Part A Mathematics 2023-24

August 21, 2023

Contents

1 Foreword

The confirmed synopses for Part A 2023-24 will be available on the course management portal `https://courses.maths.ox.ac.uk/` before the start of Michaelmas Term 2023.

Three or Four Year Course

All students who complete Parts A and B will be classified. Those who have achieved honours and who wish to graduate at this point may supplicate for a BA.

You will need to achieve overall a 2.1 or better in your second and third year assessments and a weighted average of 59.5 or above for your third year assessments to progress to Part C.

Students wishing to take the four-year course should register to do so during their third year, and will be permitted to do so if they meet the above progression criteria in the classification at the end of the third year. They will take Part C in their fourth year, be awarded a separate classification and, if successful, may supplicate for an MMath.

Masters in Theoretical and Mathematical Physics

There is a Mathematical Physics stream as an alternative to Part C (the fourth year). Students who move on to this stream and successfully complete the year will be awarded an MMathPhys.

Mathematics students interested in transferring to the MMathPhys will need to make an application during their third year. This option will not be available to students on the joint degrees.

Interested students should bring this up with their tutors. Full details relating to this masters will be in the MMathPhys handbook, including details of those second and third year options which are suggested background or recommendations for the masters and a description of the application process and deadlines. Further details available online: `http://mmathphys.physics.ox.ac.uk/`.

Pathways to Part B

Most, but not all, third year options (Part B) have certain pre-requisites from Part A. Whilst the courses that will be offered in Part B in a year's time (to follow on from the Part A courses detailed in this supplement) have not been wholly decided there is not substantial change year-on-year in the list of Part B options offered.

What follows is a list envisaging how the Part A options for 2023-24 would be pre-requisites or useful knowledge for the Part B options for 2024-25. You will note that there are also a good number of courses that have no prerequisites.

- **A3 Rings and Modules**

  Essential for B2.1: Introduction to Representation Theory

  Essential for B2.2: Commutative Algebra

  Essential for B3.1: Galois Theory

  Essential for B3.4: Algebraic Number Theory

- **A4 Integration**

Essential for B4.1 Functional Analysis I

Essential for B4.2 Functional Analysis II

Essential for B4.3: Distribution Theory

Recommended for B8.1: Probability, Measure and Martingales

Essential for B8.2: Continuous Martingales and Stochastic Calculus

Recommended for B8.3: Mathematical Models of Financial Derivatives

- **A5 Topology**

  Recommended for B3.2: Geometry of Surfaces

  Essential for B3.3: Algebraic Curves

  Essential for B3.5: Topology and Groups

- **A6 Differential Equations 2**

  Essential for B5.2: Applied Partial Differential Equations

  Recommended for B5.3: Viscous Flow

  Recommended for B5.5: Further Mathematical Biology

- **A7 Numerical Analysis**

  Recommended for B6.1 Numerical Solution of Partial Differential Equations

  Recommended for B6.2 Optimisation for Data Science

- **A8 Probability**

  Essential for B5.1: Stochastic Modelling of Biological Processes

  Recommended for B8.1: Probability, Measure and Martingales

  Essential for B8.2: Continuous Martingales and Stochastic Calculus

  Essential for B8.3: Mathematical Models of Financial Derivatives

  Recommended for B8.4: Information Theory

  Essential for SB3.1: Applied Probability

  Essential for SB1 Applied and Computational Statistics

  Useful for SB2.1 Foundations of Statistical Inference

- **A9 Statistics**

  Essential for SB1 Applied and Computational Statistics

  Essential for SB2.1 Foundations of Statistical Inference

- **A10 Fluids and Waves**

  Recommended for B5.2: Applied Partial Differential Equations

  Recommended for B5.3: Viscous Flow

  Essential for B5.4: Waves and Compressible Flow

- **A11 Quantum Theory**

  Essential for B7.3: Further Quantum Theory

- **ASO: Number Theory**
  Recommended for B3.4: Algebraic Number Theory
- **ASO: Group Theory**
  Recommended for B2.1: Introduction to Representation Theory
  Recommended for B3.1: Galois Theory
  Recommended for B3.5: Topology and Groups
- **ASO: Projective Geometry**
  Recommended for B3.3: Algebraic Curves
- **ASO: Multidimensional Analysis and Geometry**
  Useful for B3.2: Geometry of Surfaces
  Useful for B3.3: Algebraic Curves
- **ASO: Integral Transforms**
  Recommended for B4.3: Distribution Theory
  Essential for B5.1: Stochastic Modelling of Biological Processes
  Recommended for B5.2: Applied Partial Differential Equations
  Essential for B5.4: Waves and Compressible Flow
- **ASO: Calculus of Variations**
  Recommended for B5.2: Applied Partial Differential Equations
  Essential for B7.1: Classical Mechanics
- **ASO: Graph Theory**
  Recommended for B8.5: Graph Theory
- **ASO: Special Relativity**- no Part B courses explicitly require this.
- **ASO: Mathematical Modelling in Biology**
  Essential for B5.5: Further Mathematical Biology

The following Part B courses in 2022-23 have no prerequisites from Part A.

- B1.1: Logic
- B1.2: Set Theory
- B4.4 Fourier Analysis
- B6.3 Integer Programming
- B7.2: Electromagnetism
- BEE/BOE Mathematical/Other Mathematical Extended Essay
- BSP Structured Projects

- BO1.1: History of Mathematics
- OCS1 Lambda Calculus and Types
- OCS2 Computational Complexity
- N101 Early Modern Philosophy
- N102 Knowledge and Reality
- N122 Philosophy of Mathematics
- N127 Philosophical Logic

2 Syllabus

The examination syllabus, as referred to in the *Examination Regulations*, and synopses have been approved by the Mathematics Teaching Committee for examination in 2024. Please see the current edition of the *Examination Regulations* (`https://www.admin.ox.ac.uk/examregs/`) for the full regulations governing these examinations. Examination Conventions can be found at: `https://www.maths.ox.ac.uk/members/students/undergraduate-courses/examinations-assessments/examination-conventions`.

The Part A examination **syllabus** is the mathematical material of the synopses, as separately detailed by paper below. The course synopses on the course webpages also give additional detail to the syllabus (for example, showing how the material is split by lectures) and are also accompanied by lists of recommended reading.

Honour School of Mathematics - Part A

For Part A, each candidate shall be required to offer nine or ten written papers. These papers must include:

- A0 - Linear Algebra (1.5 hours)
- A1 - Differential Equations 1 (1.5 hours)
- A2 - Metric Spaces and Complex Analysis (3 hours)
- ASO - Short Options (1.5 hours)

and five or six papers from the Long Options (each 1.5 hours long).

- A3 Rings and Modules
- A4 Integration
- A5 Topology
- A6 Differential Equations 2
- A7 Numerical Analysis
- A8 Probability
- A9 Statistics
- A10 Fluids and Waves
- A11 Quantum Theory

Paper ASO will examine the nine Short Options (Number Theory, Group Theory, Projective Geometry, Multidimensional Analysis and Geometry, Integral Transforms, Calculus of Variations, Graph Theory, Special Relativity, and Mathematical Modelling in Biology). Students are recommended to take three of these Short Options.

Part A shall be taken on one occasion only (there will be no resits). At the end of the Part A examinations, a candidate will be awarded nine or ten 'University Standardised

Marks' (USMs). The USM from Paper A2 will have twice the weight of the USMs awarded for the other papers. A weighted average of these USMs will be carried forward for the classification awarded at the end of the third year, with this average from the second year papers counting for 40

Additional Long Option

Students are permitted to take an additional long option at Part A. A student taking 5 long options will still have each of them counting as a unit's weight towards their overall second year mark.

