A11473W1

**DEGREE OF MASTER OF SCIENCE**

**Mathematical Modelling and Scientific Computing**

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# Mathematical Methods I

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**Hilary TERM 2021**

**Thursday, 14 January 2021**

**Opening Time: 9:30am GMT**

***You have 2 hours and 30 minutes of writing time to complete the paper and up to 30 minutes technical time to upload your answer file.***

***This exam paper contains two sections. You may attempt as many questions as you like but you must answer at least one question in each section. Your best answer in each section will count, along with your next best two answers, making a total of four answers.***

## Applied Partial Differential Equations

1. (a) [10 marks] You are given that the smooth functions $p = p(\xi,\eta)$, $q = q(\xi,\eta)$ and $r = r(\xi,\eta)$ satisfy $F(p,q,r) = 0$, where $F$ is a smooth, differentiable function. Suppose that the functions $x = x(\xi,\eta,\tau)$, $y = y(\xi,\eta,\tau)$, $z = z(\xi,\eta,\tau)$ and $u = u(\xi,\eta,\tau)$ are such that

$$x = x_0(\xi,\eta) + \tau\frac{\partial F}{\partial p}, \quad y = y_0(\xi,\eta) + \tau\frac{\partial F}{\partial q}, \quad z = z_0(\xi,\eta) + \tau\frac{\partial F}{\partial r},$$

$$u = u_0(\xi,\eta) + \tau\left(p\frac{\partial F}{\partial p} + q\frac{\partial F}{\partial q} + r\frac{\partial F}{\partial r}\right),$$

where $x_0(\xi,\eta)$, $y_0(\xi,\eta)$, $z_0(\xi,\eta)$ and $u_0(\xi,\eta)$ satisfy

$$p\frac{\partial x_0}{\partial \xi} + q\frac{\partial y_0}{\partial \xi} + r\frac{\partial z_0}{\partial \xi} = \frac{\partial u_0}{\partial \xi}, \quad p\frac{\partial x_0}{\partial \eta} + q\frac{\partial y_0}{\partial \eta} + r\frac{\partial z_0}{\partial \eta} = \frac{\partial u_0}{\partial \eta},$$

for all $\xi$ and $\eta$. Show that

$$\frac{\partial u}{\partial \tau} = p\frac{\partial x}{\partial \tau} + q\frac{\partial y}{\partial \tau} + r\frac{\partial z}{\partial \tau},$$

and derive similar expressions for $\partial u/\partial\eta$ and $\partial u/\partial\xi$. Deduce further that $p = \partial u/\partial x$, $q = \partial u/\partial y$ and $r = \partial u/\partial z$ provided that

$$\frac{\partial x}{\partial \xi}\left(\frac{\partial y}{\partial \eta}\frac{\partial z}{\partial \tau} - \frac{\partial y}{\partial \tau}\frac{\partial z}{\partial \eta}\right) + \frac{\partial y}{\partial \xi}\left(\frac{\partial x}{\partial \tau}\frac{\partial z}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial z}{\partial \tau}\right) + \frac{\partial z}{\partial \xi}\left(\frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \tau} - \frac{\partial x}{\partial \tau}\frac{\partial y}{\partial \eta}\right) \neq 0.$$

(b) Consider the first order partial differential equation for $u(x,y)$:

$$\frac{1}{4}p^2 - q^2 - y^2 + x^2 = 0, \tag{1}$$

where $p = \frac{\partial u}{\partial x}$, $q = \frac{\partial u}{\partial y}$. Suppose that the initial data is specified on a curve in the $(x,y)$-plane so that

$$x = 0, \quad y = s, \quad u = u_0(s), \quad \text{for } s_1 \leqslant s \leqslant s_2,$$

for smooth differentiable $u_0$.

(i) [8 marks] State Charpit's equations for this problem, together with appropriate initial data for their solution. Find a solution of Charpit's equations in parametric form for the case where $p > 0$ on the initial curve. In particular, show that $u(s,\tau)$ has the form

$$u(s,\tau) = A(\tau)(u_0')^2 + B(\tau)su_0'(s) + C(\tau)s^2 + u_0(s).$$

where you are to determine $A$, $B$, $C$, which are functions only of $\tau$. (Here $' = \mathrm{d}/\mathrm{d}s$.)

