x_i} |u'''(t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}}.$$

Hence deduce that, in a suitable discrete Sobolev norm $\|\cdot\|_{1,h}$ that you should carefully define, the solution U to the finite difference scheme (1) satisfies the error bound

$$\|u - U\|_{1,h} \leqslant Ch^2\|u'''\|_{L^2((0,1))},$$

where C is a positive constant, independent of the mesh-size h, whose value you should specify in terms of c_0 and $\|p\|_{C([0,1])}$.

[The discrete Poincaré–Friedrichs inequality may be used without proof.]

4. Suppose that $\Omega = (0,1)^2$. Consider the elliptic partial differential equation

$$-\Delta u + u = -1, \qquad (x,y) \in \Omega,$$

subject to the Dirichlet boundary condition $u|_{\partial\Omega} = b$, where b is a nonnegative real number.

(a) [9 marks] On the uniform finite difference mesh

$$\overline{\Omega}_h := \{(x_i, y_j) \,:\, x_i := ih,\ y_j := jh,\ i,j = 0,\dots,N\}$$

of spacing $h := 1/N$ in both coordinate directions, where $N \geqslant 2$, state the five-point finite difference approximation to the boundary-value problem.
Assuming that $u \in C^4(\overline{\Omega})$, show that the consistency error $\varphi_{i,j}$ of the five-point scheme at the mesh-point (x_i, y_j) satisfies the inequality

$$\max_{1 \leqslant i,j \leqslant N-1} |\varphi_{i,j}| \leqslant \frac{h^2}{12}\left(\left\| \frac{\partial^4 u}{\partial x^4} \right\|_{C(\overline{\Omega})} + \left\| \frac{\partial^4 u}{\partial y^4} \right\|_{C(\overline{\Omega})} \right).$$

(b) [9 marks] Denoting by U the solution to the five-point scheme, show that

$$\max_{0 \leqslant i,j \leqslant N} U_{i,j} = b.$$

(c) [7 marks] Show further that

$$\max_{0 \leqslant i,j \leqslant N} |u(x_i, y_j) - U_{i,j}| \leqslant \max_{1 \leqslant i,j \leqslant N-1} |\varphi_{i,j}|.$$

5. Consider the initial-value problem

$$\frac{\partial u}{\partial t} + u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \qquad -\infty < x, y < \infty, \quad 0 < t \leqslant T,$$
$$u(x, y, 0) = u_0(x, y), \qquad -\infty < x, y < \infty, \tag{2}$$

where T is a fixed real number, and u_0 is a real-valued, bounded and continuous function of $x, y \in (-\infty, \infty)$.

(a) [9 marks] Consider the finite difference mesh with uniform spacings $\Delta x > 0$ and $\Delta y > 0$ in the x and y coordinate directions, respectively, and let $\Delta t := T/M$, where M is a positive integer such that $M > T$. Let $\mathbb{Z}$ denote the set of all integers. State the Crank–Nicolson scheme for the numerical solution of the initial-value problem (2), where $U_{i,j}^m$ denotes the Crank–Nicolson approximation to $u(i\Delta x, j\Delta y, m\Delta t)$ for $i, j \in \mathbb{Z}$ and $m \in \{0, 1, \ldots, M\}$.

(b) [9 marks] Suppose that

$$\|U^0\|_{\ell_2} := \left(\Delta x \, \Delta y \sum_{i,j \in \mathbb{Z}} |U_{i,j}^0|^2 \right)^{1/2}$$

is finite and that

$$\Delta t \left(\frac{1}{2} + \frac{2}{(\Delta x)^2} + \frac{2}{(\Delta y)^2} \right) \leqslant 1.$$

Show that

$$\|U^m\|_{\ell_2} \leqslant \left(\frac{2 - \Delta t}{2 + \Delta t} \right)^m \|U^0\|_{\ell_2}$$

for all m, $1 \leqslant m \leqslant M$.

(c) [7 marks] Now, suppose that

$$\Delta t \left(\frac{1}{2} + \frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} \right) \leqslant 1.$$

Show that

$$\max_{i,j \in \mathbb{Z}} |U_{i,j}^m| \leqslant \left(\frac{2 - \Delta t}{2 + \Delta t} \right)^m \max_{i,j \in \mathbb{Z}} |U_{i,j}^0|$$

for all m, $1 \leqslant m \leqslant M$.

6. Suppose that c is a nonzero real number, $T > 0$, and $u_0 \in C(\mathbb{R})$. The initial-value problem

$$\begin{aligned} u_t + c\,u_x - u &= 0, \qquad -\infty < x < \infty, \quad 0 < t \leqslant T, \\ u(x,0) &= u_0(x), \qquad -\infty < x < \infty, \end{aligned}$$

has been approximated by the finite difference scheme

$$\begin{aligned} \frac{U_j^{m+1} - U_j^m}{\Delta t} + c\,\frac{U_{j+1}^m - U_{j-1}^m}{2\Delta x} - U_j^m &= 0, \qquad j \in \mathbb{Z}, \quad 0 \leqslant m \leqslant M-1, \\ U_j^0 &:= u_0(x_j), \qquad j \in \mathbb{Z}, \end{aligned} \tag{3}$$

where $\mathbb{Z}$ denotes the set of all integers, $\Delta x > 0$, $\Delta t := T/M$, and M is a positive integer.

(a) [6 marks] Define the consistency error T_j^m of the finite difference scheme (3), and show that

$$T_j^m = \mathcal{O}((\Delta x)^2 + \Delta t) \qquad \text{as } \Delta x \to 0 \text{ and } \Delta t \to 0.$$

You may assume that u has as many continuous and bounded partial derivatives with respect to x and t as are required by your argument.

(b) [6 marks] Let $\mu := c\Delta t/\Delta x$. Denoting by $k \in [-\pi/\Delta x, \pi/\Delta x] \mapsto \hat{U}^m(k) \in \mathbb{C}$ the semidiscrete Fourier transform of the mesh function $j \in \mathbb{Z} \mapsto U_j^m \in \mathbb{R}$, show that

$$|\hat{U}^m(k)|^2 = \left[(1+\Delta t)^2 + \mu^2 \sin^2 k\Delta x\right]^m |\hat{U}^0(k)|^2 \qquad \forall\, k \in \left[-\frac{\pi}{\Delta x}, \frac{\pi}{\Delta x}\right], \quad \forall\, m \in \{1,\dots,M\}.$$

(c) [6 marks] Let $\nu := c\Delta t/(\Delta x)^2$. Show that $\mu^2 = \nu c \Delta t$. Hence deduce that if ν is held fixed as $\Delta t \to 0$, then the scheme (3) is stable in the ℓ_2 norm in von Neumann's sense.

(d) [7 marks] By choosing an initial datum $u_0 \neq 0$ such that

$$\hat{U}^0(k) = 0 \qquad \forall\, k \in \left[-\frac{\pi}{\Delta x}, \frac{\pi}{\Delta x}\right] \setminus \left(\left[-\frac{5\pi}{6\Delta x}, -\frac{\pi}{6\Delta x}\right] \cup \left[\frac{\pi}{6\Delta x}, \frac{5\pi}{6\Delta x}\right]\right),$$

show that if μ is held fixed as $\Delta t \to 0$, then the difference scheme (3) is *not* stable in the ℓ_2 norm in von Neumann's sense.