For a student taking 6 long options, their best 4 papers (following the exams) will count one unit each and their worst 2 papers will count half a unit each. Thus these 6 papers will overall still have a weight of 5 units.

The results from all 6 papers will appear on the student's exam transcript.

The aim of the above scoring system is to ensure anyone taking on an extra option will not do so lightly (all marks will be reported and all count to some extent) but also that a student will not get a lower overall mark for having taken on the extra workload (the given scoring system was a fair compromise looking at several years' data sets).

We are anticipating that most students will not wish to take on the extra workload, and that in most years it will be some subset of the students who gained distinctions in Prelims who would wish to take 6 long options.

Core Material The examination syllabi of the three core papers A0, A1 and A2 shall be the mathematical content of the synopses for the courses

- Linear Algebra
- Differential Equations 1
- Metric Spaces and Complex Analysis

Options

The examination syllabi of the options paper, A3-A11, shall be the mathematical content of the synopses for the courses

- Rings and Modules
- Integration
- Topology
- Differential Equations 2
- Numerical Analysis
- Probability
- Statistics
- Fluids and Waves

- Quantum Theory

Short Options

The examination syllabi of the short options paper ASO shall be the mathematical content of the synopses for the courses

- Number Theory
- Group Theory
- Projective Geometry
- Multidimensional Analysis and Geometry
- Integral Transforms
- Calculus of Variations
- Graph Theory
- Special Relativity
- Mathematical Modelling in Biology

3 A0: Linear Algebra

3.1 Overview

The core of linear algebra comprises the theory of linear equations in many variables, the theory of matrices and determinants, and the theory of vector spaces and linear maps. All these topics were introduced in the Prelims course. Here they are developed further to provide the tools for applications in geometry, modern mechanics and theoretical physics, probability and statistics, functional analysis and, of course, algebra and number theory. Our aim is to provide a thorough treatment of some classical theory that describes the behaviour of linear maps on a finite-dimensional vector space to itself, both in the purely algebraic setting and in the situation where the vector space carries a metric derived from an inner product.

3.2 Learning Outcomes

Students will deepen their understanding of Linear Algebra. They will be able to define and obtain the minimal and characteristic polynomials of a linear map on a finite-dimensional vector space, and will understand and be able to prove the relationship between them; they will be able to prove and apply the Primary Decomposition Theorem, and the criterion for diagonalisability. They will have a good knowledge of inner product spaces, and be able to apply the Bessel and Cauchy–Schwarz inequalities; will be able to define and use the adjoint of a linear map on a finite-dimensional inner product space, and be able to prove and exploit the diagonalisability of a self-adjoint map.

3.3 Synopsis

Definition of an abstract vector space over an arbitrary field. Examples. Linear maps. [1]

Definition of a ring. Examples to include $\mathbb{Z}$, $F[x]$, $F[A]$ (where A is a matrix or linear map), $\mathrm{End}(V)$. Division algorithm and Bezout's Lemma in $F[x]$. Ring homomorphisms and isomorphisms. Examples. [2]

Characteristic polynomials and minimal polynomials. Coincidence of roots. [1]

Quotient vector spaces. The first isomorphism theorem for vector spaces and rank-nullity. Induced linear maps. Applications: Triangular form for matrices over $\mathbb{C}$. Cayley-Hamilton Theorem. [2]

Primary Decomposition Theorem. Diagonalizability and Triangularizability in terms of minimal polynomials. Proof of existence of Jordan canonical form over $\mathbb{C}$ (using primary decomposition and inductive proof of form for nilpotent linear maps). [3]

Dual spaces of finite-dimensional vector spaces. Dual bases. Dual of a linear map and description of matrix with respect to dual basis. Natural isomorphism between a finite-dimensional vector space and its second dual. Annihilators of subspaces, dimension formula. Isomorphism between U^0 and $(V/U)'$. [3]

Recap on real inner product spaces. Definition of non-degenerate symmetric bilinear forms and description as isomorphism between V and V'. Hermitian forms on complex vector spaces. Review of Gram-Schmidt. Orthogonal Complements. [1]

Adjoints for linear maps of inner product spaces. Uniqueness. Concrete construction via matrices [1]

Definition of orthogonal/unitary maps. Definition of the groups O_n, SO_n, U_n, SU_n. Diagonalizability of self-adjoint and unitary maps. [2]

3.4 Reading List

1) Richard Kaye and Robert Wilson, *Linear Algebra* (OUP, 1998) ISBN 0-19-850237-0. Chapters 2–13. [Chapters 6, 7 are not entirely relevant to our syllabus, but are interesting.]

3.5 Further Reading

1) Paul R. Halmos, *Finite-dimensional Vector Spaces*, (Springer Verlag, Reprint 1993 of the 1956 second edition), ISBN 3-540-90093-4. sections 1–15, 18, 32–51, 54–56, 59–67, 73, 74, 79. [Now over 50 years old, this idiosyncratic book is somewhat dated but it is a great classic, and well worth reading.]

2) Seymour Lipschutz and Marc Lipson, *Schaum's Outline of Linear Algebra* (3rd edition, McGraw Hill, 2000), ISBN 0-07-136200-2. [Many worked examples.]

3) C. W. Curtis, *Linear Algebra - an Introductory Approach*, (4th edition, Springer, reprinted 1994).

4) D. T. Finkbeiner, *Elements of Linear Algebra* (Freeman, 1972). [Out of print, but available in many libraries.]

4 A1: Differential Equations 1

4.1 Overview

The aim of this course is to introduce students reading mathematics to some of the basic theory of ordinary and partial differential equations.

Much of the study of differential equations in the first year consisted of finding explicit solutions of particular ODEs or PDEs. However, for many differential equations which arise in practice one is unable to give explicit solutions and, for the most part, this course considers what information one can discover about solutions without actually finding the solution. Does a solution exist? Is it unique? Does it depend continuously on the initial data? How does it behave asymptotically? What are appropriate data? In this course some techniques will be developed for answering such questions.

The course will furnish undergraduates with the necessary skills to pursue any of the applied options in the third year and will also form the foundation for deeper and more rigorous courses in differential equations, the part B courses on Distribution Theory and on Fourier Analysis and the Part C courses Functional Analytic Methods for PDEs and Fixed Point Methods for Nonlinear PDEs.

4.2 Learning Outcomes

Students will have learnt a range of different techniques and results used in the study of ODEs and PDEs, such as: Picard's theorem proved both by successive approximation and the contraction mapping theorem; Gronwall's inequality; phase plane analysis; method of characteristics for first order semi-linear PDEs; classification of second order semi-linear PDEs and their reduction to normal form using characteristic variables; well posedness; the maximum principle and some of its consequences. They will have gained an appreciation of the importance of existence and uniqueness of solution and will be aware that explicit analytic solutions are the exception rather than the rule.

