A11473W1

DEGREE OF MASTER OF SCIENCE

Mathematical Modelling and Scientific Computing

Mathematical Methods I

Hilary TERM 2024

Thursday 11 January, 9:30am - 12:00pm

This exam paper contains two sections. You may attempt as many questions as you like but you must answer at least one question in each section. Your best answer in each section will count, along with your next best two answers, making a total of four answers.
Please begin each question in a new answer booklet.

Do not turn this page until you are told that you may do so

Applied PDEs

1. Consider a PDE for $u(x, y)$ of the form

$$u + G(p, q) = 0, \tag{1.1}$$

where $p = u_x$, $q = u_y$.

(a) [12 marks] Starting from a clear statement of Charpit's equations, derive equations for $\dot{u}$, $\dot{p}$ and $\dot{q}$, where overdot denotes derivative with respect to the characteristic parameter τ. Give explicit forms for p and q.

(b) [13 marks] Let $G = p^2 + q^2 - 2$ in (1.1), and consider the data $u(x, 0) = x$ for $0 \leqslant x \leqslant 1$. Determine, in parametric form, the solution for $q \geqslant 0$. Sketch the domain of definition for your solution, labelling clearly any curves included.

[You are not required to determine $u(x, y)$ explicitly.]

2. Consider Laplace's equation

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0,$$

with $u = u(x, y, z)$ on the domain

$$\mathcal{D} = \{(x, y, z) : x > 0,\ -\infty < y < \infty,\ z > 0\},$$

with boundary conditions $u_x = g_1(y, z)$ on $x = 0$ and $u = g(x, y)$ on $z = 0$, where g_1 and g are prescribed functions.

(a) [5 marks] State the boundary-value problem satisfied by the corresponding Green's function $G(x, y, z; \xi, \eta, \zeta)$ on $\mathcal{D}$.

(b) [9 marks] Derive the solution $u(\xi, \eta, \zeta)$ as integrals in terms of g, g_1, G and G_z with respect to x, y and z. Specify the limits of each integration explicitly.

(c) [11 marks] Obtain G explicitly.

[*You may assume without proof that*

$$H(x, y, z; \xi, \eta, \zeta) = \frac{1}{4\pi\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}}$$

is the Green's function for the Laplace operator in $\mathbb{R}^3$.]

3. Consider solutions $h(x,t)$ of the partial differential equation (subscripts x and t denote derivatives)

$$(h^2)_t + (h^2 h_{xxx})_x = 0 \qquad \text{on } -s(t) < x < s(t), \tag{3.1a}$$

that satisfy the following boundary conditions at $x = \pm s(t)$,

$$h(\pm s(t), t) = 0, \tag{3.1b}$$

$$h_x(\pm s(t), t) = 0, \tag{3.1c}$$

$$h^2 h_{xxx} \to 0 \qquad \text{for } x \to \pm s(t), \tag{3.1d}$$

for $t > 0$, where $s(t)$ is also unknown.

(a) [6 marks] Show that solutions $h(x,t)$ and $s(t)$ of (3.1a)-(3.1d) conserve

$$I = \int_{-s(t)}^{s(t)} h^2(x,t)\,\mathrm{d}x.$$

(b) [10 marks] Suppose that $I = 1$. Show that, for a suitable choice of a, b and c, equations (3.1a)-(3.1d) are invariant under the scalings

$$t = \varepsilon^a \bar{t}, \quad x = \varepsilon^b \bar{x}, \quad s(t) = \varepsilon^b \bar{s}(\bar{t}), \quad h(x,t) = \varepsilon^c \bar{h}(\bar{x}, \bar{t}),$$

for all $\varepsilon > 0$.

(c) [9 marks] Use the result in (b) to determine constants α and β so that (3.1a)-(3.1d) with $I = 1$ have a self-similar solution of the form

$$h(x,t) = t^\alpha H(\xi) \quad \text{with } \xi = x/t^\beta \text{ and } s(t) = \sigma t^\beta.$$

State the third-order BVP solved by $H(\xi)$ and the integral constraint that determines σ.

4. You are given the first order quasilinear partial differential equation

$$u_t + \frac{1}{2}(u^2)_x = 0, \qquad (4.1)$$

on the domain $t > 0$ and $-\infty < x < \infty$.

