Part A Mathematics & Philosophy 2024-25

September 25, 2024

Contents

1 Foreword

The confirmed synopses for Part A 2024-25 will be available on the course management portal `https://courses.maths.ox.ac.uk/` before the start of Michaelmas Term 2024.

Three or Four Year Course

All students who complete Parts A and B will be classified. Those who have achieved honours and who wish to graduate at this point may supplicate for a BA.

Students wishing to take the four-year course should register to do so during their third year, and will be permitted to do so on the basis of an upper second class performance, or better, in the third year classification. They will take Part C in their fourth year, be awarded a separate classification and, if successful, may supplicate for an MMathPhil.

2 Syllabus

The syllabus details below are those referred to in the *Examination Regulations* and have been approved by the Mathematics Teaching Committee for examination in Trinity Term 2025. Please see the current edition of the *Examination Regulations* (`https://www.admin.ox.ac.uk/examregs/`) for the full regulations governing these examinations. Examination Conventions can be found at: `https://www.maths.ox.ac.uk/members/students/undergraduate-courses/examinations-assessments/examination-conventions`.

The Part A examination **syllabus** is the mathematical material of the synopses, as separately detailed by paper below. The course synopses on the course webpages also give additional detail to the syllabus (for example, showing how the material is split by lectures) and are also accompanied by lists of recommended reading.

Honour School of Mathematics and Philosophy - Part A In Part A each candidate shall be required to offer four written papers in Mathematics from the schedule of papers for Part A (given below).

These papers must include:

- A0 - Linear Algebra (1.5 hours)
- A2 - Metric Spaces and Complex Analysis (3 hours)

and two of the following papers (each 1.5 hours long).

- A3 Rings and Modules
- A4 Integration
- A5 Topology
- A8 Probability
- ASO Short Options

Paper ASO will examine the seven Short Options (Number Theory, Group Theory, Projective Geometry, Integral Transforms, Calculus of Variations, Graph Theory, and Mathematical Modelling in Biology). Students taking ASO are recommended to take three of these Short Options.

Part A shall be taken on one occasion only (there will be no resits). At the end of the Part A examination a candidate will be awarded a 'University Standardised Mark' (USM) for each Mathematics paper in Part A. A weighted average of these USMs will be carried forward into the classification awarded at the end of the third year. In the calculation of any averages used to arrive at the final classification, USMs for A2 will have twice the weight of the USMs awarded for A0, the Long Options and Paper ASO.

Core Material

The examination syllabi of the two core papers A0, A2.1 and A2.2 shall be the mathematical content of the synopses for the courses

- Linear Algebra,
- Metric Spaces
- Complex Analysis.

Options

The examination syllabi of the four papers of options, A3, A4, A5 and A8, shall be the mathematical content of the synopses for the courses

- Rings and Modules,
- Integration,
- Topology,
- Probability,

and such other options from Mathematics Part A as are approved by the Joint Committee of Mathematics and Philosophy.

Short Options

The examination syllabi of the short options paper ASO shall be the mathematical content of the synopses for the courses

- Number Theory,
- Group Theory,
- Projective Geometry,
- Integral Transforms,
- Calculus of Variations,
- Graph Theory,
- Mathematical Modelling in Biology.

Candidates may also, with the support of their college tutors, apply to the Joint Committee for Mathematics and Philosophy for approval of other Optional Subjects as listed for Part A of the Honour School of Mathematics.

Procedure for seeking approval of other options where this is required

You may, if you have the support of your Mathematics tutor, apply to the Chair of the Joint Committee for Mathematics and Philosophy for approval of one or more other options from the list of Mathematics Department options for Part A. This list can be found in the schedule of units for Part A Mathematics 2024-25.

Applications for special approval must be made through the candidate's college and sent to the Chair of the Joint Committee for Mathematics and Philosophy, c/o Haleigh Bellamy, Mathematical Institute, to arrive by **Monday of Week 2 of Hilary Term**. Be sure to consult your college tutors if you are considering asking for approval to offer one of these other options.

Given that each of these other options, which are all in applied mathematics, presume facility with some or other results and techniques covered in first or second year core Mathematics courses not taken by Mathematics & Philosophy candidates, such applications will be exceptional.

3 A0: Linear Algebra

3.1 Overview

The core of linear algebra comprises the theory of linear equations in many variables, the theory of matrices and determinants, and the theory of vector spaces and linear maps. All these topics were introduced in the Prelims course. Here they are developed further to provide the tools for applications in geometry, modern mechanics and theoretical physics, probability and statistics, functional analysis and, of course, algebra and number theory. Our aim is to provide a thorough treatment of some classical theory that describes the behaviour of linear maps on a finite-dimensional vector space to itself, both in the purely algebraic setting and in the situation where the vector space carries a metric derived from an inner product.

