Handbook for the Undergraduate Mathematics Courses
Supplement to the Handbook
Honour School of Mathematics
Syllabus and Synopses for Part A 2009–10
for examination in 2010

Contents

1 Foreword

Notice of misprints or errors of any kind, and suggestions for improvements in this booklet should be addressed to the Academic Assistant in the Mathematical Institute.

1.1 Honour School of Mathematics

[See the current edition of the *Examination Regulations* for the full regulations governing these examinations.]

In Part A each candidate shall be required to offer the 4 written papers from the schedule of papers for Part A (below).

Part A shall be taken on one occasion only (there will be no resits). At the end of the Part A examinations, a candidate will be awarded four 'University Standardised Marks' (USMs) for their performance in Part A. These will be carried forward into the classification awarded at the end of the third year. In this classification, the marks in Part A will be given a 'weighting' of 2, and the marks in Part B will be given a 'weighting' of 3. All students who complete the first three years of the course will be classified, and those wishing to graduate at this point may supplicate for a BA.

Students wishing to take the four-year course should register to do so at the beginning of their third year. They will take Part C in their fourth year, be awarded a separate classification and, if successful, be allowed to supplicate for an MMath.

1.1.1 Extract from Examination Conventions The Schedule of Papers

Altogether, these papers will include 1 short question and 1 longer question for each 8 hour lecture course; 2 short questions and 2 longer questions for each 16 hour lecture course; 3 short questions and 3 longer questions for each 24 hour lecture course.

Paper AC1 Algebra, Analysis and Differential Equations

This paper will contain questions set on the CORE material, and will contain 9 short questions (3 for each course), attracting 10 marks each, all of which should be answered.

Paper AC2 Algebra, Analysis and Differential Equations

The paper AC2 contains 9 questions in total with 3 questions in each section, namely 3 on Algebra, 3 on Analysis and 3 on Differential Equations. Candidates may submit answers to as many questions as they wish. The best 4 questions will count for the total mark for this paper, with at least 1 from each section. That is, the best answer from each section together with the next best answer will count for the total mark for this paper.

Paper AO1 Options

This paper will contain questions set on the OPTIONAL material, and will contain 19 short questions, attracting 10 marks each. Candidates may submit answers to as many questions as they wish. The best 9 answers will count for the total mark for this paper.

Paper AO2 Options

This paper will contain questions set on the OPTIONAL material, and will contain 19 longer questions, attracting 25 marks each. Candidates may submit answers to as many question as they wish. The best 4 answers will count for the total mark for this paper.

Marking of Papers

Mark schemes for questions out of 10 will aim to ensure that the following qualitative criteria hold:

- 9-10 marks: a completely or almost completely correct answer, showing good understanding of the concepts and skill in carrying through arguments and calculations; minor slips or omissions only.
- 5-8 marks: a good though not complete answer, showing understanding of the concepts and competence in handling the arguments and calculations.

Mark schemes for questions out of 25 will aim to ensure that the following qualitative criteria hold:

- 20-25 marks: a completely or almost completely correct answer, showing excellent understanding of the concepts and skill in carrying through the arguments and/or calculations; minor slips or omissions only.
- 13-19 marks: a good though not complete answer, showing understanding of the concepts and competence in handling the arguments and/or calculations. In this range, an answer might consist of an excellent answer to a substantial part of the question, or a good answer to the whole question which nevertheless shows some flaws in calculation or in understanding or in both.

This should be regarded only as a guide conveying the intentions of the examiners.

Parts B and C

Where not otherwise stated, an overview of the course and form of the papers for each unit and half unit is given in the lecture synopsis.

The Examination Papers

For Mathematics Examinations in Parts B and C the following apply. Examinations for whole unit papers are of three hours duration and half unit papers are of one and a half hour duration. Each half unit examination will contain 3 questions and each full paper 6 questions, 3 on each section. The rubrics are given below.

The rubrics

For Parts B and C, a whole unit paper rubric states "candidates may submit answers to as many questions as they wish; the best two from each section will count for the total mark for this paper." For the half unit the rubric states "candidates may submit answers to as many questions as they wish; the best two will count for the total mark for this paper."

Analysis of marks

Part A

At the end of the Part A examination, a candidate will be awarded a University Standardised Mark (USM) for each of the four papers. The Examiners will recalibrate the raw marks to arrive at the USMs reported to candidates. In arriving at this recalibration, the examiners will principally take into account the total sum over all four papers of the marks for each question, subject to the rules above on numbers of questions answered.

The Examiners aim to ensure that all papers and all subjects within a paper are fairly and equally rewarded, but if in any case a paper, or a subject within a paper, appears to have been problematical, then the Examiners may take account of this in calculating USMs.

The USMs awarded to a candidate for papers in Part A will be carried forward into a classification as described below.

Part B

The Board of Examiners in Part B will assign USMs for full unit and half unit papers taken in Part B and may recalibrate the raw marks to arrive at the USMs reported to candidates. The full unit papers are designed so that the raw marks sum to 100, however, Examiners will take into account the relative difficulty of papers when assigning USMs. In order to achieve this, Examiners may use information on candidates' performances on the Part A examination when recalibrating the raw marks. They may also use other statistics to check that the USMs assigned fairly reflect the students' performances on a paper.

The USMs awarded to a candidate for papers in Part B will be aggregated with the USMs from Part A to arrive at a classification.

Part C

The Board of Examiners in Part C will assign USMs for full unit and half unit papers taken in Part C and may recalibrate the raw marks to arrive at the USMs reported to candidates. The full unit papers are designed so that the raw marks sum to 100, however, Examiners will take into account the relative difficulty of papers when assigning USMs. In order to achieve this, Examiners may use information on candidates' performances on the earlier parts of the degree when recalibrating the raw marks. They may also use other statistics to check that the USMs assigned fairly reflect the students' performances on a paper.

The USMs awarded to a candidate for papers in Part C will be averaged to arrive at a classification for Year 4.