*[You might find the following identities helpful:* $\cos^2\tau = \frac{1+\cos(2\tau)}{2}$, $\sin^2\tau = \frac{1-\cos(2\tau)}{2}$, $\sin(2\tau) = 2\sin\tau\cos\tau$.*]*

(ii) [7 marks] Now consider the specific initial data

$$u(0, y) = u_0(y) = 0, \qquad y > 0. \tag{2}$$

Explain why the point $y = 0$ is excluded in this specification. For this initial data, determine the resulting domain of definition for $u$. Give the solution $u(x, y)$ of the initial value problem (1), (2) on $x \geqslant 0$, $y > 0$.
*[You are not required to simplify the final answer.]*

2. You are given the first order quasilinear partial differential equation in conservation form

$$u_t + \frac{1}{2}(u^2)_x = 0, \tag{3}$$

on the domain $t > 0$.

(a) Consider the initial data

$$u(x,0) = \frac{1}{1+x} \quad \text{for } x \geqslant 0. \tag{4}$$

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(i) [5 marks] State the characteristic equations for (3) with this initial data, and obtain the solution in parametric form.

(ii) [11 marks] Determine the envelope of the characteristic projections. Hence, or otherwise, find the domain of definition of the classical solution $u(x,t)$ and sketch it, including a couple of characteristic projections. Determine $u(x,t)$.

(b) Now consider the initial data

$$u(x,0) = \begin{cases} 1 & \text{for } x < 0, \\ \frac{1}{1+x} & \text{for } x \geqslant 0. \end{cases} \tag{5}$$

(i) [2 marks] State conditions for the speed $s$ of a shock

$$u(x,t) = \begin{cases} u_- & \text{for } x \leqslant st, \\ u_+ & \text{for } x > st, \end{cases}$$

of (3) and for its causality in terms of $u_-$ and $u_+$.

(ii) [7 marks] For the initial data (5), obtain a weak solution $u(x,t)$ of (3) in $t > 0$ for all $x$, by including a shock propagating along a trajectory $x = \xi(t)$ for $t \geqslant t_0$. Define $\xi(t)$ through an initial value problem of the form

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = f(\xi, t), \qquad \xi(t_0) = \xi_0, \tag{6}$$

where you are to specify $f(\xi,t)$ and the values for $t_0$ and $\xi_0$. *[You are not required to solve* (6)*.]*

3. (a) [10 marks] Construct the Green's function for the following system

$$\begin{aligned} \nabla^2 u &= 0 \qquad \text{in } x > 0,\ y > 0, \\ \frac{\partial u}{\partial x}(0, y) &= g(y), \\ u(x, 0) &= h(x), \end{aligned}$$

where $g$ and $h$ are given smooth functions with appropriate decay at infinity, using the method of images. Express the Green's function in the form $G(x, y; \xi, \eta) = \frac{1}{4\pi} \log\left(f(x, y; \xi, \eta)\right)$, where $f$ is a function that you should define.

(b) We consider the partial differential equation

$$u_t + \left[u^3\right]_x = \varepsilon u_{xx}, \tag{7}$$

with constant $\varepsilon \geqslant 0$. Let $u_l$ and $u_r$ be constant parameters with $u_l > 0$ and $u_r < u_l$.

(i) [5 marks] For the case $\varepsilon = 0$, give the speed $s$ of a shock

$$u(x, t) = \begin{cases} u_l & \text{for } x < st \\ u_r & \text{for } x > st. \end{cases}$$

in terms of the values $u_l$ and $u_r$. Show that shocks are only causal for $u_r > a(u_l)$, where you are to determine $a(u_l)$.

(ii) [10 marks] For $\varepsilon > 0$, assume you are given a *travelling wave solution* of (7), that is, a solution

$$u(x, t) = V(\xi), \qquad \xi = x - st,$$

where $s$ is the constant wave speed. Assume further that $V$ satisfies the far-field conditions

$$\begin{aligned} &V \to u_l \text{ and } V' \to 0 \text{ for } \xi \to -\infty, \\ &V \to u_r \text{ and } V' \to 0 \text{ for } \xi \to \infty, \end{aligned}$$

($' = \mathrm{d}/\mathrm{d}\xi$). Then show that $V$ solves the ordinary differential equation

$$V' = \frac{g(V)}{\varepsilon}, \quad \text{with } g(V) = \frac{1}{3}(V^3 - u_l^3) - s(V - u_l). \tag{8}$$

and that the wave speed $s$ must be the same as for the shock in part (i). You are now given that $g$ can be factored as

$$g(V) = \frac{1}{3}(V - u_l)(V - u_r)(V - \alpha),$$

where you are to determine $\alpha$. Use this to show that for a travelling wave solution $V$ to exist, $\alpha$ must lie outside of the interval $[u_l, u_r]$. Now show that $u_r > a(u_l)$ with the same $a(u_l)$ as in (i).