3.2 Learning Outcomes

Students will deepen their understanding of Linear Algebra. They will be able to define and obtain the minimal and characteristic polynomials of a linear map on a finite-dimensional vector space, and will understand and be able to prove the relationship between them; they will be able to prove and apply the Primary Decomposition Theorem, and the criterion for diagonalisability. They will have a good knowledge of inner product spaces, and be able to apply the Bessel and Cauchy–Schwarz inequalities; will be able to define and use the adjoint of a linear map on a finite-dimensional inner product space, and be able to prove and exploit the diagonalisability of a self-adjoint map and prove and use the singular value decomposition.

3.3 Synopsis

Definition of an abstract vector space over an arbitrary field. Examples. Linear maps. [1]

Definition of a ring. Examples to include $\mathbb{Z}$, $F[x]$, $F[A]$ (where A is a matrix or linear map), $\mathrm{End}(V)$. Division algorithm and Bézout Lemma in $F[x]$. Ring homomorphisms and isomorphisms. Examples. [2]

Characteristic polynomials and minimal polynomials. Coincidence of roots. [1]

Quotient vector spaces. The first isomorphism theorem for vector spaces and rank-nullity. Induced linear maps. Applications: Triangular form for matrices over $\mathbb{C}$. Cayley-Hamilton Theorem. [2]

Primary Decomposition Theorem. Diagonalizability and Triangularizability in terms of minimal polynomials. Proof of existence of Jordan canonical form over $\mathbb{C}$ (using primary decomposition and inductive proof of form for nilpotent linear maps). [3]

Dual spaces of finite-dimensional vector spaces. Dual bases. Dual of a linear map and description of matrix with respect to dual basis. Natural isomorphism between a finite-dimensional vector space and its second dual. Annihilators of subspaces, dimension formula. Isomorphism between U^0 and $(V/U)'$. [3]

Recap on real inner product spaces. Definition of non-degenerate symmetric bilinear forms and description as isomorphism between V and V'. Hermitian forms on complex vector spaces. Review of Gram-Schmidt. Orthogonal Complements. [1]

Adjoints for linear maps of inner product spaces. Uniqueness. Concrete construction via matrices [1]

Definition of orthogonal/unitary maps. Definition of the groups O_n, SO_n, U_n, SU_n. Diagonalizability of self-adjoint and unitary maps. Singular value decomposition. [2]

3.4 Reading List

1) Richard Kaye and Robert Wilson, *Linear Algebra* (OUP, 1998) ISBN 0-19-850237-0. Chapters 2–13. [Chapters 6, 7 are not entirely relevant to our syllabus, but are interesting.]

3.5 Further Reading

1) Paul R. Halmos, *Finite-dimensional Vector Spaces*, (Springer Verlag, Reprint 1993 of the 1956 second edition), ISBN 3-540-90093-4. sections 1–15, 18, 32–51, 54–56, 59–67, 73, 74, 79. [Now over 50 years old, this idiosyncratic book is somewhat dated but it is a great classic, and well worth reading.]

2) Seymour Lipschutz and Marc Lipson, *Schaum's Outline of Linear Algebra* (3rd edition, McGraw Hill, 2000), ISBN 0-07-136200-2. [Many worked examples.]

3) C. W. Curtis, *Linear Algebra - an Introductory Approach*, (4th edition, Springer, reprinted 1994).

4) D. T. Finkbeiner, *Elements of Linear Algebra* (Freeman, 1972). [Out of print, but available in many libraries.]

4 A2.1 Metric Spaces

4.1 Overview

The theory of metric spaces is foundational to much of pure and applied mathematics, for it axiomatises the notion of a sensible distance defined on a set and builds a rich theory surrounding this. The development of this theory forms the central part of the course. Before and after this though, we consider two related topics of a slightly different nature. We begin the course by extending the idea of differentiability from one to several real variables. This is somewhat more subtle than one might expect, and of central importance in real analysis. And we end the course by studying some aspects of the geometry of the complex plane. Some of the topics considered will be touched on again in the companion course Complex Analysis, which runs in parallel to this one.

4.2 Learning Outcomes

Students will have become familiar with the central concepts and results in metric spaces, and thus some of the basic ideas of point-set topology. They will have a deeper understanding of real differentiation, by developing it in higher dimensions, and will have greater appreciation of the beautiful geometry of the complex plane.