Aggregation of marks for award of Part B

All successful candidates will be awarded a classification at the end of three years, after the Part B examination. This classification will be based on the following rules (agreed by the Mathematics Teaching Committee).

A *Strong Paper rule* is adopted for classification.

By the *nth class strong paper rule* we mean that for a candidate to be classified at the nth class standard, at least 3 papers from Parts A and B must lie in the nth class and at least one of these must be at Part B. For example, for a first class award, a candidate would need at least 3 of their whole unit paper USMs to be first class marks (with at least 1 first class whole unit at Part B) together with a weighted average score of parts A and B over 70.

In effect we are looking at a *marks profile.*

The Part A USMs are given a weighting of 2, and the Part B USMs a weighting of 3 for a full unit and 1.5 for a half unit.

Let $AvUSM - PartA\&B = Average\ weighted\ USM\ in\ Parts\ A\ and\ B\ together$ (rounded up to whole number);

- First Class: $AvUSM - PartA\&B \geq 70$ and the first class strong paper rule satisfied.
- Upper Second Class: $AvUSM - PartA\&B \geq 70$ and not satisfying the first class strong paper rule **OR** $70 > AvUSM - PartA\&B \geq 60$ and the upper second strong paper rule satisfied.
- Lower Second Class: $70 > AvUSM - PartA\&B \geq 60$ and not satisfying the upper second strong paper rule **OR** $60 > AvUSM - PartA\&B \geq 50$ and the lower second strong paper rule satisfied.
- Third Class: $50 > AvUSM - PartA\&B \geq 40$ **OR** $60 > AvUSM - PartA\&B \geq 50$ and not satisfying the lower second strong paper rule
- Pass: $40 > AvUSM - PartA\&B \geq 30$
- Fail: $AvUSM - PartA\&B < 30$

[Note: Half unit papers count as half a paper when determining the average USM, or determining the number of strong papers.]

BA in Mathematics

All candidates who wish to leave at the end of their third year and who satisfy the Examiners may supplicate for a classified BA in Mathematics at the end of Part B based on the above classification.

MMath in Mathematics

In order to proceed to Part C, a candidate must minimally achieve lower second standard in Part A and Part B together.

Candidates successfully studying for a fourth year will receive a separate classification based on their University Standardised Marks in Part C papers, according to the following rules (agreed by the Mathematics Teaching Committee).

$AvUSM - PartC = Average\ USM\ in\ Part\ C$ (rounded up to whole number)

- First Class: $AvUSM - PartC \geq 70$
- Upper Second Class: $70 > AvUSM - PartC \geq 60$
- Lower Second Class: $60 > AvUSM - PartC \geq 50$
- Third Class: $50 > AvUSM - PartC \geq 40$

A 'Pass' will not be awarded for Year 4. Candidates achieving:

$$AvUSM - PartC < 40,$$

may supplicate for a BA.

[Note: Half unit papers count as half a paper when determining the average USM.]

Candidates leaving after four years who satisfy the Examiners may supplicate for an MMath. in Mathematics, with two associated classifications; for example:
MMath. in Mathematics: Years 2 and 3 together - First class; Year 4 - First class.

Note that successful candidates may supplicate for one degree only - either a BA or an MMath. The MMath. has two classifications associated with it but a candidate will not be awarded a BA degree and an MMath. degree.

Syllabus and Synopses

The **syllabus** details in this booklet are those referred to in the *Examination Regulations* and have been approved by the Mathematics Teaching Committee in Trinity Term 2009 for examination in 2010.

The **synopses** in this booklet give some additional detail, and show how the material is split between the different lecture courses. They also include details of recommended reading.

1.2 Honour School of Mathematics & Philosophy

See the current edition of the *Examination Regulations* for the full regulations governing these examinations. For the Schedule of Mathematics Papers for Part A see the Supplement to the Undergraduate Handbook for the Honour School of Mathematics & Philosophy:

http://www.maths.ox.ac.uk/current-students/undergraduates/handbooks-synopses/mathsphil.

1.3 Honour School of Mathematics & Statistics

See the current edition of the *Examination Regulations* for the full regulations governing these examinations, and the details published by the Statistics Department:

http://www.stats.ox.ac.uk/current_students/bammath/course_handbooks.

Papers AC1 and AC2 under the schedule of papers here are taken by candidates in Mathematics & Statistics.

1.4 Honour Schools of Computer Science and Mathematics & Computer Science

See the current edition of the *Examination Regulations* for the full regulations governing these examinations, and the details published in a handbook by the Computing Laboratory:

http://web.comlab.ox.ac.uk/teaching/handbooks.html.

2 CORE MATERIAL

2.1 Syllabus

This section contains the examination syllabi for the two core papers AC1 and AC2.

2.1.1 Algebra

Vector spaces over an arbitrary field, subspaces, direct sums; quotient spaces; projection maps and their characterisation as idempotent operators.

Dual spaces of finite-dimensional spaces; annihilators; the natural isomorphism between a space and its second dual; dual transformations and their matrix representation with respect to dual bases.

Some theory of a single linear map on a finite-dimensional space: characteristic polynomial, minimal polynomial, Primary Decomposition Theorem, the Cayley–Hamilton Theorem; diagonalisability; triangular form. Statement of the Jordan normal form.

Real and complex inner product spaces. Orthogonal complements, orthonormal sets; the Gram–Schmidt process. Bessel's inequality; the Cauchy–Schwarz inequality.

The adjoint of a linear map on a finite-dimensional inner product space to itself. Eigenvalues and diagonalisability of self-adjoint linear maps.

Commutative rings with unity, integral domains, fields; units, irreducible elements, primes.

Ideals and quotient rings; isomorphism theorems. The Chinese Remainder Theorem [classical case of $\mathbb{Z}$ only].

Maximal ideals and their quotient rings.

Euclidean rings and their properties: polynomial rings as examples, theorem that their ideals are principal; theorem that their irreducible elements are prime; uniqueness of factorisation (proof non-examinable).

Gauss' Lemma; Eisenstein's criterion.

