DEGREE OF MASTER OF SCIENCE

Mathematical Modelling and Scientific Computing

Paper A2: Mathematical Methods II

TRINITY TERM 2021

Thursday, 22 April 2021

Opening Time: 9:30am BST

Mode of Completion: Handwritten

You have 2 hours and 30 minutes of writing time to complete the paper and up to 30 minutes technical time to upload your answer file.

This exam paper contains three sections. You may attempt as many questions as you like but you must answer at least one question in each section. Your best answer in each section will count, along with your next best answer, making a total of four answers.

Nonlinear Systems

1. Consider the system

$$\begin{aligned}\dot{x} &= -y^3 + x - x^3,\\ \dot{y} &= xy^2,\end{aligned}$$

with associated flow $\varphi_t : \mathbb{R}^2 \mapsto \mathbb{R}^2$.

(a) [2 marks] Show that the line $y = 0$, the halfspace $y > 0$, and the halfspace $y < 0$ are invariant sets.

(b) [9 marks] Determine the stable, unstable and/or centre linear subspaces for each of the fixed points $(0, 0)$, $(-1, 0)$ and $(1, 0)$. Which fixed points are non-hyperbolic? What does the linearisation tell you about the asymptotic stability of the fixed points?

(c) [7 marks] Find a cubic approximation to the centre manifold in the vicinity of the fixed points $(\pm 1, 0)$, and use it to determine the local dynamics. Are these fixed points stable?

(d) [7 marks] For $c > 0$ let D denote the open set bounded by the curves

$$\begin{aligned}
&\Gamma_1 = \{(x, y) : x + y = c,\ -2 < x < 2\},\\
&\Gamma_2 = \{(x, y) : x^2 + y^2 = 4 + (c-2)^2,\ x > 2,\ y > 0\},\\
&\Gamma_3 = \{(x, y) : x^2 + y^2 = 4 + (c+2)^2,\ x < -2,\ y > 0\},\\
&\Gamma_4 = \{(x, y) : y = 0,\ -\sqrt{4 + (c+2)^2} < x < \sqrt{4 + (c-2)^2}.
\end{aligned}$$

Show that D is an invariant set for c sufficiently large.

Deduce that $\varphi_t(\mathbf{x}) \to (-1, 0)$ as $t \to \infty$ for all $\mathbf{x} \in D$. Is $\{(-1, 0)\}$ an attracting set? *[You may assume that there are no limit cycles.]*

2. Consider the system

$$\begin{aligned}\dot{x} &= (\mu-1)x+(\mu+1)y-x^3,\\ \dot{y} &= (\mu+1)x+(\mu-1)y+\delta x^2.\end{aligned}$$

(a) [10 marks] Consider the case $\delta = 0$. For what value of μ does a bifurcation of the steady state $(0,0)$ occur? Find the local dynamics on the extended centre manifold, and describe the type of bifurcation.

(b) [8 marks] Now consider the case $\delta > 0$. Find the local dynamics on the extended centre manifold, and describe the type of bifurcation.

(c) [7 marks] Suppose now that $0 < \delta \ll 1$. Find a cubic approximation to the local dynamics on the centre manifold. Sketch the bifurcation diagram, including the stability of the branches.

Further Mathematical Methods

3. (a) [7 marks] Consider the boundary value problem

$$y''(x) + 2\varepsilon y'(x) + y(x) = \sin(x), \quad y(0) = y(2\pi), \quad y'(0) = y'(2\pi). \tag{1}$$

Explicitly use the solvability condition to show that a regular perturbation solution of the form

$$y(x) \sim y_0(x) + \varepsilon y_1(x) + O(\varepsilon^2),$$

does not work. Propose an alternative expansion, and determine the first term.

(b) [18 marks] Solve the inhomogeneous Fredholm equation,

$$y(x) = f(x) + \lambda \int_0^{2\pi} (\cos(x)\sin(t) + xt)\, y(t)dt,$$

determining the value(s) of λ such that the solution is unique. For any value(s) of λ where a unique solution does not exist, use the Fredholm Alternative to describe the form of non-unique solutions, noting any solvability conditions on f. Write the solution for $f(x) = \cos(x)$, noting carefully what happens as λ varies.

4. (a) Suppose the function $y(x)$ minimizes the functional,

$$J[y] = \int_a^b F(x, y, y')dx,$$

over all functions $y \in C^2([a, b])$ with $y(a)$ and $y(b)$ undetermined, where F and its derivatives are continuously differentiable in all arguments.

(i) [10 marks] Derive the Euler-Lagrange equations and boundary conditions that y must satisfy.

(ii) [6 marks] Determine the curve that minimizes the functional,

$$\int_0^1 \frac{y'^2}{2} + yy' + y' + y + f(x)dx,$$

when $y(0)$ and $y(1)$ are undetermined. Explain why the curve $y(x)$ does not depend on the function $f(x)$, and what impact it has on the value of the functional.

(b) [9 marks] Minimize the functional,

$$\int_1^2 x^2 y'^2 dx, \quad \text{subject to} \quad \int_1^2 y dx = A,$$

over all $y(x)$ with $y(1) = 0$ and $y(2) = 0$.