4.3 Synopsis

Real differentiation in $\mathbf{R}^2$. Continuous partial derivatives imply differentiability and differentiability implies continuity. Some topology on $\mathbf{R}^2$. Continuous functions on compact sets achieve their maximum. Different examples of distance. The statement (no proof) of the inverse function theorem on $\mathbf{R}^2$. [4]

The definition of a metric space. Examples of metric spaces in various parts of mathematics. Norms, and metrics derived from a norm on a real vector space, particularly $\ell^1, \ell^2, \ell^\infty$-norms on $\mathbf{R}^n$. Metrics on product spaces. Open and closed balls, and bounded sets. Limits, continuity, and uniform continuity (ε-δ definition). Function spaces, of continuous real-valued functions and of bounded functions on a metric space. Isometries and homeomorphisms. Open and closed sets and their basic properties. Continuity in terms of pre-images of open and closed sets. A subset of a metric space inherits a metric. Interiors, closures and limit points. [3]

Completeness (but not completion). Completeness of the space of bounded real-valued functions on a set, equipped with the supremum norm, and the completeness of the space of bounded continuous real-valued functions on a metric space. Lipschitz maps and contractions. Contraction Mapping Theorem. [2.5]

Connected metric spaces, path-connectedness. Closure of a connected space is connected, union of connected sets is connected if there is a non-empty intersection, continuous image of a connected space is connected. Path-connectedness implies connectedness. Connected open subset of a normed vector space is path-connected. [2]

Definition of sequential compactness and proof of basic properties of sequentially compact sets. Preservation of sequential compactness under continuous maps, equivalence of continuity and uniform continuity for functions on a sequentially compact set. Equivalence of sequential compactness with being complete and totally bounded. Open cover definition of compactness. Statement of Heine-Borel for $\mathbf{R}^n$ and proof that compactness implies sequential compactness (statement of the converse only). [2.5]

The Riemann sphere and the isometry with the extended plane via stereographic projection. Möbius transformations: definition and examples. Conformal maps. [2]

4.4 Reading List

- W. A. Sutherland, *Introduction to Metric and Topological Spaces* (Second Edition, OUP, 2009).

5 A2.2 Complex Analysis

5.1 Overview

The theory of functions of a complex variable is a rewarding branch of mathematics to study at the undergraduate level with a good balance between general theory and examples. It occupies a central position in mathematics with links to analysis, algebra, number theory, potential theory, geometry, and topology, and generates a number of powerful techniques (for example, evaluation of integrals) with applications in many aspects of both pure and applied mathematics and other disciplines, particularly the physical sciences.

In these lectures, we begin by introducing students to the theory of (holomorphic) functions of a complex variable. The central aim of the lectures is to present Cauchy's Theorem and its consequences, particularly series expansions of holomorphic functions, the calculus of residues and its applications.

5.2 Learning Outcomes

Students will have become familiar with the main concepts of Complex Analysis. They will have grasped a deeper understanding of differentiation and integration in this setting and will know the tools and results of Complex Analysis including Cauchy's Theorem, Cauchy's integral formula, Liouville's Theorem, Laurent's expansion and the theory of residues.

5.3 Synopsis

Complex differentiation. Holomorphic functions. Cauchy-Riemann equations (different versions). Real and imaginary parts of a holomorphic function are harmonic.

Recap on power series and differentiation of power series. Exponential function and logarithm function. Fractional powers — examples of multifunctions. The use of cuts as a method of defining a branch of a multifunction.

Path integration. Winding numbers. Cauchy's Theorem (partial proof only). Homology form of Cauchy's Theorem (sketch of proof only — students referred to various texts for proof.) Fundamental Theorem of Calculus in the path integral/holomorphic situation.

Cauchy's Integral formulae. Taylor expansion. Liouville's Theorem. Morera's Theorem. Identity Theorem.

Laurent's expansion. Classification of isolated singularities. Calculation of principal parts, particularly residues. The argument principle and applications.

Residue Theorem. Evaluation of integrals by the method of residues (main examples only but to include the use of Jordan's Lemma and simple poles on the contour of integration).

5.4 Reading List

- H. A. Priestley, *Introduction to Complex Analysis* (Second edition, OUP, 2003).

5.5 Further Reading

- L. Ahlfors, *Complex Analysis* (McGraw-Hill, 1979).
- Reinhold Remmert, *Theory of Complex Functions* (Springer, 1989) (Graduate Texts in Mathematics 122).