2.1.2 Analysis

The topology of Euclidean space and its subsets, particularly $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$: open sets, closed sets, subspace topology; continuous functions and their characterisation in terms of pre-images of open or closed sets; connected sets, path-connected sets; compact sets, Heine-Borel Theorem.

The algebra and geometry of the complex plane. Complex differentiation. Holomorphic functions. Cauchy-Riemann equations (including z, $\bar{z}$ version). Real and imaginary parts of a holomorphic function are harmonic.

Path integration. Fundamental Theorem of Calculus in the path integral/holomorphic function setting. Power series and differentiation of power series. Exponential function, logarithm function, fractional powers - examples of multifunctions.

Cauchy's Theorem (proof excluded). Cauchy's Integral formulae. Taylor expansion. Liouville's Theorem. Identity Theorem. Morera's Theorem. Laurent's expansion. Classification of singularities. Calculation of principal parts and residues. Residue Theorem. Evaluation of integrals by the method of residues (straight forward examples including the use of simple estimates, and examples with simple poles on contour of integration).

Conformal mapping: Möbius functions, exponential functions, fractional powers; mapping regions (not Christoffel transformations or Jowkowski's transformation).

2.1.3 Differential Equations

Picard's Theorem for first-order scalar ODEs with proof. Statement, without proof, of extension to systems. Examples with blow-up and non-uniqueness.

Second-order ODEs: variation of parameters, Wronskian and Green's function.

Phase planes, critical points, Poincaré-Bendixson criterion. Examples including conservative nonlinear oscillators, Van der Pol's equation and Lotka-Volterra equations. Stability of periodic solutions.

Characteristic methods for first-order quasilinear PDEs. Examples from conservation laws. Multi-Valued solutions and shocks.

Classification of second-order linear PDEs. Ideas of uniqueness and well-posedness for Laplace, Wave and Heat equations. Illustration of suitable boundary conditions by example. Multi-dimensional Laplacian operator giving rise to Bessel's and Legendre's equations.

Theory of Fourier and Laplace transforms, inversion, convolution. Inversion of some standard Fourier and Laplace transforms via contour integration. Use of Fourier transform in solving Laplace's equation and the Heat equation. Use of Laplace transform in solving the Heat equation.

2.2 Synopses of Lectures

This section contains the lecture synopses associated with the two core papers AC1 and AC2.

2.2.1 Algebra — Dr Stewart & Dr Papazoglou — 24 lectures MT

Overview

Linear Algebra
The core of linear algebra comprises the theory of linear equations in many variables, the theory of matrices and determinants, and the theory of vector spaces and linear maps. All these topics were introduced in the Moderations course. Here they are developed further to provide the tools for applications in geometry, modern mechanics and theoretical physics, probability and statistics, functional analysis and, of course, algebra and number theory. Our aim is to provide a thorough treatment of some classical theory that describes the behaviour of linear maps on a finite-dimensional vector space to itself, both in the purely algebraic setting and in the situation where the vector space carries a metric derived from an inner product.

Rings
The rings part of the course introduces the student to some classic ring theory which is basic for other parts of abstract algebra, for linear algebra and for those parts of number theory that lead ultimately to applications in cryptography. The first-year algebra course contains a treatment of the Euclidean Algorithm in its classical forms for integers and for polynomial rings over a field; here the idea is developed *in abstracto*.

Learning Outcomes

Linear Algebra
Students will deepen their understanding of Linear Algebra. They will be able to define and obtain the minimal and characteristic polynomials of a linear map on a finite-dimensional vector space, and will understand and be able to prove the relationship between them; they will be able to prove and apply the Primary Decomposition Theorem, and the criterion for diagonalisability. They will have a good knowledge of inner product spaces, and be able to apply the Bessel and Cauchy–Schwarz inequalities; will be able to define and use the adjoint of a linear map on a finite-dimensional inner product space, and be able to prove and exploit the diagonalisability of a self-adjoint map.

Rings
By the end of the course students will have extended their knowledge of abstract algebra to include the key elements of classical ring theory. They will understand and be able to prove and use the Isomorphism Theorem. They will have a good knowledge of Euclidean rings, and be able to apply it.

Synopsis

1. **Linear Algebra**

   MT (17 lectures)
   Vector spaces over an arbitrary field, subspaces, direct sums; quotient vector spaces; induced linear map; projection maps and their characterisation as idempotent operators.

   [2 Lectures]

   Dual spaces of finite-dimensional spaces; annihilators; the natural isomorphism between a finite-dimensional space and its second dual; dual transformations and their matrix representation with respect to dual bases.

   [2-3 Lectures]

   Some theory of a single linear map on a finite-dimensional space: characteristic polynomial, minimal polynomial, Primary Decomposition Theorem, the Cayley-Hamilton Theorem (economically); diagonalisability; triangular form. Statement of the Jordan normal form.

   [4-5 Lectures]

   Real and complex inner product spaces: examples, including function spaces [but excluding completeness and L^2]. Orthogonal complements, orthonormal sets; the Gram–Schmidt process. Bessel's inequality; the Cauchy–Schwarz inequality.

   [4 Lectures]

   Some theory of a single linear map on a finite-dimensional inner product space: the adjoint; eigenvalues and diagonalisability of a self-adjoint linear map.

   [4 Lectures]

2. **Rings**

   MT (7 Lectures)

   Review of commutative rings with unity, integral domains, ideals, fields, polynomial rings and subrings of $\mathbb{R}$ and $\mathbb{C}$. The Chinese Remainder Theorem; the quotient ring by a maximal ideal is a field.

   [2 lectures]

   Euclidean rings and their properties : units, associates, irreducible elements, primes. The Euclidean Algorithm for a Euclidean ring; $\mathbb{Z}$ and $F[x]$ as prototypes; their ideals are principal; their irreducible elements are prime; factorisation is unique (proof not examinable).

   [3 Lectures]

   Examples for applications: Gauss's Lemma and factorisation in $\mathbb{Z}[x]$; Eisenstein's criterion.