4.3 Synopsis

Picard's Existence Theorem: Picard's Theorem for first-order scalar ODEs with proof. Gronwall's inequality leading to uniqueness and continuous dependence on the initial data. Examples of blow-up and nonuniqueness, discussion of continuation and global existence. Proof of Picard's Theorem via Contraction Mapping (Theorem CMT to be covered in Metric Spaces course) and relationship between the two proofs; extension to systems. Application to scalar second order ODEs, with particular reference to linear equations. (5 lectures)

Phase plane analysis: Phase planes, critical points, definition of stability, classification of critical points and linearisation, Bendixson-Dulac criterion. (4 lectures)

PDEs in two independent variables: First order semi-linear PDEs (using parameterisation). Classification of second order, semilinear PDEs; Normal form; Ideas of uniqueness and well-posedness. Illustration of suitable boundary conditions by example. Poisson's Equation and the Heat Equation: Maximum Principle leading to uniqueness and continuous dependence on the initial data. (7 lectures)

4.4 Reading List

The best single text is:

P. J. Collins, *Differential and Integral Equations* (O.U.P., 2006), Chapters 1-7, 14,15.

4.5 Further Reading

W. E. Boyce & R. C. DiPrima, *Elementary Differential Equations and Boundary Value Problems* (7th edition, Wiley, 2000).

Erwin Kreyszig, *Advanced Engineering Mathematics* (8th Edition, Wiley, 1999).

W. A. Strauss, *Partial Differential Equations: an Introduction* (Wiley, 1992).

G. F. Carrier & C E Pearson, *Partial Differential Equations - Theory and Technique* (Academic, 1988).

J. Ockendon, S. Howison, A. Lacey & A. Movchan, *Applied Partial Differential Equations* (Oxford, 1999). [More advanced.]

5 A2: Metric Spaces and Complex Analysis

5.1 Overview

The theory of functions of a complex variable is a rewarding branch of mathematics to study at the undergraduate level with a good balance between general theory and examples. It occupies a central position in mathematics with links to analysis, algebra, number theory, potential theory, geometry, topology, and generates a number of powerful techniques (for example, evaluation of integrals) with applications in many aspects of both pure and applied mathematics, and other disciplines, particularly the physical sciences.

In these lectures we begin by introducing students to the language of topology before using it in the exposition of the theory of (holomorphic) functions of a complex variable. The central aim of the lectures is to present Cauchy's Theorem and its consequences, particularly series expansions of holomorphic functions, the calculus of residues and its applications.

The course concludes with an account of the conformal properties of holomorphic functions and applications to mapping regions.

5.2 Learning Outcomes

Students will have been introduced to point-set topology and will know the central importance of complex variables in analysis. They will have grasped a deeper understanding of differentiation and integration in this setting and will know the tools and results of complex analysis including Cauchy's Theorem, Cauchy's integral formula, Liouville's Theorem, Laurent's expansion and the theory of residues.

5.3 Synopsis

Metric Spaces (10 lectures)

Basic definitions: metric spaces, isometries, continuous functions ($\varepsilon - \delta$ definition), homeomorphisms, open sets, closed sets. Examples of metric spaces, including metrics derived from a norm on a real vector space, particularly l^1, l^2, l^∞ norms on $\mathbb{R}^n$, the sup norm on the bounded real-valued functions on a set, and on the bounded continuous real-valued functions on a metric space. The characterisation of continuity in terms of the pre-image of open sets or closed sets. The limit of a sequence of points in a metric space. A subset of a metric space inherits a metric. Discussion of open and closed sets in subspaces. The closure of a subset of a metric space. [3]

Completeness (but not completion). Completeness of the space of bounded real-valued functions on a set, equipped with the norm, and the completeness of the space of bounded continuous real-valued functions on a metric space, equipped with the metric. Lipschitz maps and contractions. Contraction Mapping Theorem. [2.5]

Connected metric spaces, path-connectedness. Closure of a connected space is connected, union of connected sets is connected if there is a non-empty intersection, continuous image of a connected space is connected. Path-connectedness implies connectedness. Connected open subset of a normed vector space is path-connected. [2]

Definition of sequential compactness and proof of basic properties of compact sets. Preservation of compactness under continuous maps, equivalence of continuity and uniform continuity for functions on a compact set. Equivalence of sequential compactness with being complete and totally bounded. The Arzela-Ascoli theorem (proof non-examinable). Open cover definition of compactness. Heine-Borel (for the interval only) and proof that compactness implies sequential compactness (statement of the converse only). [2.5]

Complex Analysis (22 lectures)

Basic geometry and topology of the complex plane, including the equations of lines and circles. Extended complex plane, Riemann sphere, stereographic projection. Möbius transformations acting on the extended complex plane. Möbius transformations take circlines to circlines. [3]

Complex differentiation. Holomorphic functions. Cauchy-Riemann equations (including $z, \bar{z}$ version). Real and imaginary parts of a holomorphic function are harmonic. [2]

Recap on power series and differentiation of power series. Exponential function and logarithm function. Fractional powers — examples of multifunctions. The use of cuts as method of defining a branch of a multifunction. [3]

Path integration. Cauchy's Theorem. (Sketch of proof only — students referred to various texts for proof.) Fundamental Theorem of Calculus in the path integral/holomorphic situation. [2]

Cauchy's Integral formulae. Taylor expansion. Liouville's Theorem. Identity Theorem. Morera's Theorem. [4]

Laurent's expansion. Classification of isolated singularities. Calculation of principal parts, particularly residues. [2]

Residue Theorem. Evaluation of integrals by the method of residues (straightforward examples only but to include the use of Jordan's Lemma and simple poles on contour of integration). [3]

Conformal mappings. Riemann mapping theorem (no proof), Möbius transformations, exponential functions, fractional powers; mapping regions (not Christoffel transformations or Joukowski's transformation). [3]

5.4 Reading List

- W. A. Sutherland, *Introduction to Metric and Topological Spaces* (Second Edition, OUP, 2009).
- H. A. Priestley, *Introduction to Complex Analysis* (Second edition, OUP, 2003).

5.5 Further Reading

- L. Ahlfors, *Complex Analysis* (McGraw-Hill, 1979).
- Reinhold Remmert, *Theory of Complex Functions* (Springer, 1989) (Graduate Texts in Mathematics 122).

6 A3: Rings and Modules

6.1 Overview

The first abstract algebraic objects which are normally studied are groups, which arise naturally from the study of symmetries. The focus of this course is on rings, which generalise the kind of algebraic structure possessed by the integers: a ring has two operations, addition and multiplication, which interact in the usual way. The course begins by studying the fundamental concepts of rings (already met briefly in core Algebra): what are maps between them, when are two rings isomorphic etc. much as was done for groups. As an application, we get a general procedure for building fields, generalising the way one constructs the complex numbers from the reals. We then begin to study the question of factorization in rings, and find a class of rings, known as Unique Factorization Domains, where any element can be written uniquely as a product of prime elements generalising the case of the integers. Finally, we study modules, which roughly means we study linear algebra over certain rings rather than fields. This turns out to have powerful applications to ordinary linear algebra and to abelian groups.

6.2 Learning Outcomes

Students should become familiar with rings and fields, and understand the structure theory of modules over a Euclidean domain along with its implications. The material underpins many later courses in algebra and number theory, and thus should give students a good background for studying these more advanced topics.

6.3 Synopsis

Recap on rings (not necessarily commutative) and examples: $\mathbb{Z}$, fields, polynomial rings (in more than one variable), matrix rings. Zero-divisors, integral domains. Units. The characteristic of a ring. Discussion of fields of fractions and their characterization (proofs non-examinable) [2]

Homomorphisms of rings. Quotient rings, ideals and the first isomorphism theorem and consequences, e.g. Chinese remainder theorem. Relation between ideals in R and R/I. Prime ideals and maximal ideals, relation to fields and integral domains. Examples of ideals. Application of quotients to constructing fields by adjunction of elements; examples to include $\mathbb{C} = \mathbb{R}[x]/(x^2 + 1)$ and some finite fields. Degree of a field extension, the tower law. [4]

Euclidean Domains. Examples. Principal Ideal Domains. EDs are PIDs. Unique factorisation for PIDs. Gauss's Lemma and Eisenstein's Criterion for irreducibility. [3]

Modules: Definition and examples: vector spaces, abelian groups, vector spaces with an endomorphism. Submodules and quotient modules and direct sums. The first isomorphism theorem. [2]

Row and column operations on matrices over a ring. Equivalence of matrices. Smith Normal form of matrices over a Euclidean Domain. [1.5]

Free modules and presentations of finitely generated modules. Structure of finitely generated modules of a Euclidean domain. [2]

Application to rational canonical form and Jordan normal form for matrices, and structure of finitely generated Abelian groups. [1.5]

6.4 Reading List

1) M. E. Keating, A First Course in Module Theory, Imperial College Press (1998) Covers almost all material of the course. Out of print but many libraries should have it and second hand copies readily available.