[*Hint: At which values of $V$ does $g(V)$ change sign? What are the implications for $V'$?*]

4. Consider the partial differential equation (PDE)

$$4s^3 v_{rr} - s v_{ss} + v_s - \frac{8s^3}{r^2} v = g. \tag{9a}$$

for $v = v(r, s)$, $r > 1/2$, $s > 0$, with initial data

$$v(1/2, s) = 0, \ v_r(1/2, s) = 0, \quad \text{for } s > 0, \tag{9b}$$

where $g = g(r, s)$ is a given function.

(a) [12 marks] Classify the PDE (9a). By introducing suitable new coordinates $r = r(x, y)$, $s = s(x, y)$, show that it transforms into the canonical form

$$a_1 u_{xx} + a_2 u_{xy} + a_3 u_{yy} - \frac{2u}{(x + y)^2} = f(x, y),$$

for the new dependent variable $u(x, y) = v(r, s)$, where you are to specify the function $f(x, y)$ in terms of $x$ and $y$ as well as the constants $a_1$, $a_2$, $a_3$. State the initial condition (9b) in canonical coordinates.

(b) [5 marks] Obtain a solution of the canonical form of the PDE for the case $f = 0$, using the ansatz

$$u(x, y) = 1 + \alpha \frac{(x - \xi)(y - \eta)}{(\xi + \eta)(x + y)},$$

where $\xi$ and $\eta$ are arbitrary, while $\alpha$ is a constant that you are to determine.

(c) [8 marks] Hence determine the solution $v(r, s)$ of (9) for general $g$, using the result from part (b).
*[You are not required to evaluate any integrals.]*

## Supplementary Applied Mathematics

5. Consider the boundary value problem

$$Ly = -\frac{d^2y}{dx^2} + v\frac{dy}{dx} = f(x), \quad x \in [0, R], \tag{10}$$

where $v, R > 0$ are constants, along with the boundary conditions,

$$\frac{dy}{dx}(0) - vy(0) = 0, \quad \frac{dy}{dx}(R) - vy(R) = 0. \tag{11}$$

(a) [6 marks] Using Sturm-Liouville theory, show that the eigenvalues $\lambda$ in

$$Ly = \lambda y \tag{12}$$

must be real.

(b) [9 marks] Find the eigenfunctions $y_n$ and eigenvalues $\lambda_n$ of the operator $L$.

(c) [5 marks] Compute the adjoint operator, and the associated adjoint eigenfunctions directly or by considering the transformation $v \to -v$.

(d) [5 marks] Determine the general solution $y$ for the boundary value problem

$$Ly = f(x), \quad \frac{dy}{dx}(0) - vy(0) = -1, \quad -\frac{dy}{dx}(R) + vy(R) = 1,$$

and state any conditions on $f$ for such a solution to exist.

6. (a) [8 marks] Consider the boundary-value problem,

$$Lu = -x^2\frac{d^2u}{dx^2} - x\frac{du}{dx} + u = f(x), \quad u(1) = 0, \quad u(2) = 0. \tag{13}$$

Determine the equation and matching conditions satisfied by the Green's function, $g$, associated to the operator $L$ in (13). Determine solutions to the homogeneous equation $Lu = 0$, and use these to construct $g$. Finally show how to construct the solution to the original problem (13) for a given $f(x)$.

(b) [5 marks] The derivative of a distribution $T : C_0^\infty \to \mathbb{R}$, i.e. $T'$, is defined in terms of its action on a test function $\phi$ by $\langle T', \phi\rangle = -\langle T, \phi'\rangle$ for all test functions $\phi$. In terms of scalar multiplication, translation, and convergence of distributions, state an equivalent definition of $T'$.

Let $A \subset \mathbb{R}$ be an (at most countable) union of disjoint bounded intervals. Compute $\langle \mathbf{1}'_{x\in A}, \phi\rangle$, where $\mathbf{1}_{x\in A}$ is the indicator function defined by,

$$\mathbf{1}_{x\in A} = \begin{cases} 1 & x \in A, \\ 0 & x \notin A. \end{cases}$$

and $\phi \in C_0^\infty$.

(c) [12 marks] Consider the boundary-value problem,

$$Lu = \frac{d^4u}{dx^4} = f(x), \quad u(0) = 0, \quad u(1) = 0, \quad u''(0) = 0, \quad u''(1) = 0. \tag{14}$$

Determine the equation and matching conditions satisfied by the Green's function, $g$, associated to the operator $L$ in (14). Determine solutions to the homogeneous equation $Lu = 0$, and use these to construct $g$. Show how to construct the solution to the original problem (14) for a given $f(x)$.

*[Hint: You can simplify solving for the matching conditions by combining two homogeneous solutions which satisfy the left and right boundary conditions respectively.]*