   [2 lectures]

Reading

Richard Kaye and Robert Wilson, *Linear Algebra* (OUP, 1998) ISBN 0-19-850237-0. Chapters 2–13. [Chapters 6, 7 are not entirely relevant to our syllabus, but are interesting.]

Peter J. Cameron, *Introduction to Algebra* (OUP, 1998) ISBN 0-19-850194-3. Chapter 2.

Alternative and further reading:

Joseph J. Rotman, *A First Course in Abstract Algebra* (Second edition, Prentice Hall, 2000), ISBN 0-13-011584-3. Chapters 1, 3.

I. N. Herstein, *Topics in Algebra* (Second edition, Wiley, 1975), ISBN 0-471-02371-X. Chapter 3. [Harder than some, but an excellent classic. Widely available in Oxford libraries; still in print.]

P. M. Cohn, *Classic Algebra* (Wiley, 2000), ISBN 0-471-87732-8. Various sections. [This is the third edition of his book previously called *Algebra I*.]

David Sharpe, *Rings and Factorization* (CUP, 1987), ISBN 0-521-33718-6. [An excellent little book, now sadly out of print; available in some libraries, though.]

Paul R. Halmos, *Finite-dimensional Vector Spaces*, (Springer Verlag, Reprint 1993 of the 1956 second edition), ISBN 3-540-90093-4. §§1–15, 18, 32–51, 54–56, 59–67, 73, 74, 79. [Now over 50 years old, this idiosyncratic book is somewhat dated but it is a great classic, and well worth reading.]

Seymour Lipschutz and Marc Lipson, *Schaum's Outline of Linear Algebra* (3rd edition, McGraw Hill, 2000), ISBN 0-07-136200-2. [Many worked examples.]

C. W. Curtis, *Linear Algebra—an Introductory Approach* (4th edition, Springer, reprinted 1994).

D. T. Finkbeiner, *Elements of Linear Algebra* (Freeman, 1972). [Out of print, but available in many libraries.]

J. A. Gallian, *Contemporary Abstract Algebra* (Houghton Mifflin Company, 2006).

There are very many other such books on abstract and linear algebra in Oxford libraries.

2.2.2 Analysis — Dr Qian — 24 lectures MT

Overview

The theory of functions of a complex variable is a rewarding branch of mathematics to study at the undergraduate level with a good balance between general theory and examples. It occupies a central position in mathematics with links to analysis, algebra, number theory, potential theory, geometry, topology, and generates a number of powerful techniques (for example, evaluation of integrals) with applications in many aspects of both pure and applied mathematics, and other disciplines, particularly the physical sciences.

In these lectures we begin by introducing students to the language of topology before using it in the exposition of the theory of (holomorphic) functions of a complex variable. The central aim of the lectures is to present Cauchy's Theorem and its consequences, particularly series expansions of holomorphic functions, the calculus of residues and its applications.

The course concludes with an account of the conformal properties of holomorphic functions and applications to mapping regions.

Learning Outcomes

Students will have been introduced to point-set topology and will know the central importance of complex variables in analysis. They will have grasped a deeper understanding of differentiation and integration in this setting and will know the tools and results of complex analysis including Cauchy's Theorem, Cauchy's integral formula, Lioville's Theorem, Laurent's expansion and the theory of residues.

Synopsis

(1-4) Topology of Euclidean space and its subsets, particularly $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$. Open sets, closed sets, subspace topology; continuous functions and their characterisation in terms of preimages of open or closed sets; connected sets, path-connected sets; compact sets, Heine-Borel Theorem (covered in Chapter 3 of Apostol).

(5-7) Review of algebra and geometry of the complex plane. Complex differentiation. Holomorphic functions. Cauchy-Riemann equations. Real and imaginary parts of a holomorphic function are harmonic.

(8-11) Path integration. Power series and differentiation of power series. Exponential function and logarithm function. Fractional powers - examples of multifunctions.

(12-13) Cauchy's Theorem. (Sketch of proof only - students referred to various texts for proof.) Fundamental Theorem of Calculus in the path integral/holomorphic situation.

(14-16) Cauchy's Integral formulae. Taylor expansion. Liouville's Theorem. Identity Theorem. Morera's Theorem

(17-18) Laurent's expansion. Classification of singularities. Calculation of principal parts, particularly residues.

(19-21) Residue theorem. Evaluation of integrals by the method of residues (straight forward examples only but to include the use of Jordan's Lemma and simple poles on contour of integration).

(22-23) Conformal mapping: Möbius functions, exponential functions, fractional powers; mapping regions (not Christoffel transformations or Jowkowski's transformation).

(24) Summary and Outlook.

Reading

Main texts

H. A. Priestley, *Introduction to Complex Analysis* (second edition, Oxford Science Publications, 2003).

T. M. Apostol, *Mathematical Analysis* (Addison–Wesley, 1974)(Chapter 3 for the topology).

J.B. Conway, *Functions of One Complex Variable* (Springer-Verlag, 1986).

Mark J. Ablowitz, Athanassios S. Focas, *Complex Variables, Introduction and Applications*(2nd edition, Cambridge Texts in Applied Mathematics, 2003).

Further Reading

L. Ahlfors, *Complex Analysis* (McGraw-Hill, 1979).

Theodore Gamelin, *Complex Analysis* (Springer, 2000).

Reinhold Remmert, *Theory of Complex Functions* (Springer, 1989) (Graduate Texts in Mathematics 122).

E. C. Titchmarsh, *The Theory of Functions* (2nd edition, Oxford University Press).

I. Stewart and D. Tall, *Complex Analysis*, (CUP, 1983).

2.2.3 Differential Equations — Prof. Mason — 24 lectures MT

Overview

The aim of this course is to introduce all students reading mathematics to the basic theory of ordinary and partial differential equations.

The course will be example-led and will concentrate on equations that arise in practice rather than those constructed to illustrate a mathematical theory. The emphasis will be on solving equations and understanding the possible behaviours of solutions, and the analysis will be developed as a means to this end.