2) Joseph Gallian, *Contemporary Abstract Algebra* (9th edition, CENGAGE 2016) (Excellent text covering material on groups, rings and fields).

3) B. Hartley, T. O. Hawkes, Chapman and Hall, *Rings, Modules and Linear Algebra.* (Out of print, but many libraries should have it. Relatively concise and covers all the material in the course).

4) Neils Lauritzen, *Concrete Abstract Algebra*, CUP (2003) (Excellent on groups, rings and fields, and covers topics in the Number Theory course also. Does not cover material on modules).

5) Michael Artin, *Algebra* (2nd ed. Pearson, (2010). (Excellent but highly abstract text covering everything in this course and much more besides).

7 A4: Integration

7.1 Overview

The course will exhibit Lebesgue's theory of integration in which integrals can be assigned to a huge range of functions on the real line, thereby greatly extending the notion of integration presented in Prelims. The theory will be developed in such a way that it can be easily extended to a wider framework, but measures other than Lebesgue's will only be lightly touched.

Operations such as passing limits, infinite sums, or derivatives, through integral signs, or reversing the order of double integrals, are often taken for granted in courses in applied mathematics. Actually, they can occasionally fail. Fortunately, there are powerful convergence and other theorems allowing such operations to be justified under conditions which are widely applicable. The course will display these theorems and a wide range of their applications.

This is a course in rigorous applications. Its principal aim is to develop understanding of the statements of the theorems and how to apply them carefully. Knowledge of technical proofs concerning the construction of Lebesgue measure will not be an essential part of the course, and only outlines will be presented in the lectures.

7.2 Learning Outcomes

By the end of the course, students will be able to - use the definition of a measure space and prove results using this definition, including about key examples of measures - use the definitions of measurable functions and integrable functions and prove results using these definitions - analyse and explain the connection between integrability as defined in this course, and Riemann integrability from Prelims Analysis III - state, prove and apply the Monotone Convergence Theorem, Fatou's Lemma and the Dominated Convergence Theorem - state and apply Fubini's Theorem and Tonelli's Theorem - use the definition of L^p-spaces and prove results about them

7.3 Synopsis

Measure spaces. Outer measure, null set, measurable set. The Cantor set. Lebesgue measure on the real line. Counting measure. Probability measures. Construction of a non-measurable set (non-examinable). Simple function, measurable function, integrable function. Reconciliation with the integral introduced in Prelims.

A simple comparison theorem. Integrability of polynomial and exponential functions over suitable intervals. Monotone Convergence Theorem. Fatou's Lemma. Dominated Convergence Theorem. Corollaries and applications of the Convergence Theorems (including term-by-term integration of series).

Theorems of Fubini and Tonelli (proofs not examinable). Differentiation under the integral sign. Change of variables.

Brief introduction to L^p spaces. Hölder and Minkowski inequalities.

7.4 Reading List

1. M. Capinski & E. Kopp, *Measure, Integral and Probability* (Second Edition, Springer, 2004).
2. E. M. Stein & R. Shakarchi, *Real Analysis: Measure Theory, Integration and Hilbert Spaces* (Princeton Lectures in Analysis III, Princeton University Press, 2005).
3. D.J.H. Garling, *A Course in Mathematical Analysis, III* (CUP, 2014).

7.5 Further Reading

1. D. S. Kurtz & C. W. Swartz, *Theories of Integration* (Series in Real Analysis Vol.9, World Scientific, 2004).
2. H. A. Priestley, *Introduction to Integration* (OUP, 1997). [Useful for worked examples, although adopts a different approach to construction of the integral].
3. H. L. Royden, *Real Analysis* (various editions; 4th edition has P. Fitzpatrick as co author).
4. R. L. Schilling, *Measures, Integrals and Martingales* (CUP first ed. 2005, or second ed. 2017).

8 A5: Topology

8.1 Overview

Topology is the study of 'spatial' objects. Many key topological concepts were introduced in the Metric Spaces course, such as the open subsets of a metric space, and the continuity of a map between metric spaces. More advanced concepts such as connectedness and compactness were also defined and studied. Unlike in a metric space, there is no notion of distance between points in a topological space. Instead, one keeps track only of the open subsets, but this is enough to define continuity, connectedness and compactness. By dispensing with a metric, the fundamentals of proofs are often clarified and placed in a more general setting. In the first part of the course, these topological concepts are introduced and studied. In the second part of the course, simplicial complexes are defined; these are spaces that are obtained by gluing together triangles and their higher-dimensional analogues in a suitable way. This is a very general construction: many spaces admit a homeomorphism to a simplicial complex, which is known as a triangulation of the space. At the end of the course, the proof of one of the earliest and most famous theorems in topology is sketched. This is the classification of compact triangulated surfaces.

8.2 Learning Outcomes

By the end of the course, a student should be able to understand and construct abstract arguments about topological spaces. Their topological intuition should also be sufficiently well-developed to be able to reason about concrete topological spaces such as surfaces.

8.3 Synopsis

Axiomatic definition of an abstract topological space in terms of open sets. Basic definitions: closed sets, continuity, homeomorphism, convergent sequences, connectedness and comparison with the corresponding definitions for metric spaces. Examples to include metric spaces (definition of topological equivalence of metric spaces), discrete and indiscrete topologies, cofinite topology. The Hausdorff condition. Subspace topology. [2 lectures] Accumulation points of sets. Closure of a set. Interior of a set. Continuity if and only if$f\left(\overline{A}\right) \subseteq \overline{f\left(A\right)}$. [2 lectures] Basis of a topology. Product topology on a product of two spaces and continuity of projections. [2 lectures] Compact topological spaces, closed subset of a compact set is compact, compact subset of a Hausdorff space is closed. Product of two compact spaces is compact. A continuous bijection from a compact space to a Hausdorff space is a homeomorphism. Proof that sequential compactness implies compactness in metric spaces. [2 lectures] Quotient topology. Quotient maps. Characterisation of when quotient spaces are Hausdorff in terms of saturated sets. Examples, including the torus, Klein bottle and real projective plane. [2 lectures] Abstract simplicial complexes and their topological realisation. A triangulation of a space. Any compact triangulated surface is homeomorphic to the sphere with g handles ($g \geqslant 0$) or the sphere with h cross-caps ($h \geqslant 1$). (No proof that these surfaces are not homeomorphic, but a brief informal discussion of Euler characteristic.) [6 lectures]

8.4 Reading List

The lectures will follow:

1. J.D. Jackson, *Classical Electrodynamics* (John Wiley, 1962), chapters 1 to 8.

8.5 Further Reading

1. R. Feynman, *Lectures in Physics*, Vol.2. Electromagnetism, Addison Wesley.
2. L.D. Landau and E.M. Lifshitz, *A classical theory of fields* Volume 2.

9 A6: Differential Equations 2

9.1 General Prerequisites

It is recommended to take Integral Transforms in parallel with Differential Equations 2.