6 A3: Rings and Modules

6.1 Overview

The first abstract algebraic objects which are normally studied are groups, which arise naturally from the study of symmetries. The focus of this course is on rings, which generalise the kind of algebraic structure possessed by the integers: a ring has two operations, addition and multiplication, which interact in the usual way. The course begins by studying the fundamental concepts of rings (already met briefly in core Algebra): what are maps between them, when are two rings isomorphic etc. much as was done for groups. As an application, we get a general procedure for building fields, generalising the way one constructs the complex numbers from the reals. We then begin to study the question of factorization in rings, and find a class of rings, known as Unique Factorization Domains, where any element can be written uniquely as a product of prime elements generalising the case of the integers. Finally, we study modules, which roughly means we study linear algebra over certain rings rather than fields. This turns out to have powerful applications to ordinary linear algebra and to abelian groups.

6.2 Learning Outcomes

Students should become familiar with rings and fields, and understand the structure theory of modules over a Euclidean domain along with its implications. The material underpins many later courses in algebra and number theory, and thus should give students a good background for studying these more advanced topics.

6.3 Synopsis

Recap on rings (not necessarily commutative) and examples: $\mathbb{Z}$, fields, polynomial rings (in more than one variable), matrix rings. Zero-divisors, integral domains. Units. The characteristic of a ring. Discussion of fields of fractions and their characterization (proofs non-examinable) [2]

Homomorphisms of rings. Quotient rings, ideals and the first isomorphism theorem and consequences, e.g. Chinese remainder theorem. Relation between ideals in R and R/I. Prime ideals and maximal ideals, relation to fields and integral domains. Examples of ideals. Application of quotients to constructing fields by adjunction of elements; examples to include $\mathbb{C} = \mathbb{R}[x]/(x^2 + 1)$ and some finite fields. Degree of a field extension, the tower law. [4]

Euclidean Domains. Examples. Principal Ideal Domains. EDs are PIDs. Unique factorisation for PIDs. Gauss's Lemma and Eisenstein's Criterion for irreducibility. [3]

Modules: Definition and examples: vector spaces, abelian groups, vector spaces with an endomorphism. Submodules and quotient modules and direct sums. The first isomorphism theorem. [2]

Row and column operations on matrices over a ring. Equivalence of matrices. Smith Normal form of matrices over a Euclidean Domain. [1.5]

Free modules and presentations of finitely generated modules. Structure of finitely generated modules of a Euclidean domain. [2]

Application to rational canonical form and Jordan normal form for matrices, and structure of finitely generated Abelian groups. [1.5]

6.4 Reading List

1) M. E. Keating, A First Course in Module Theory, Imperial College Press (1998) Covers almost all material of the course. Out of print but many libraries should have it and second hand copies readily available.

2) Joseph Gallian, *Contemporary Abstract Algebra* (9th edition, CENGAGE 2016) (Excellent text covering material on groups, rings and fields).

3) B. Hartley, T. O. Hawkes, Chapman and Hall, *Rings, Modules and Linear Algebra.* (Out of print, but many libraries should have it. Relatively concise and covers all the material in the course).

4) Neils Lauritzen, *Concrete Abstract Algebra*, CUP (2003) (Excellent on groups, rings and fields, and covers topics in the Number Theory course also. Does not cover material on modules).

5) Michael Artin, *Algebra* (2nd ed. Pearson, (2010). (Excellent but highly abstract text covering everything in this course and much more besides).

7 A4: Integration

7.1 Overview

The course will exhibit Lebesgue's theory of integration in which integrals can be assigned to a huge range of functions on the real line, thereby greatly extending the notion of integration presented in Prelims. The theory will be developed in such a way that it can be easily extended to a wider framework, but measures other than Lebesgue's will only be lightly touched.

Operations such as passing limits, infinite sums, or derivatives, through integral signs, or reversing the order of double integrals, are often taken for granted in courses in applied mathematics. Actually, they can occasionally fail. Fortunately, there are powerful convergence and other theorems allowing such operations to be justified under conditions which are widely applicable. The course will display these theorems and a wide range of their applications.

This is a course in rigorous applications. Its principal aim is to develop understanding of the statements of the theorems and how to apply them carefully. Knowledge of technical proofs concerning the construction of Lebesgue measure will not be an essential part of the course, and only outlines will be presented in the lectures.