The course will furnish undergraduates with the necessary skills to pursue any of the applied options in the third year and will also form the foundation for a deeper and more rigorous course in partial differential equations.

Learning Outcomes

On completion of the course, students will have acquired a sound knowledge of a range of techniques for solving linear ordinary and partial differential equations. They will have gained an appreciation of the importance of existence and uniqueness of solution and will be aware that explicit analytic solutions are the exception rather than the rule.

Synopsis

(1–4) Picard's Theorem for $\mathrm{d}y/\mathrm{d}x = f(x, y)$ with proof. Extension to systems stated but not proved. Examples with blow-up and non-uniqueness.
(Collins section 2.1. Boyce & DiPrima section 2.12. Kreyszig section 1.9.)

(5–7) Second-order ODEs: variation of parameters, Wronskian and Green's function.
(Collins chapters 3, 4. Boyce & DiPrima sections 3.1–3.5, 3.6, 3.6.2. Hildebrand chapter 3.

Kreyszig sections 2.1, 2.7–2.10.)

(8–11) Phase planes, critical points, Poincaré-Bendixson criterion. Examples including conservative nonlinear oscillators, Van der Pol's equation and Lotka-Volterra equations. Stability of periodic solutions.
(Collins chapters 3, 4. Boyce & DiPrima sections 9.1–9.4. Kreyszig sections 3.3–3.5.)

(12–14) Characteristic methods for first-order quasilinear PDEs (using parameterisation). Examples from conservation laws. Multivalued solutions and shocks. Charpit's method and artificial examples excluded.
(Collins Chapter 5. Carrier & Pearson Chapters 6, 13. Ockendon *et al.* Chapter 1.)

(15–18) Classification of second-order linear PDEs. Ideas of uniqueness and well-posedness for Laplace, Wave and Heat equations. Revision of separation of variables from Mods and illustration of suitable boundary conditions by example. Multi-dimensional Laplacian operator giving rise to Bessel's and Legendre's equations.
(Collins chapters 6, 7. Carrier & Pearson Chapters 1, 3, 4, 5, 7. Strauss Chapter 1. Kreyszig sections 11.7–11.11.)

(19–24) Theory of Fourier and Laplace transforms, inversion, convolution. Inversion of some standard Fourier and Laplace transforms via contour integration. Use of Fourier transform in solving Laplace's equation and the Heat equation. Use of Laplace transform in solving the Heat equation.
(Collins chapter 14. Carrier & Pearson chapters 2, 15. Kreyszig chapter 5, sections 10.8–10.11, 11.6. Priestley chapter 9.)

Reading

The best single text is:

P. J. Collins, *Differential and Integral Equations* (O.U.P., 2006), Chapters 1-7, 14,15.

Alternatives

W. E. Boyce & R. C. DiPrima, *Elementary Differential Equations and Boundary Value Problems* (7th edition, Wiley, 2000).

Erwin Kreyszig, *Advanced Engineering Mathematics* (8th Edition, Wiley, 1999).

F. B. Hildebrand, *Methods of Applied Mathematics* (Dover, 1992).

W. A. Strauss, *Partial Differential Equations: an Introduction* (Wiley, 1992).

G. F. Carrier & C E Pearson, *Partial Differential Equations — Theory and Technique* (Academic, 1988).

H. A. Priestley, *Introduction to Complex Analysis* (Second edition, Oxford, 2003).

J. Ockendon, S. Howison, A. Lacey & A. Movchan, *Applied Partial Differential Equations* (Oxford, 1999). [More advanced.]

3 OPTIONS

3.1 Syllabus

This section contains the examination syllabi for the two papers AO1 and AO2.

3.1.1 Introduction to Fields

Fields, subfields, finite extensions; examples. Degree of an extension, the Tower Theorem. Simple algebraic extensions; splitting fields, uniqueness (proof not to be examined); examples. Characteristic of a field. Finite fields: existence; uniqueness (proof not to be examined). Subfields. The multiplicative group of a finite field. The Frobenius automorphism.

3.1.2 Group Theory

Groups: subgroups, normal subgroups and quotient groups; elementary results concerning symmetric and alternating groups; important examples of groups, including the general Linear groups. Isomorphism theorems for groups. Simplicity; composition series. Finite soluble groups. Actions of groups on sets; examples, including coset spaces, groups acting on themselves by translation and conjugation, the Möbius groups. Orbits, transitivity, stabilisers, equivalence of a transitive space with a coset space, kernels of such actions, examples. Symmetry groups of geometric objects including regular polyhedra.

3.1.3 Number Theory

The ring of integers; congruences; rings of integers modulo n; the Chinese Remainder Theorem. Wilson's Theorem; Fermat's Little Theorem for prime modulus. Euler's phi-function; Euler's generalisation of Fermat's Little Theorem to arbitrary modulus. Quadratic residues modulo primes. Quadratic reciprocity. Factorisation of large integers; basic version of the RSA encryption method.

3.1.4 Integration

Measure spaces. Outer measure, null set, measurable set. The Cantor set. Lebesgue measure on the real line. Counting measure. Probability measures. Construction of a non-measurable set (non-examinable). Measurable function, simple function, integrable function. Reconciliation with the integral introduced in Moderations.

A simple comparison theorem. Integrability of polynomial and exponential functions over suitable intervals. Changes of variable. Fatou's Lemma (proof not examinable). Monotone Convergence Theorem (proof not examinable). Dominated Convergence Theorem. Corollaries and applications of the Convergence Theorems (including term-by-term integration of series).

Theorems of Fubini and Tonelli (proofs not examinable). Differentiation under the integral sign. Change of variables.

Brief introduction to L^p spaces. Hölder and Minkowski inequalities (proof not examinable).