9.2 Overview

This course continues the Differential equations 1 course, with the focus on boundary value problems. The course aims to develop a number of techniques for solving boundary value problems and for understanding solution behaviour. The course concludes with an introduction to asymptotic theory and how the presence of a small parameter can affect solution construction and form.

9.3 Learning Outcomes

Students will acquire a range of techniques for solving second order ODE's and boundary value problems. They will gain a familiarity with ideas that are applicable beyond the direct content of the course, such as the Fredholm alternative, Bessel functions, and asymptotic expansions.

9.4 Synopsis

Models leading to two point boundary value problems for second order ODEs

Inhomogeneous two point boundary value problems ($Ly = f$); Wronskian and variation of parameters. Green's functions.

Adjoints. Self-adjoint operators. Eigenfunction expansions (issues of convergence and completeness noted but full treatment deferred to later courses). Sturm-Liouville theory. Fredholm alternative.

Series solutions. Method of Frobenius. Special functions.

Asymptotic sequences. Approximate roots of algebraic equations. Regular perturbations in ODE's. Introduction to boundary layer theory.

9.5 Reading List

K. F. Riley, M. P. Hobson and S. J. Bence, *Mathematical Methods for Physics and Engineering*, (3rd Ed. Cambridge University Press, 2006).

W. E. Boyce & R. C. DiPrima, *Elementary Differential Equations and Boundary Value Problems* (7th edition, Wiley, 2000).

P. J. Collins, *Differential and Integral Equations* (O.U.P., 2006).

Erwin Kreyszig, *Advanced Engineering Mathematics* (8th Edition, Wiley, 1999).

E. J. Hinch, *Perturbation Methods* (Cambridge University Press, Cambridge, 1991).

J. D. Logan, *Applied Mathematics*, (3rd Ed. Wiley Interscience, 2006).

10 A7: Numerical Analysis

10.1 Overview

Scientific computing pervades our lives: modern buildings and structures are designed using it, medical images are reconstructed for doctors using it, the cars and planes we travel on are designed with it, the pricing of "Instruments" in the financial market is done using it, tomorrow's weather is predicted with it. The derivation and study of the core, underpinning algorithms for this vast range of applications defines the subject of Numerical Analysis. This course gives an introduction to that subject. It covers the basics of three key fields in the subject: Numerical Linear Algebra, Approximation Theory, and Numerical Solution of Differential Equations.

Through studying the material of this course students should gain an understanding of numerical methods, their derivation, analysis and applicability. They should be able to solve certain mathematically posed problems using numerical algorithms. This course is designed to introduce numerical methods - i.e. techniques which lead to the (approximate) solution of mathematical problems which are usually implemented on computers. The course covers derivation of useful methods and analysis of their accuracy and applicability.

The course begins with a study of methods and errors associated with approximation of functions which are described by data values (interpolation or data fitting). Following this we turn to numerical methods of linear algebra, which form the basis of a large part of computational mathematics, science, and engineering. Key ideas here include algorithms for linear equations, least squares, and eigenvalues built on LU and QR matrix factorizations. We also treat the singular value decomposition (SVD), a fundamental matrix decomposition of major importance in data science.

We then return to approximation of functions, and discuss key concepts in orthogonal polynomials and best L2 polynomial approximation, and cover the beautiful method of Gauss quadrature for numerical integration.

The final part of this course treats the numerical solution of initial value problems for ordinary differential equations (ODEs), which directly complements the theoretical study of these problems in Part A Differential Equations I. We discuss basic and advanced methods for their numerical solution, the fundamental concept of numerical stability, and culminate with Dahlquist's theorem that consistency and stability imply convergence.

The course requires elementary knowledge of functions, calculus, linear algebra and ordinary differential equations. Although there are no assessed practicals for this course, the tutorial work involves a mix of written work and computational experiments. Knowledge of a programming language such as Matlab or Python would be desirable, but many examples will be provided.

10.2 Learning Outcomes

At the end of the course the student will know how to:

Find the solution of linear systems of equations. Compute eigenvalues and eigenvectors of matrices. Compress matrices vie the SVD. Approximate functions of one variable by polynomials and piecewise polynomials (splines). Compute good approximations to one-dimensional integrals. Numerically solve initial value problems for ODEs. Understand the stability of the numerical methods employed. Use computing to achieve these goals.

10.3 Synopsis

Lagrange interpolation [1 lecture]

Gaussian elimination, LU, QR factorisations, least-squares problems [3.5 lectures]

Eigenvalues: Gershgorin's Theorem, symmetric QR algorithm, polynomial rootfinding via eigenvalues [3.5 lectures]

SVD and low-rank matrix approximation [2 lectures]

Best approximation in inner product spaces, orthogonal polynomials, Gauss quadrature [3 lectures]

Forward and backward Euler, trapezium rule, leapfrog, Runge-Kutta methods [3 lectures]

Linear multi-step methods and Dahlquist's theorem [2 lectures]

10.4 Reading List

The main recommended book for this course are:

1) L. N. Trefethen and D. Bau, Numerical Linear Algebra (SIAM, 1997). (For Numerical Linear Algebra)

2) L. N. Trefethen, Approximation Theory and Approximation Practice (SIAM, 2012; extended edition 2020). (Highly recommended for Function Approximation)

3) E. Süli and D. F. Mayers, An Introduction to Numerical Analysis (CUP, 2003). Of which the relevant chapters are: 6, 7, 2, 5, 9, 11. (For ODEs; covers the subject broadly)

11 A8: Probability

11.1 Overview

The first half of the course takes further the probability theory that was developed in the first year. The aim is to build up a range of techniques that will be useful in dealing with mathematical models involving uncertainty. The second half of the course is concerned with Markov chains in discrete time and Poisson processes in one dimension, both with developing the relevant theory and giving examples of applications.

11.2 Synopsis

Continuous random variables. Jointly continuous random variables, independence, conditioning, functions of one or more random variables, change of variables. Examples including some with later applications in statistics.

Moment generating functions and applications. Statements of the continuity and uniqueness theorems for moment generating functions. Characteristic functions (definition only). Convergence in distribution and convergence in probability. Weak law of large numbers and central limit theorem for independent identically distributed random variables. Strong law of large numbers (proof not examinable).

Discrete-time Markov chains: definition, transition matrix, n-step transition probabilities, communicating classes, absorption, irreducibility, periodicity, calculation of hitting probabilities and mean hitting times. Recurrence and transience. Invariant distributions, mean return time, positive recurrence, convergence to equilibrium (proof not examinable), ergodic theorem (proof not examinable). Reversibility, detailed balance equations. Random walks (including symmetric and asymmetric random walks on Z, and symmetric random walks on Z^d).

11.3 Reading List

G. R. Grimmett and D. R. Stirzaker, *Probability and Random Processes* (3rd edition, OUP, 2001). Chapters 4, 6.1-6.5, 6.8.

G. R. Grimmett and D. R. Stirzaker, *One Thousand Exercises in Probability* (OUP, 2001).

G. R. Grimmett and D J A Welsh, *Probability: An Introduction* (OUP, 1986). Chapters 6, 7.4, 8, 11.1-11.3.

J. R. Norris, *Markov Chains* (CUP, 1997). Chapter 1.

D. R. Stirzaker, *Elementary Probability* (Second edition, CUP, 2003). Chapters 7-9 excluding 9.9.

12 A9: Statistics

12.1 General Prerequisites

Part A Probability is recommended for this course, but is not essential. If you are not doing Part A Probability then you should make sure that you are familiar with Prelims work on Probability, and you may also need to familiarise yourself with a couple of lectures' worth of material from Part A Probability. Should you be interested in taking courses involving statistics in Parts B or C, then it would be strongly advisable to take Part A Probability.