(a) [4 marks] State conditions for the speed σ of a shock

$$u(x,t) = \begin{cases} u_- & \text{for } x \leqslant \sigma t, \\ u_+ & \text{for } x > \sigma t, \end{cases}$$

of (4.1) and for its causality in terms of u_- and u_+.

(b) [8 marks] Consider the initial data

$$u(x,0) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } x \geqslant 0. \end{cases} \qquad (4.2)$$

Using the method of characteristics, derive the causal solution of (4.1).

(c) Now consider the initial data

$$u(x,0) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } 0 \leqslant x < 1, \\ 0 & \text{for } x \geqslant 1. \end{cases} \qquad (4.3)$$

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(i) [6 marks] Determine the causal solution $u(x,t)$ for $t < 2$. Sketch the solution $u(x,t)$ at $t = 2$. In a separate diagram, sketch the characteristic projections in the (x,t) plane, clearly indicating any shock trajectory. What happens as $t \to 2$?

(ii) [7 marks] Continue the solution $u(x,t)$ for $t \geqslant 2$, and determine its limit as $t \to \infty$.

Supplementary Applied Mathematics

5. (a) [5 marks] Find the general real solution of the linear differential equation

$$Ly \equiv \frac{d^2y}{dx^2} + 2\,\frac{dy}{dx} + 2y = 0, \tag{5.1}$$

for $0 < x < \pi/2$.

(b) [10 marks] Consider the boundary value problem

$$Ly(x) = f(x) \quad \text{for } 0 < x < \frac{\pi}{2}, \quad \text{with } y(0) = 0 \text{ and } \frac{dy}{dx}\left(\frac{\pi}{2}\right) = 0, \tag{5.2}$$

with Ly as defined in equation (5.1).

State two equivalent problems for the Green's function $g(x, \xi)$:

I. using the delta function $\delta(x)$;

II. using only classical functions and with appropriate conditions at $x = \xi$.

Determine $g(x, \xi)$ explicitly.

(c) [10 marks] The set C_0^∞ denotes the set of "test" functions, i.e. all functions that have compact support and have derivatives of arbitrary order.

Define what it means to say that the map $T : C_0^\infty \to \mathbb{R}$ is a *distribution*.

Let

$$f(x) = \begin{cases} x^2 & -1 \leqslant x \leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

State the definition of the *derivative of a distribution* and use it to show that

$$D^2 f = \alpha + \beta_1 \delta(x-1) + \beta_2 \delta(x+1) + \gamma_1 \delta'(x-1) + \gamma_2 \delta'(x+1),$$

in the distributional sense, with constants α, β_1, β_2, γ_1 and γ_2 that you should determine.

6. (a) [15 marks] Consider the following third-order, boundary value problem:

$$Ly \equiv a_3(x)y'''(x) + a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0, \quad a < x < b, \tag{6.1}$$

$$\text{with } \alpha y(a) + \beta y''(a) = 0, \quad y'(a) = 0, \quad y(b) = 0, \tag{6.2}$$

where $\alpha > 0$, $\beta > 0$ and $a_i(x) > 0$ for all $x \in [a, b]$.

(i) Determine the associated adjoint problem.

(ii) Derive conditions on the functions $a_i(x)$ $(i = 0, 1, 2, 3)$ for which the adjoint differential operator L^* is such that $L^* = -L$, noting the minus sign.

(b) [10 marks] You are given that

$$Ly = y''' = f(x), \quad 2y(0) + y''(0) = 0, \quad y'(0) = 0, \quad y(1) = 0. \tag{6.3}$$

(i) Use the results from (a) to determine whether $L^* = -L$. Also state the boundary conditions associated with the adjoint operator L^*.

(ii) State clearly how the eigenvalues λ_k and eigenfunctions $y_k(x)$ of boundary value problem (6.3) are defined. State also how the eigenfunctions w_k of the boundary value problem associated with L^* are defined. [*Note: you do not need to determine expressions for λ_k, $y_k(x)$ or $w_k(x)$.*]

(iii) Determine the eigenfunctions $y_0(x)$ and $w_0(x)$ associated with $\lambda_0 = 0$. Hence, or otherwise, determine conditions on $f(x)$ for which (6.3) has a solution. Is the solution unique?