3.1.5 Topology

Metric spaces. Examples to include metrics derived from a norm on a real vector space, particularly l^1, l^2, l^∞ norms on $\mathbb{R}^n$, the *sup* norm on the bounded real-valued functions on a set, and on the bounded continuous real-valued functions on a metric space. Continuous functions (ϵ, δ definition). Uniformly continuous functions; examples include Lipschitz functions and contractions. Open balls, open sets, accumulation points of a set. Completeness (but not completion). Contraction Mapping Theorem. Completeness of the space of bounded real-valued functions on a set, equipped with the *sup* norm, and the completeness of the space of bounded continuous real-valued functions on a metric space, equipped with the *sup* metric.

Axiomatic definition of an abstract topological space in terms of open sets. Continuous functions, homeomorphisms. Closed sets. Accumulation points of sets. Closure of a set ($\bar{A} = A$ together with its accumulation points). Interior of a set. Continuity if $f(\bar{A}) \subseteq \overline{f(A)}$. Examples to include metric spaces (definition of topological equivalence of metric spaces), discrete and indiscrete topologies, subspace topology, cofinite topology, quotient topology. Base of a topology. Product topology on a product of two spaces and continuity of projections. Hausdorff topology.

Connected spaces: closure of a connected space is connected, union of connected sets is connected if there is a non-empty intersection, continuous image of a connected space is connected. Path-connectedness implies connectedness. Connected open subset of a normed vector space is path-connected.

Compact sets, closed subset of a compact set is compact, compact subset of a Hausdorff space is closed. Heine-Borel Theorem in $\mathbb{R}^n$. Product of two compact spaces is compact. A continuous bijection from a compact space to a Hausdorff space is a homeomorphism. Equivalence of sequential compactness and abstract compactness in metric spaces.

Further discussion of quotient spaces: simple classical geometric spaces such as the torus and Klein bottle.

3.1.6 Multivariable Calculus

Definition of a derivative of a function from $\mathbb{R}^m$ to $\mathbb{R}^n$; examples; elementary properties; partial derivatives; the chain rule; the gradient of a function from $\mathbb{R}^m$ to $\mathbb{R}$; Jacobian. Continuous partial derivatives imply differentiability, Mean Value Theorems. Higher order derivatives.

The Implicit Function Theorem (proof for special case, non-examinable), the Inverse Function Theorem (proof non-examinable).

The definition of a submanifold of $\mathbb{R}^m$, its tangent space at a point. Examples, defined parametrically and implicitly, including curves and surfaces in $\mathbb{R}^3$.

Lagrange multipliers.

3.1.7 Calculus of Variations

The basic variational problem and Euler's equation. Examples, including axisymmetric soap films. Extension to several dependent variables. Hamilton's principle for free particles and particles subject to holonomic constraints. Equivalence with Newton's second law. Geodesics on surfaces. Extension to several independent variables. Examples including Laplace's equation. Lagrange multipliers and variations subject to constraint. The Rayleigh-Ritz method and eigenvalue problems for Sturm-Liouville equations.

3.1.8 Classical Mechanics

Angular momentum of a system of particles about a fixed point and about the centre of mass. The description of the motion of a rigid body with one fixed point in terms of a time-dependent rotation matrix. Definition of angular velocity. Moments of inertia, kinetic energy, and angular momentum of a rigid body with axial symmetry. Lagrangian equations of motion; holonomic constraints [derivation non-examinable]. Gyroscopes and the classical integrable cases of rigid body motion. Oscillations near equilibrium; normal frequencies, normal modes.

3.1.9 Quantum Theory

The Schrödinger equation; stationary states, quantum states of a particle in a box; interpretation of the wave function, probability density and current. Boundary conditions; conservation of current, tunnelling, parity.

Expectation values of observable eigenvalues and eigenfunctions.

The one-dimensional harmonic oscillator, higher-dimensional oscillators and normal modes.

The rotationally symmetric and general radial states of the hydrogen atom with fixed nucleus.

3.1.10 Fluid Dynamics and Waves

Incompressible flow. Convective derivative, streamlines and particle paths. Euler's equations of motion for an inviscid fluid. Bernoulli's Theorem. Vorticity, circulation and Kelvin's Theorem.

Irrotational incompressible flow; velocity potential. Two-dimensional flow, stream function and complex potential. Line sources and vortices. Method of images, circle theorem and Blasius's Theorem.

Uniform flow past a circular cylinder. Circulation, lift. Use of conformal mapping to determine flow past a flat wing. Water waves, including effects of finite depth and surface tension. Dispersion, simple introduction to group velocity. The vorticity equation and vortex motion.

3.1.11 Probability

Random variables and their distribution; joint distribution, conditional distribution; functions of one or more random variables. Generating functions and applications. Characteristic functions, definition only. Statements of the continuity and uniqueness theorems for moment generating functions. Chebychev and Markov inequalities. The weak law of large numbers and central limit theorem for independent identically distributed variables with a second moment. Discrete-time Markov chains: definition, transition matrix, n-step transition probabilities, communicating classes, absorption, irreducibility, calculation of hitting probabilities and mean hitting times, recurrence and transience. Invariant distributions, mean return time, positive recurrence, convergence to equilibrium (proof not examinable). Examples of applications in areas such as: genetics, branching processes, Markov chain Monte Carlo. Poisson processes in one dimension: exponential spacings, Poisson counts, thinning and superposition.

3.1.12 Statistics

Estimation: observed and expected information, statement of large sample properties of maximum likelihood estimators in the regular case, methods for calculating maximum likelihood estimates, large sample distribution of sample estimators using the delta method.

Hypothesis testing: simple and composite hypotheses, size, power and p-values, Neyman-Pearson Lemma, distribution theory for testing means and variances in the normal model, generalized likelihood ratio, statement of its large sample distribution under the null hypothesis, analysis of count data.

Confidence intervals: exact intervals, approximate intervals using large sample theory, relationship to hypothesis testing.

Regression: correlation, least squares and maximum likelihood estimation, use of matrices, distribution theory for the normal model, hypothesis tests and confidence intervals for linear regression problems, examining assumptions by plotting residuals.

3.1.13 Numerical Analysis

Lagrange interpolation, Newton-Cotes quadrature, Gaussian elimination and LU factorization, QR factorization. Eigenvalues: Gershgorin's theorem, symmetric QR algorithm. Best approximation in inner product spaces, least squares, orthogonal polynomials. Piecewise polynomials, splines, Richardson Extrapolation.