12.2 Overview

Building on the first year course, this course develops statistics for mathematicians, emphasising both its underlying mathematical structure and its application to the logical interpretation of scientific data. Advances in theoretical statistics are generally driven by the need to analyse new and interesting data which come from all walks of life.

12.3 Learning Outcomes

At the end of the course students should have an understanding of: the use of probability plots to investigate plausible probability models for a set of data; maximum likelihood estimation and large sample properties of maximum likelihood estimators; hypothesis tests and confidence intervals (and the relationship between them). They should have a corresponding understanding of similar concepts in Bayesian inference.

12.4 Synopsis

Order statistics, probability plots.

Estimation: observed and expected information, statement of large sample properties of maximum likelihood estimators in the regular case, methods for calculating maximum likelihood estimates, large sample distribution of sample estimators using the delta method.

Hypothesis testing: simple and composite hypotheses, size, power and p-values, Neyman-Pearson lemma, distribution theory for testing means and variances in the normal model, generalized likelihood ratio, statement of its large sample distribution under the null hypothesis, analysis of count data.

Confidence intervals: exact intervals, approximate intervals using large sample theory, relationship to hypothesis testing.

Probability and Bayesian Inference. Posterior and prior probability densities. Constructing priors including conjugate priors, subjective priors, Jeffreys priors. Bayes estimators and credible intervals. Statement of asymptotic normality of the posterior. Model choice via posterior probabilities and Bayes factors.

Examples: statistical techniques will be illustrated with relevant datasets in the lectures.

12.5 Reading List

F. Daly, D.J. Hand, M.C. Jones, A.D. Lunn and K.J. McConway, *Elements of Statistics* (Addison Wesley, 1995) Chapters 7-10 (and Chapters 1-6 for background).

J. A. Rice, *Mathematical Statistics and Data Analysis* (2nd edition, Wadsworth, 1995) Sections 8.5, 8.6, 9.1-9.7, 9.9, 10.3-10.6, 11.2, 11.3, 12.2.1, 13.3, 13.4.

T Leonard and J.S.J. Hsu, *Bayesian Methods* (CUP, 1999), Chapters 2 and 3.

12.6 Further Reading

G. Casella and R. L. Berger, *Statistical Inference* (2nd edition, Wadsworth, 2001).

A. C. Davison, *Statistical Models* (Cambridge University Press, 2003), Chapter 11.

13 A10: Fluids and Waves

13.1 Overview

This course introduces students to the mathematical theory of inviscid fluids. The theory provides insight into physical phenomena such as flight, vortex motion, and water waves. The course also explains important concepts such as conservation laws and dispersive waves and, thus, serves as an introduction to the mathematical modelling of continuous media.

13.2 Synopsis

Incompressible flow. Convective derivative, streamlines and particle paths. Euler's equations of motion for an inviscid fluid. Bernoulli's Theorem. Vorticity, circulation and Kelvin's Theorem. The vorticity equation and vortex motion.

Irrotational incompressible flow; velocity potential. Two-dimensional flow, stream function and complex potential. Line sources and vortices. Method of images, circle theorem and Blasius's Theorem.

Uniform flow past a circular cylinder. Circulation, lift. Use of conformal mapping to determine flow past a flat wing. Water waves, including effects of finite depth and surface tension. Dispersion, simple introduction to group velocity.

13.3 Reading List

D. J. Acheson, *Elementary Fluid Dynamics* (OUP, 1997). Chapters 1, 3.1-3.5, 4.1-4.8, 4.10-4.12, 5.1, 5.2, 5.6, 5.7.

R. P. Feynman, R.B. Leighton, M. Sands, *The Feynman Lectures on Physics*, volume II, (Addison Wesley 1964) Chapter 40 (`http://www.feynmanlectures.caltech.edu/II_40.html`)

M. van Dyke, *An Album of Fluid Motion* (Parabolic Press, 1982).

14 A11: Quantum Theory

14.1 Overview

Quantum theory was born out of the attempt to understand the interactions between radiation, described by Maxwell's theory of electromagnetism, and matter, described by Newton's mechanics.

Although there remain deep mathematical and physical questions at the frontiers of the subject, the resulting theory encompasses not just the mechanical but also the electrical and chemical properties of matter. Many of the key components of modern technology such as transistors and lasers were developed using quantum theory.

In quantum theory particles also have some wave-like properties. This introductory course explores some of the consequences of this culminating in a treatment of the hydrogen atom.

14.2 Learning Outcomes

By the end of this course, students will be able to solve the Schroedinger equation in a range of simple situations and understand its significance. In particular they will be able to calculate the energy levels of hydrogen-like atoms. They will also learn the abstract, algebraic formulation of quantum mechanics which complements and sometimes replaces the solution of the Schroedinger equation.

14.3 Synopsis

Wave-particle duality; Schrödinger's equation; stationary states; quantum states of a particle in a box (infinite squarewell potential).

Interpretation of the wave function; boundary conditions; probability density and conservation of current; parity.

The one-dimensional harmonic oscillator; higher-dimensional oscillators and normal modes; degeneracy. The rotationally symmetric states of the hydrogen atom with fixed nucleus.

The mathematical structure of quantum mechanics and the postulates of quantum mechanics.

Commutation relations. Heisenberg's uncertainty principle.

Creation and annihilation operators for the harmonic oscillator. Measurements and the collapse of the wave function.

Schrödinger's cat. Angular momentum in quantum mechanics. The particular case of spin-1/2. Particle in a central potential. General states of the hydrogen atom. Quantum key distribution.

14.4 Reading List

B.H. Bransden and C.J Joachain *Quantum Mechanics* (Second edition, Pearson Education Limited, 2000). Chapters 1-4.

P.C.W. Davies and D.S. Betts, *Quantum Mechanics (Physics and its Applications)* (2nd edition, Taylor & Francis Ltd, 1994). Chapters 1,2,4.

R.P Feynman, R.B Leighton, M. Sands *The Feynman Lectures on Physics, Volume 3*, (Addison-Wesley, 1998). Chapters 1,2 (for physical background).

K.C Hannabuss, *An Introduction to Quantum Theory* (Oxford University Press 1997). Chapters 1-4.

A.I.M. Rae, *Quantum Mechanics* (4th Edition, Taylor & Francis Ltd, 2002). Chapters 1-3.

15 ASO: Number Theory

15.1 Overview

Number theory is one of the oldest parts of mathematics. For well over two thousand years it has attracted professional and amateur mathematicians alike. Although notoriously 'pure' it has turned out to have more and more applications as new subjects and new technologies have developed. Our aim in this course is to introduce students to some classical and important basic ideas of the subject.

15.2 Learning Outcomes

Students will learn some of the foundational results in the theory of numbers due to mathematicians such as Fermat, Euler and Gauss. They will also study a modern application of this ancient part of mathematics.

15.3 Synopsis

The ring of integers; congruences; ring of integers modulo n; the Chinese Remainder Theorem.

Wilson's Theorem; Fermat's Little Theorem for prime modulus; Euler's phi-function. Euler's generalisation of Fermat's Little Theorem to arbitrary modulus; primitive roots.

Quadratic residues modulo primes. Quadratic reciprocity.

Factorisation of large integers; basic version of the RSA encryption method.