Further Partial Differential Equations

5. Consider the partial differential equation

$$\frac{\partial h}{\partial t} = \frac{\partial^2 h}{\partial x^2} + \frac{x}{2t^{3/2}} h. \tag{1}$$

(a) [3 marks] Suppose that the following boundary conditions are satisfied:

$$h(0,t) = 1 \qquad \text{and} \qquad h \to 0 \quad \text{as} \quad x \to \infty. \tag{2}$$

Use a scaling-law analysis or otherwise to show that a similarity solution exists of the form $h(x,t) = f(\eta)$ with $\eta = x/t^{\alpha}$ for some value of α that you should determine.

(b) [3 marks] Find the ordinary differential equation satisfied by f and state the boundary conditions on f.

(c) [3 marks] Use the fact that the solution for f is

$$f = \left(1 - \frac{\eta}{4}\right) \exp\left(-\frac{\eta^2}{4} + \eta\right)$$

to show that the function h that solves (1) subject to (2) has two turning points and find the location in x of each of these.

(d) [2 marks] By finding the value of $f'(0)$ or otherwise, find the nature of each of the turning points found in (c).

(e) [3 marks] Now suppose that instead of the boundary conditions (2) we have

$$h(0,t) = t \qquad \text{and} \qquad h \to 0 \quad \text{as } x \to \infty. \tag{3}$$

Show that the similarity form is now of the form $h = t^{\beta} F(\xi)$ with $\xi = x/t^{\gamma}$ for some value of β and γ that you should determine.

(f) [3 marks] Find the ordinary differential equation satisfied by F and state the boundary conditions on F.

(g) [5 marks] Consider the behaviour of F for large ξ by rescaling $\xi = \lambda z$ for $\lambda \gg 1$ and $z = O(1)$. Assuming that F takes the form

$$F(z) = A\exp\left(-\lambda k(z)\right), \tag{4}$$

for some constant A, find a differential equation for k correct to leading order in $1/\lambda$, and show that the solution is $k(z) = z + B$ for some constant B.

(h) [3 marks] Show by direct substitution or otherwise that the solution for F as $\xi \to \infty$, given by equation (4), with the function k found in part (g), is actually the solution for all η for some value of A that you should determine. Use this result to state the similarity solution of (1) subject to the boundary conditions (3).

6. Consider a substance located in $0 \leqslant x \leqslant h(t)$, with a solid–liquid interface at $x = s(t)$ and a time-dependent boundary at $h(t)$, described by the following system.

$$\frac{\partial^2 T_1}{\partial x^2} = 0, \qquad 0 \leqslant x \leqslant s(t), \tag{5a}$$

$$\frac{\partial^2 T_2}{\partial x^2} = 0, \qquad s(t) \leqslant x \leqslant h(t), \tag{5b}$$

$$T_1(0,t) = 1, \tag{5c}$$

$$T_1(s(t),t) = T_2(s(t),t) = 0, \tag{5d}$$

$$\frac{\partial T_2(s(t),t)}{\partial x} - \frac{\partial T_1(s(t),t)}{\partial x} = \frac{\mathrm{d}s}{\mathrm{d}t}, \tag{5e}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -T_2(h(t),t), \tag{5f}$$

$$\frac{\partial T_2(h(t),t)}{\partial x} = \beta(s(t)-1), \tag{5g}$$

$$s(0) = 0, \qquad h(0) = 1. \tag{5h}$$

Here, T_1 denotes the temperature in the liquid, T_2 denotes the temperature in the solid and β is a constant.

(a) [7 marks] Use the system (5) to find expressions for the temperatures T_1 and T_2 in terms of s, h and the parameter β and generate ordinary differential equations $\mathrm{d}s/\mathrm{d}t = f(s)$ and $\mathrm{d}h/\mathrm{d}t = g(s,h)$ for some functions f and g (which may also depend on β) that you should find.

(b) [5 marks] Consider the steady-state solution. Show that this corresponds to a case in which the entire substance has melted i.e., $s = h = \text{constant} = s^*$ and find the possible values of s^*. State any restrictions on β that are required to obtain a real solution.

(c) [3 marks] By drawing an approximate sketch of $f(s)$ and looking near the points where $f(s) = 0$, determine the stability of the steady states you have found.

(d) [5 marks] Now consider the case in which the system is not in steady state. By scaling $t = \epsilon T$ for $\epsilon \ll 1$ along with $s = \epsilon^a S$ and $h = 1 + \epsilon^b H$ in the ordinary differential equations for s and h derived in (a), find values for a and b that lead to a dominant balance in ϵ and hence find differential equations for s and h that are accurate to leading order in ϵ. Hence, show that $s \approx \sqrt{2t}$ and find an expression for $h(t)$ in this early-time limit.

(e) [5 marks] Now consider the system close to the steady state, so that almost all of the substance has melted. By scaling s appropriately, show that $(s - s^*) \propto \exp(-(1/s^{*2} - \beta)t)$ in this limit.