7.2 Learning Outcomes

By the end of the course, students will be able to - use the definition of a measure space and prove results using this definition, including about key examples of measures - use the definitions of measurable functions and integrable functions and prove results using these definitions - analyse and explain the connection between integrability as defined in this course, and Riemann integrability from Prelims Analysis III - state, prove and apply the Monotone Convergence Theorem, Fatou's Lemma and the Dominated Convergence Theorem - state and apply Fubini's Theorem and Tonelli's Theorem - use the definition of L^p-spaces and prove results about them

7.3 Synopsis

Measure spaces. Outer measure, null set, measurable set. The Cantor set. Lebesgue measure on the real line. Counting measure. Probability measures. Construction of a non-measurable set (non-examinable). Simple function, measurable function, integrable function. Reconciliation with the integral introduced in Prelims.

A simple comparison theorem. Integrability of polynomial and exponential functions over suitable intervals. Monotone Convergence Theorem. Fatou's Lemma. Dominated Convergence Theorem. Corollaries and applications of the Convergence Theorems (including term-by-term integration of series).

Theorems of Fubini and Tonelli (proofs not examinable). Differentiation under the integral sign. Change of variables.

Brief introduction to L^p spaces. Hölder and Minkowski inequalities.

7.4 Reading List

1. M. Capinski & E. Kopp, *Measure, Integral and Probability* (Second Edition, Springer, 2004).

2. E. M. Stein & R. Shakarchi, *Real Analysis: Measure Theory, Integration and Hilbert Spaces* (Princeton Lectures in Analysis III, Princeton University Press, 2005).

3. D.J.H. Garling, *A Course in Mathematical Analysis, III* (CUP, 2014).

7.5 Further Reading

1. D. S. Kurtz & C. W. Swartz, *Theories of Integration* (Series in Real Analysis Vol.9, World Scientific, 2004).

2. H. A. Priestley, *Introduction to Integration* (OUP, 1997). [Useful for worked examples, although adopts a different approach to construction of the integral].

3. H. L. Royden, *Real Analysis* (various editions; 4th edition has P. Fitzpatrick as co author).

4. R. L. Schilling, *Measures, Integrals and Martingales* (CUP first ed. 2005, or second ed. 2017).

8 A5: Topology

8.1 Overview

Topology is the study of 'spatial' objects. Many key topological concepts were introduced in the Metric Spaces course, such as the open subsets of a metric space, and the continuity of a map between metric spaces. More advanced concepts such as connectedness and compactness were also defined and studied. Unlike in a metric space, there is no notion of distance between points in a topological space. Instead, one keeps track only of the open subsets, but this is enough to define continuity, connectedness and compactness. By dispensing with a metric, the fundamentals of proofs are often clarified and placed in a more general setting. In the first part of the course, these topological concepts are introduced and studied. In the second part of the course, simplicial complexes are defined; these are spaces that are obtained by gluing together triangles and their higher-dimensional analogues in a suitable way. This is a very general construction: many spaces admit a homeomorphism to a simplicial complex, which is known as a triangulation of the space. At the end of the course, the proof of one of the earliest and most famous theorems in topology is sketched. This is the classification of compact triangulated surfaces.

8.2 Learning Outcomes

By the end of the course, a student should be able to understand and construct abstract arguments about topological spaces. Their topological intuition should also be sufficiently well-developed to be able to reason about concrete topological spaces such as surfaces.

8.3 Synopsis

Axiomatic definition of an abstract topological space in terms of open sets. Basic definitions: closed sets, continuity, homeomorphism, convergent sequences, connectedness and comparison with the corresponding definitions for metric spaces. Examples to include metric spaces (definition of topological equivalence of metric spaces), discrete and indiscrete topologies, cofinite topology. The Hausdorff condition. Subspace topology. [2 lectures] Accumulation points of sets. Closure of a set. Interior of a set. Continuity if and only if$f\left(\overline{A}\right) \subseteq \overline{f\left(A\right)}$. [2 lectures] Basis of a topology. Product topology on a product of two spaces and continuity of projections. [2 lectures] Compact topological spaces, closed subset of a compact set is compact, compact subset of a Hausdorff space is closed. Product of two compact spaces is compact. A continuous bijection from a compact space to a Hausdorff space is a homeomorphism. Proof that sequential compactness implies compactness in metric spaces. [2 lectures] Quotient topology. Quotient maps. Characterisation of when quotient spaces are Hausdorff in terms of saturated sets. Examples, including the torus, Klein bottle and real projective plane. [2 lectures] Abstract simplicial complexes and their topological realisation. A triangulation of a space. Any compact triangulated surface is homeomorphic to the sphere with g handles ($g \geqslant 0$) or the sphere with h cross-caps ($h \geqslant 1$). (No proof that these surfaces are not homeomorphic, but a brief informal discussion of Euler characteristic.) [6 lectures]

8.4 Reading List

The lectures will follow:

1. J.D. Jackson, *Classical Electrodynamics* (John Wiley, 1962), chapters 1 to 8.

8.5 Further Reading

1. R. Feynman, *Lectures in Physics*, Vol.2. Electromagnetism, Addison Wesley.
2. L.D. Landau and E.M. Lifshitz, *A classical theory of fields* Volume 2.