3.2 Synopses of Lectures

This section contains the lecture synopses associated with the two papers AO1 and AO2.

3.2.1 Introduction to Fields — Dr Kremnizer — 8 lectures HT

Weeks 1 to 4 in Hilary Term.

Overview

Informally, finite fields are generalisations of systems of real numbers such as the rational or the real numbers— systems in which the usual rules of arithmetic (including those for division) apply. Formally, fields are commutative rings with unity in which division by non-zero elements is always possible. It is a remarkable fact that the finite fields may be completely classified. Furthermore, they have classical applications in number theory, algebra, geometry, combinatorics, and coding theory, and they have newer applications in other areas. The aim of this course is to show how their structure may be elucidated, and to present the main theorems about them that lead to their various applications.

Learning Outcomes

Students will have a sound knowledge of field theory including the classification of finite fields. They will have an appreciation of the applications of this theory.

Synopsis

Fields, subfields and their intersections. Statement of the Fundamental Theorem of Algebra; the splitting field for a rational polynomial as the minimal subfield of $\mathbb{C}$ that contains all its roots, its Galois group over $\mathbb{Q}$ (basic concept only). The link between the structure of the Galois group and the solubility of equations (not examinable). [$1\frac{1}{2}$ lectures]

The characteristic of a field, prime subfields. [$\frac{1}{2}$ lecture]

Extensions of fields; examples. Degree of a finite extension, the Tower Theorem. [1 lecture]

Simple algebraic extensions; splitting fields, uniqueness (proof sketched but not examinable); examples. [2 lectures]

Finite fields: existence and uniqueness (proof sketched but not examinable), subfields. The multiplicative group of a finite field, the Frobenius automorphism. [3 lectures]

Reading

P.J. Cameron *Introduction to Algebra* (2nd. ed., OUP, 2008) pp. 99-103, 220-223, 268-276.

Joseph J. Rotman, *A First Course in Abstract Algebra* (Second edition, Prentice Hall, 2000), ISBN 0-13-011584-3. Chapters 1,3.

Further Reading

I. N. Herstein, *Topics in Algebra* (Wiley, 1975). ISBN 0-471-02371-X 5.1, 5.3, 7.1. [Harder than some, but an excellent classic. Widely available in Oxford libraries; still in print.]

P. M. Cohn, *Classic Algebra* (Wiley, 2000), ISBN 0-471-87732-8, parts of Chapter 6. [This is the third edition of his book on abstract algebra, in Oxford libraries.]

There are many other such books on abstract algebra in Oxford libraries.

3.2.2 Group Theory — Dr Szendroi — 8 lectures HT

Weeks 5 to 8 in Hilary Term.

Overview

This group theory course develops the theory of finite groups begun in Mods. In this course we will present an introduction to general "structural" theory via the Jordan-Hölder Theorem for finite groups and a basic study of finite soluble groups. This will be followed by a discussion of the concept of a "group acting on a set" which lies at the heart of the application to solving quadratic, cubic and quartic equations over the rationals but which appears wherever groups are studied throughout mathematics.

Learning Outcomes

Students will begin to have a deeper knowledge of group structure and theory, particularly finite groups. They will have an appreciation of some of the important properties of groups including simplicity, solubility and actions of a group on a set. Examples include S_n, A_n, the Möbius group, and the symmetry group of Platonic solids.

Synopsis

Brief revision of group theory: homomorphisms, normal subgroups and quotient groups, First Isomorphism Theorem. The groups S_4 and S_5 and their normal subgroups. Simplicity of A_5. [1 lecture]

Second and Third Isomorphism Theorems. Automorphisms. Semidirect products. [1 lecture]

Simplicity, composition series and Jordan-Hölder Theorem (finite groups only); examples. [See Peter J. Cameron, *Introduction to Algebra* (Oxford University Press, 1998), section 7.1.3 and 7.1.4, pages 185-187.] [1 lecture]

Finite soluble groups; subgroups, quotients and extensions. Insolubility of S_n for $n > 4$. [1 lecture]

Actions of groups on sets, equivalence of actions, examples including coset spaces and conjugation actions.[$1\frac{1}{2}$ lectures]

Orbits, transitivity, stabilisers, kernels of actions, equivalence of a transitive action with a coset space. Examples, including Möbius groups and symmetry groups of Platonic solids. [2 $\frac{1}{2}$ lectures]

Reading

P.J. Cameron *Introduction to Algebra* (2nd. ed., OUP, 2008) pp. 124-146, 237-250.

Further Reading

Peter M. Neumann, G. A. Stoy, E. C. Thompson, *Groups and Geometry* (OUP, 1994, reprinted 2002), ISBN 0-19-853451-5. Chapters 1-9, 15.

Geoff Smith, Olga Tabachnikova, *Topics in Groups Theory* (Springer Undergraduate Mathematics Series, 2002) ISBN 1-85233-2. Chapter 3.

M. A. Armstrong, *Groups and Symmetry* (Springer, 1988), ISBN 0-387-96675-7. Chapters 1-19.

Joseph J. Rotman, *A First Course in Algebra* (Second Edition, Prentice Hall, 2000). Chapter 2.

3.2.3 Number Theory — Dr Testa — 8 lectures TT

Overview

Number theory is one of the oldest parts of mathematics. For well over two thousand years it has attracted professional and amateur mathematicians alike. Although notoriously 'pure' it has turned out to have more and more applications as new subjects and new technologies have developed. Our aim in this course is to introduce students to some classical and important basic ideas of the subject.

Synopsis

The ring of integers; congruences; ring of integers modulo n; the Chinese Remainder Theorem. [2 lectures]

Wilson's Theorem; Fermat's Little Theorem for prime modulus; Euler's generalisation of Fermat's Little Theorem to arbitrary modulus; primitive roots. [2 lectures]

Quadratic residues modulo primes. Quadratic reciprocity. [2 lectures]

Factorisation of large integers; basic version of the RSA encryption method. [2 lectures]

Reading

Alan Baker, *A Concise Introduction to the Theory of Numbers* (Cambridge University Press, 1984) ISBN: 0521286549 Chapters 1,3,4.