15.4 Reading List

1) H. Davenport, *The Higher Arithmetic* (Cambridge University Press, 1992) ISBN 0521422272 2) G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers* (OUP, 1980) ISBN 0198531710 3) P. Erdős and J. Surányi, *Topics in the Theory of Numbers* (Springer, 2003) ISBN 0387953205

All these books contain some elementary material but go way beyond what is in the course. Full printed notes will be provided.

16 ASO: Group Theory

16.1 Overview

This group theory course develops the theory begun in prelims, and this course will build on that. After recalling basic concepts, the focus will be on two circles of problems.

1. The concept of free group and its universal property allow to define and describe groups in terms of generators and relations.

2. The notion of composition series and the Jordan-Hölder Theorem explain how to see, for instance, finite groups as being put together from finitely many simple groups. This leads to the problem of finding and classifying finite simple groups. Conversely, it will be explained how to put together two given groups to get new ones.

Moreover, the concept of symmetry will be formulated in terms of group actions and applied to prove some group theoretic statements.

16.2 Learning Outcomes

Students will learn to construct and describe groups. They will learn basic properties of groups and get familiar with important classes of groups. They will understand the crucial concept of simple groups. They will get a better understanding of the notion of symmetry by using group actions.

16.3 Synopsis

Free groups. Uniqueness of reduced words and universal mapping property. Normal subgroups of free groups and generators and relations for groups. Examples. [2]

Review of the First Isomorphism Theorem and proof of Second and Third Isomorphism Theorems. Simple groups, statement that A_n is simple (proof for $n = 5$). Definition and proof of existence of composition series for finite groups. Statement of the Jordan-Hölder Theorem. Examples. The derived subgroup and solvable groups. [3]

Discussion of semi-direct products and extensions of groups. Examples. [1]

Sylow's three theorems. Applications including classification of groups of small order. [2]

16.4 Reading List

1) Humphreys, J. F. *A Course in Group Theory*, Oxford, 1996

2) Armstrong, M. A. *Groups and Symmetry*, Springer-Verlag, 1988

17 ASO: Projective Geometry

17.1 Overview

Projective Geometry might be viewed as the geometry of perspective. Two observers of a painting – one looking obliquely, one straight on – will not agree on angles and distances but will both sees lines as lines and will agree on whether they meet. So projective transformations (such as relate the two observers' views) are less rigid than Euclidean, or even affine, transformations. Projective geometry also introduces the idea of points at infinity – points where parallel lines meet. These points fill in the missing gaps/address some special cases of geometry in a similar way to which complex numbers resolve such problems in algebra. From this point of view ellipses, parabolae and hyperbolae are all projectively equivalent and just happen to include no, one or two points at infinity. The study of such conics also has applications to the study of quadratic Diophantine equations.

17.2 Learning Outcomes

Students will be familiar with the idea of projective space and the linear geometry associated to it, including examples of duality and applications to Diophantine equations.

17.3 Synopsis

1-2: Projective Spaces (as $P(V)$ of a vector space V). Homogeneous Co-ordinates. Linear Subspaces.

3-4: Projective Transformations. General Position. Desargues Theorem. Cross-ratio.

5: Dual Spaces. Duality.

6-7: Symmetric Bilinear Forms. Conics. Singular conics, singular points. Projective equivalence of non-singular conics.

7-8: Correspondence between P^1 and a non-singular conic. Simple applications to Diophantine Equations.

17.4 Reading List

1) N.J. Hitchin, *Maths Institute notes on Projective Geometry* (found under 'Teaching')

2) M. Reid and B. Szendrői, *Geometry and topology*, Cambridge University Press, 2005 (Chapter 5).

3) R. Casse, Projective Geometry, An Introduction, Oxford University Press (2006)

18 ASO: Multidimensional Analysis and Geometry

18.1 Overview

In this course, the notion of the total derivative for a function $f\colon \mathbb{R}^m \to \mathbb{R}^n$ is introduced. Roughly speaking, this is an approximation of the function near each point in $\mathbb{R}^n$ by a linear transformation. This is a key concept which pervades much of mathematics, both pure and applied. It allows us to transfer results from linear theory locally to nonlinear functions. For example, the Inverse Function Theorem tells us that if the derivative is an invertible linear mapping at a point then the function is invertible in a neighbourhood of this point. Another example is the tangent space at a point of a surface in $\mathbb{R}^3$, which is the plane that locally approximates the surface best.

18.2 Learning Outcomes

Students will understand the concept of derivative in n dimensions and the implict and inverse function theorems which give a bridge between suitably nondegenerate infinitesimal information about mappings and local information. They will understand the concept of manifold and see some examples such as matrix groups.

18.3 Synopsis

Definition of a derivative of a function from $\mathbb{R}^m$ to $\mathbb{R}^n$; examples; elementary properties; partial derivatives; the chain rule; the gradient of a function from $\mathbb{R}^n$ to $\mathbb{R}$; Jacobian. Continuous partial derivatives imply differentiability, Mean Value Theorems. [3 lectures]

The Inverse Function Theorem and the Implicit Function Theorem (proofs non-examinable). [2 lectures]

The definition of a submanifold of $\mathbb{R}^m$. Its tangent and normal space at a point, examples, including two-dimensional surfaces in $\mathbb{R}^3$. [2 lectures]

Lagrange multipliers. [1 lecture]

18.4 Reading List

Theodore Shifrin, *Multivariable Mathematics* (Wiley, 2005). Chapters 3-6.

T. M. Apostol, *Mathematical Analysis: Modern Approach to Advanced Calculus (World Students)* (Addison Wesley, 1975). Chapters 6 and 7.

S. Dineen, *Multivariate Calculus and Geometry* (Springer, 2001). Chapters 1-4.

J. J. Duistermaat and J A C Kolk, *Multidimensional Real Analysis I, Differentiation* (Cambridge University Press, 2004).

M. Spivak, *Calculus on Manifolds: A modern approach to classical theorems of advanced calculus*, W. A. Benjamin, Inc., New York-Amsterdam, 1965.

18.5 Further Reading

William R. Wade, *An Introduction to Analysis* (Second Edition, Prentice Hall, 2000). Chapter 11.

M. P. Do Carmo, *Differential Geometry of Curves and Surfaces* (Prentice Hall, 1976).

Stephen G. Krantz and Harold R. Parks, *The Implicit Function Theorem: History, Theory and Applications* (Birkhaeuser, 2002).

19 ASO: Integral Transforms

19.1 General Prerequisites

This course is highly recommended for Differential Equations 2.

19.2 Overview

The Laplace and Fourier Transforms aim to take a differential equation in a function and turn it into an algebraic equation involving its transform. Such an equation can then be solved by algebraic manipulation, and the original solution determined by recognizing its transform or applying various inversion methods.

The Dirac *delta*-function, which is handled particularly well by transforms, is a means of rigorously dealing with ideas such as instantaneous impulse and point masses, which cannot be properly modelled using functions in the normal sense of the word. δ is an example of a *distribution* or *generalized function* and the course provides something of an introduction to these generalized functions and their calculus.

19.3 Learning Outcomes

Students will gain a range of techniques employing the Laplace and Fourier Transforms in the solution of ordinary and partial differential equations. They will also have an appreciation of generalized functions, their calculus and applications.

19.4 Synopsis

Motivation for a "function" with the properties the Dirac *delta*-function. Test functions. Continuous functions are determined by $\phi \to \int f\phi$. Distributions and δ as a distribution. Differentiating distributions. (3 lectures)

Theory of Fourier and Laplace transforms, inversion, convolution. Inversion of some standard Fourier and Laplace transforms via contour integration.

Use of Fourier and Laplace transforms in solving ordinary differential equations, with some examples including δ.