9 A8: Probability

9.1 Overview

The first half of the course takes further the probability theory that was developed in the first year. The aim is to build up a range of techniques that will be useful in dealing with mathematical models involving uncertainty. The second half of the course is concerned with Markov chains in discrete time and Poisson processes in one dimension, both with developing the relevant theory and giving examples of applications.

9.2 Synopsis

Continuous random variables. Jointly continuous random variables, independence, conditioning, functions of one or more random variables, change of variables. Examples including some with later applications in statistics.

Moment generating functions and applications. Statements of the continuity and uniqueness theorems for moment generating functions. Characteristic functions (definition only). Convergence in distribution and convergence in probability. Weak law of large numbers and central limit theorem for independent identically distributed random variables. Strong law of large numbers (proof not examinable).

Discrete-time Markov chains: definition, transition matrix, n-step transition probabilities, communicating classes, absorption, irreducibility, periodicity, calculation of hitting probabilities and mean hitting times. Recurrence and transience. Invariant distributions, mean return time, positive recurrence, convergence to equilibrium (proof not examinable), ergodic theorem (proof not examinable). Reversibility, detailed balance equations. Random walks (including symmetric and asymmetric random walks on Z, and symmetric random walks on Z^d).

9.3 Reading List

G. R. Grimmett and D. R. Stirzaker, *Probability and Random Processes* (3rd edition, OUP, 2001). Chapters 4, 6.1-6.5, 6.8.

G. R. Grimmett and D. R. Stirzaker, *One Thousand Exercises in Probability* (OUP, 2001).

G. R. Grimmett and D J A Welsh, *Probability: An Introduction* (OUP, 1986). Chapters 6, 7.4, 8, 11.1-11.3.

J. R. Norris, *Markov Chains* (CUP, 1997). Chapter 1.

D. R. Stirzaker, *Elementary Probability* (Second edition, CUP, 2003). Chapters 7-9 excluding 9.9.

10 ASO: Number Theory

10.1 Overview

Number theory is one of the oldest parts of mathematics. For well over two thousand years it has attracted professional and amateur mathematicians alike. Although notoriously 'pure' it has turned out to have more and more applications as new subjects and new technologies have developed. Our aim in this course is to introduce students to some classical and important basic ideas of the subject.

10.2 Learning Outcomes

Students will learn some of the foundational results in the theory of numbers due to mathematicians such as Fermat, Euler and Gauss. They will also study a modern application of this ancient part of mathematics.

10.3 Synopsis

The ring of integers; congruences; ring of integers modulo n; the Chinese Remainder Theorem.

Wilson's Theorem; Fermat's Little Theorem for prime modulus; Euler's phi-function. Euler's generalisation of Fermat's Little Theorem to arbitrary modulus; primitive roots.

Quadratic residues modulo primes. Quadratic reciprocity.

Factorisation of large integers; basic version of the RSA encryption method.

10.4 Reading List

1) H. Davenport, *The Higher Arithmetic* (Cambridge University Press, 1992) ISBN 0521422272 2) G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers* (OUP, 1980) ISBN 0198531710 3) P. Erdős and J. Surányi, *Topics in the Theory of Numbers* (Springer, 2003) ISBN 0387953205

All these books contain some elementary material but go way beyond what is in the course. Full printed notes will be provided.

11 ASO: Group Theory

11.1 Overview

This group theory course develops the theory begun in prelims, and this course will build on that. After recalling basic concepts, the focus will be on two circles of problems.

1. The concept of free group and its universal property allow to define and describe groups in terms of generators and relations.

2. The notion of composition series and the Jordan-Hölder Theorem explain how to see, for instance, finite groups as being put together from finitely many simple groups. This leads to the problem of finding and classifying finite simple groups. Conversely, it will be explained how to put together two given groups to get new ones.

Moreover, the concept of symmetry will be formulated in terms of group actions and applied to prove some group theoretic statements.

11.2 Learning Outcomes

Students will learn to construct and describe groups. They will learn basic properties of groups and get familiar with important classes of groups. They will understand the crucial concept of simple groups. They will get a better understanding of the notion of symmetry by using group actions.

11.3 Synopsis

Free groups. Uniqueness of reduced words and universal mapping property. Normal subgroups of free groups and generators and relations for groups. Examples. [2]

Review of the First Isomorphism Theorem and proof of Second and Third Isomorphism Theorems. Simple groups, statement that A_n is simple (proof for $n = 5$). Definition and proof of existence of composition series for finite groups. Statement of the Jordan-Hölder Theorem. Examples. The derived subgroup and solvable groups. [3]

Discussion of semi-direct products and extensions of groups. Examples. [1]

Sylow's three theorems. Applications including classification of groups of small order. [2]

11.4 Reading List

1) Humphreys, J. F. *A Course in Group Theory*, Oxford, 1996

2) Armstrong, M. A. *Groups and Symmetry*, Springer-Verlag, 1988

12 ASO: Projective Geometry

12.1 Overview

Projective Geometry might be viewed as the geometry of perspective. Two observers of a painting – one looking obliquely, one straight on – will not agree on angles and distances but will both sees lines as lines and will agree on whether they meet. So projective transformations (such as relate the two observers' views) are less rigid than Euclidean, or even affine, transformations. Projective geometry also introduces the idea of points at infinity – points where parallel lines meet. These points fill in the missing gaps/address some special cases of geometry in a similar way to which complex numbers resolve such problems in algebra. From this point of view ellipses, parabolae and hyperbolae are all projectively equivalent and just happen to include no, one or two points at infinity. The study of such conics also has applications to the study of quadratic Diophantine equations.

12.2 Learning Outcomes

Students will be familiar with the idea of projective space and the linear geometry associated to it, including examples of duality and applications to Diophantine equations.

12.3 Synopsis

1-2: Projective Spaces (as $P(V)$ of a vector space V). Homogeneous Co-ordinates. Linear Subspaces.

3-4: Projective Transformations. General Position. Desargues Theorem. Cross-ratio.

5: Dual Spaces. Duality.

6-7: Symmetric Bilinear Forms. Conics. Singular conics, singular points. Projective equivalence of non-singular conics.

7-8: Correspondence between P^1 and a non-singular conic. Simple applications to Diophantine Equations.

12.4 Reading List

1) N.J. Hitchin, *Maths Institute notes on Projective Geometry* (found under 'Teaching')

2) M. Reid and B. Szendrői, *Geometry and topology*, Cambridge University Press, 2005 (Chapter 5).

3) R. Casse, Projective Geometry, An Introduction, Oxford University Press (2006)

13 ASO: Integral Transforms

13.1 General Prerequisites

This course is highly recommended for Differential Equations 2.

13.2 Overview

The Laplace and Fourier Transforms aim to take a differential equation in a function and turn it into an algebraic equation involving its transform. Such an equation can then be solved by algebraic manipulation, and the original solution determined by recognizing its transform or applying various inversion methods.

The Dirac *delta*-function, which is handled particularly well by transforms, is a means of rigorously dealing with ideas such as instantaneous impulse and point masses, which cannot be properly modelled using functions in the normal sense of the word. δ is an example of a *distribution* or *generalized function* and the course provides something of an introduction to these generalized functions and their calculus.

13.3 Learning Outcomes

Students will gain a range of techniques employing the Laplace and Fourier Transforms in the solution of ordinary and partial differential equations. They will also have an appreciation of generalized functions, their calculus and applications.

13.4 Synopsis

Motivation for a "function" with the properties the Dirac *delta*-function. Test functions. Continuous functions are determined by $\phi \to \int f\phi$. Distributions and δ as a distribution. Differentiating distributions. (3 lectures)

Theory of Fourier and Laplace transforms, inversion, convolution. Inversion of some standard Fourier and Laplace transforms via contour integration.

Use of Fourier and Laplace transforms in solving ordinary differential equations, with some examples including δ.

Use of Fourier and Laplace transforms in solving partial differential equations; in particular, use of Fourier transform in solving Laplace's equation and the Heat equation. (5 lectures)

13.5 Reading List

S. Howison, *Practical Applied Mathematics* (CUP 2005), Chapters 9 & 10 (for distributions).

P. J. Collins, *Differential and Integral Equations* (OUP, 2006), Chapter 14.

W. E. Boyce & R. C. DiPrima, *Elementary Differential Equations and Boundary Value Problems*, there are many editions, most recently in 2017; in all of them Chapter 6 covers Laplace Transforms.