Use of Fourier and Laplace transforms in solving partial differential equations; in particular, use of Fourier transform in solving Laplace's equation and the Heat equation. (5 lectures)

19.5 Reading List

S. Howison, *Practical Applied Mathematics* (CUP 2005), Chapters 9 & 10 (for distributions).

P. J. Collins, *Differential and Integral Equations* (OUP, 2006), Chapter 14.

W. E. Boyce & R. C. DiPrima, *Elementary Differential Equations and Boundary Value Problems*, there are many editions, most recently in 2017; in all of them Chapter 6 covers Laplace Transforms.

K. F. Riley & M. P. Hobson, *Essential Mathematical Methods for the Physical Sciences* (CUP 2011) Chapter 5.

H. A. Priestley, *Introduction to Complex Analysis* (2nd edition, OUP, 2003) Chapters 21 and 22.

19.6 Further Reading

L. Debnath & P. Mikusinski, *Introduction to Hilbert Spaces with Applications*, (3rd Edition, Academic Press. 2005) Chapter 6

20 ASO: Calculus of Variations

20.1 Overview

The calculus of variations concerns problems in which one wishes to find the minima or extrema of some quantity over a system that has functional degrees of freedom. Many important problems arise in this way across pure and applied mathematics and physics. They range from the problem in geometry of finding the shape of a soap bubble, a surface that minimizes its surface area, to finding the configuration of a piece of elastic that minimises its energy. Perhaps most importantly, the principle of least action is now the standard way to formulate the laws of mechanics and basic physics.

In this course it is shown that such variational problems give rise to a system of differential equations, the Euler-Lagrange equations. Furthermore, the minimizing principle that underlies these equations leads to direct methods for analysing the solutions to these equations. These methods have far reaching applications and will help develop students technique.

20.2 Learning Outcomes

Students will be able to formulate variational problems and analyse them to deduce key properties of system behaviour.

20.3 Synopsis

The basic variational problem and Euler's equation. Examples, including axi-symmetric soap films.

Extension to several dependent variables. Hamilton's principle for free particles and particles subject to holonomic constraints. Equivalence with Newton's second law. Geodesics on surfaces. Extension to several independent variables.

Examples including Laplace's equation. Lagrange multipliers and variations subject to constraint. Eigenvalue problems for Sturm-Liouville equations. Legendre Polynomials.

20.4 Reading List

Arfken Weber, *Mathematical Methods for Physicists* (5th edition, Academic Press, 2005). Chapter 17.

20.5 Further Reading

N. M. J. Woodhouse, *Introduction to Analytical Dynamics* (1987). Chapter 2 (in particular 2.6). (This is out of print, but still available in most College libraries.)

M. Lunn, *A First Course in Mechanics* (OUP, 1991). Chapters 8.1, 8.2.

P. J. Collins, *Differential and Integral Equations* (O.U.P., 2006). Chapters 11, 12.

21 ASO: Graph Theory

21.1 Overview

This course introduces some central topics in graph theory.

21.2 Learning Outcomes

By the end of the course, students should have an appreciation of the methods and results of graph theory. They should have a good understanding of the basic objects in graph theory, such as trees, Euler circuits and matchings, and they should be able to reason effectively about graphs.

21.3 Synopsis

Introduction. Paths, walks, cycles and trees. Euler circuits. Hamiltonian cycles. Hall's theorem. Application and analysis of algorithms for minimum cost spanning trees, shortest paths, bipartite matching and the Chinese Postman Problem.

21.4 Reading List

R. J. Wilson, *Introduction to Graph Theory*, 5th edition, Prentice Hall, 2010.

D.B. West, *Introduction to Graph Theory*, 2nd edition, Prentice Hall, 2001.

22 ASO: Special Relativity

22.1 Overview

The unification of space and time into a four-dimensional space-time is essential to the modern understanding of physics. This course will build on first-year algebra, geometry, and applied mathematics to show how this unification is achieved. The results will be illustrated throughout by reference to the observed physical properties of light and elementary particles.

22.2 Learning Outcomes

Students will be able to describe the fundamental phenomena of relativistic physics within the algebraic formalism of four-vectors. They will be able to solve simple problems involving Lorentz transformations. They will acquire a basic understanding of how the four-dimensional picture completes and supersedes the physical theories studied in first-year work.

22.3 Synopsis

Constancy of the speed of light. Lorentz transformations; time dilation, length contraction, the relativistic Doppler effect.

Index notation, four-vectors, four-velocity and four-momentum; equivalence of mass and energy; particle collisions and four-momentum conservation; equivalence of mass and energy: $E = mc^2$; four-acceleration and four-force, the example of the constant-acceleration world-line.

22.4 Reading List

N. M. J. Woodhouse, *Special Relativity*, (Springer, 2002).

23 ASO: Mathematical Modelling in Biology

23.1 Overview

Mathematical Modelling in Biology introduces the applied mathematician to practical applications in an area that is growing very rapidly. The course focuses on how to model various processes in ecology, epidemiology, chemistry, biology and medicine, using ordinary differential equation models, as well as an introduction to discrete models. It demonstrates how mathematical techniques such as linear stability analysis, phase planes, perturbation and asymptotic analysis can enable us to predict the behaviour of living systems.

23.2 Learning Outcomes

Students will have developed a sound knowledge and appreciation of the ideas and concepts related to modelling biological and ecological systems using continuous-time non-spatial models.

23.3 Synopsis

Continuous population models for a single species including hysteresis and harvesting.

Discrete time models for single species – linear stability analysis and cobwebbing.

Modelling interacting populations, including predator-prey and the principle of competitive exclusion.

Enzyme-substrate kinetics, the quasi-steady state approximation and perturbation analysis.

Modelling of neuronal signalling using the Hodgkin-Huxley model and excitable kinetics.

Infectious disease modelling including SIR models.

23.4 Reading List

J. D. Murray, *Mathematical Biology*, Volume I: An Introduction. 3rd Edition, Springer (2002).

23.5 Further Reading

N. F. Britton, *Essential Mathematical Biology.* Springer (2003).

G. de Vries, T. Hillen, M. Lewis, J. Müller, B. Schönfisch. *A Course in Mathematical Biology: Quantitative Modelling with Mathematical and Computational Methods.* SIAM (2006).

24 An Introduction to LaTeX

24.1 General Prerequisites

There are no prerequisites. The course is mainly intended for students writing a Part B Extended Essay or a Part C Dissertation but any students are welcome to attend the two lectures given in Michaelmas Term. Note that there is no assessment associated with this course, nor credit for attending the course.

24.2 Overview

This short lecture series provides an introduction to LaTeX.

LaTeX is a markup language, released by Donald Knuth in 1984 and freely sourced, for the professional typesetting of mathematics. (It is based on the earlier TeX released in

1978.) A markup language provides the means for rendering text in various ways - such as bold, italicized or Greek symbols - with the main focus of LaTeX being the rendering of mathematics so that even complicated expressions involving equations, integrals and matrices and images can be professionally typeset.

24.3 Learning Outcomes

Following these introductory lectures, a student should feel comfortable writing their own LaTeX documents, and producing professionally typset mathematics. The learning curve to producing a valid LaTeX document is shallow, and students will further become familiar will some of the principal features of LaTeX such as chapters, item lists, typesetting mathematics, including equations, tables, bibliographies and images. Then, with the aid of a good reference manual, a student should feel comfortable researching out for themselves further features and expanding their LaTeX vocabulary

24.4 Reading List

The Department has a page of LaTeX resources at `https://www.maths.ox.ac.uk/members/it/faqs/latex` which has various free introductory guides to LaTeX.