K. F. Riley & M. P. Hobson, *Essential Mathematical Methods for the Physical Sciences* (CUP 2011) Chapter 5.

H. A. Priestley, *Introduction to Complex Analysis* (2nd edition, OUP, 2003) Chapters 21 and 22.

13.6 Further Reading

L. Debnath & P. Mikusinski, *Introduction to Hilbert Spaces with Applications*, (3rd Edition, Academic Press. 2005) Chapter 6

14 ASO: Calculus of Variations

14.1 Overview

The calculus of variations concerns problems in which one wishes to find the minima or extrema of some quantity over a system that has functional degrees of freedom. Many important problems arise in this way across pure and applied mathematics and physics. They range from the problem in geometry of finding the shape of a soap bubble, a surface that minimizes its surface area, to finding the configuration of a piece of elastic that minimises its energy. Perhaps most importantly, the principle of least action is now the standard way to formulate the laws of mechanics and basic physics.

In this course it is shown that such variational problems give rise to a system of differential equations, the Euler-Lagrange equations. Furthermore, the minimizing principle that underlies these equations leads to direct methods for analysing the solutions to these equations. These methods have far reaching applications and will help develop students technique.

14.2 Learning Outcomes

Students will be able to formulate variational problems and analyse them to deduce key properties of system behaviour.

14.3 Synopsis

The basic variational problem and Euler's equation. Examples, including axi-symmetric soap films.

Extension to several dependent variables. Hamilton's principle for free particles and particles subject to holonomic constraints. Equivalence with Newton's second law. Geodesics on surfaces. Extension to several independent variables.

Examples including Laplace's equation. Lagrange multipliers and variations subject to constraint. Eigenvalue problems for Sturm-Liouville equations. Legendre Polynomials.

14.4 Reading List

Arfken Weber, *Mathematical Methods for Physicists* (5th edition, Academic Press, 2005). Chapter 17.

14.5 Further Reading

N. M. J. Woodhouse, *Introduction to Analytical Dynamics* (1987). Chapter 2 (in particular 2.6). (This is out of print, but still available in most College libraries.)

M. Lunn, *A First Course in Mechanics* (OUP, 1991). Chapters 8.1, 8.2.

P. J. Collins, *Differential and Integral Equations* (O.U.P., 2006). Chapters 11, 12.

15 ASO: Graph Theory

15.1 Overview

This course introduces some central topics in graph theory.

15.2 Learning Outcomes

By the end of the course, students should have an appreciation of the methods and results of graph theory. They should have a good understanding of the basic objects in graph theory, such as trees, Euler circuits and matchings, and they should be able to reason effectively about graphs.

15.3 Synopsis

Introduction. Paths, walks, cycles and trees. Euler circuits. Hamiltonian cycles. Hall's theorem. Application and analysis of algorithms for minimum cost spanning trees, shortest paths, bipartite matching and the Chinese Postman Problem.

15.4 Reading List

R. J. Wilson, *Introduction to Graph Theory*, 5th edition, Prentice Hall, 2010.

D.B. West, *Introduction to Graph Theory*, 2nd edition, Prentice Hall, 2001.

16 ASO: Mathematical Modelling in Biology

16.1 Overview

Mathematical Modelling in Biology introduces the applied mathematician to practical applications in an area that is growing very rapidly. The course focuses on how to model various processes in ecology, epidemiology, chemistry, biology and medicine, using ordinary differential equation models, as well as an introduction to discrete models. It demonstrates how mathematical techniques such as linear stability analysis, phase planes, perturbation and asymptotic analysis can enable us to predict the behaviour of living systems.

16.2 Learning Outcomes

Students will have developed a sound knowledge and appreciation of the ideas and concepts related to modelling biological and ecological systems using continuous-time non-spatial models.

16.3 Synopsis

Continuous population models for a single species including hysteresis and harvesting.

Discrete time models for single species – linear stability analysis and cobwebbing.

Modelling interacting populations, including predator-prey and the principle of competitive exclusion.

Enzyme-substrate kinetics, the quasi-steady state approximation and perturbation analysis.

Modelling of neuronal signalling using the Hodgkin-Huxley model and excitable kinetics.

Infectious disease modelling including SIR models.

16.4 Reading List

J. D. Murray, *Mathematical Biology*, Volume I: An Introduction. 3rd Edition, Springer (2002).

16.5 Further Reading

N. F. Britton, *Essential Mathematical Biology.* Springer (2003).

G. de Vries, T. Hillen, M. Lewis, J. Müller, B. Schönfisch. *A Course in Mathematical Biology: Quantitative Modelling with Mathematical and Computational Methods.* SIAM (2006).