David Burton, *Elementary Number Theory* (McGraw-Hill, 2001).

Dominic Welsh, *Codes and Cryptography*, (Oxford University Press, 1988), ISBN 0-19853-287-3. Chapter 11.

3.2.4 Integration — Prof. Etheridge — 16 lectures HT

Overview

The course will exhibit Lebesgue's theory of integration in which integrals can be assigned to a huge range of functions on the real line, thereby greatly extending the notion of integration presented in Mods. The theory will be developed in such a way that it can be easily extended to a wider framework including summation of series and probability theory (although no knowledge of probability will be required), but measures other than Lebesgue's will only be lightly touched.

Operations such as passing limits, infinite sums, or derivatives, through integral signs, or reversing the order of double integrals, are often taken for granted in courses in applied mathematics. Actually, they can occasionally fail. Fortunately, there are powerful convergence and other theorems allowing such operations to be justified under conditions which are widely applicable. The course will display these theorems and a wide range of their applications.

This is a course in rigorous applications. Its principal aim is to develop understanding of the statements of the theorems and how to apply them carefully. Knowledge of technical proofs concerning the construction of Lebesgue measure and the integral will not be an essential part of the course, and such proofs will usually be omitted from the lectures.

Synopsis

Motivation: Why do we need a more general theory of integration?
The notion of measure.
Key examples: Lebesgue measure, probability measure, counting measure.
Measurable functions, integrable functions (via simple functions). Reconciliation with Mods Analysis III. Changes of variable.
Comparison Theorem.
Fatou's Lemma.
Monotone Convergence Theorem.
Dominated Convergence Theorem.
Corollaries and applications of the Convergence Theorems (term-by-term integration of series etc). Differentiation under the integral sign.
Double integrals, theorems of Fubini and Tonelli, changes of variable.
A very brief introduction to L^p spaces. Hölder and Minkowski inequalities.

Reading

A. Etheridge, *Integration,* Mathematical Institute Lecture Notes

M. Capinski & E. Kopp, *Measure, Integral and Probability* (Second Edition, Springer, 2004).

F. Jones, *Lebesgue Integration on Euclidean Space* (Second Edition, Jones & Bartlett, 2000).

Further Reading

R. G. Bartle, *The Elements of Integration* (Wiley, 1966).

D. S. Kurtz & C. W. Swartz, *Theories of Integration* (Series in Real Analysis Vol.9, World Scientific, 2004).

H. A. Priestley, *Introduction to Integration* (OUP 1997).

[Useful for worked examples, although adopts a different approach to construction of the integral].

H. L. Royden, *Real Analysis* (Third Edition, Macmillan, 1988).

E. M. Stein & R. Shakarchi, *Real Analysis: Measure Theory, Integration and Hilbert Spaces* (Princeton Lectures in Analysis III, Princeton University Press,2005).

3.2.5 Topology — Dr Drutu — 16 lectures HT

Overview

The ideas, concepts and constructions in general topology arose from extending the notions of continuity and convergence on the real line to more general spaces. The first class of general spaces to be studied in this way were metric spaces, a class of spaces which includes many of the spaces used in analysis and geometry. Metric spaces have a distance function which allows the use of geometric intuition and gives them a concrete feel. They allow us to introduce much of the vocabulary used later and to understand the formulation of continuity which motivates the axioms in the definition of an abstract topological space.

The axiomatic formulation of a topology leads to topological proofs of simplicity and clarity often improving on those given for metric spaces using the metric and sequences. There are many examples of topological spaces which do not admit metrics and it is an indication of the naturality of the axioms that the theory has found so many applications in other branches of mathematics and spheres in which mathematical language is used.

Learning Outcomes

The outcome of the course is that a student should understand and appreciate the central results of general topology and metric spaces, sufficient for the main applications in geometry, number theory, analysis and mathematical physics, for example.

Synopsis

Metric spaces. Examples to include metrics derived from a norm on a real vector space, particularly l^1, l^2, l^∞ norms on $\mathbb{R}^n$, the *sup* norm on the bounded real-valued functions on a set, and on the bounded continuous real-valued functions on a metric space. Continuous functions (ϵ, δ definition). Uniformly continuous functions; examples include Lipschitz functions and contractions. Open balls, open sets, accumulation points of a set. Completeness (but not completion). Contraction Mapping Theorem. Completeness of the space of bounded real-valued functions on a set, equipped with the *sup* norm, and the completeness of the space of bounded continuous real-valued functions on a metric space, equipped with the *sup* metric. Example: completion of a metric space. [3 lectures].

Axiomatic definition of an abstract topological space in terms of open sets. Continuous functions, homeomorphisms. Closed sets. Accumulation points of sets. Closure of a set ($\bar{A} = A$ together with its accumulation points). Interior of a set. Continuity if $f(\bar{A}) \subseteq \overline{f(A)}$. Examples to include metric spaces (definition of topological equivalence of metric spaces), discrete and indiscrete topologies, subspace topology, cofinite topology, quotient topology. Base of a topology. Product topology on a product of two spaces and continuity of projections. Hausdorff topology. [5 lectures]

Connected spaces: closure of a connected space is connected, union of connected sets is connected if there is a non-empty intersection, continuous image of a connected space is connected. Path-connectedness implies connectedness. Connected open subset of a normed vector space is path-connected. [2 lectures]

Compact sets, closed subset of a compact set is compact, compact subset of a Hausdorff space is closed. Heine-Borel Theorem in $\mathbb{R}^n$. Product of two compact spaces is compact. A continuous bijection from a compact space to a Hausdorff space is a homeomorphism. Equivalence of sequential compactness and abstract compactness in metric spaces. [4 lectures]

Further discussion of quotient spaces explaining some simple classical geometric spaces such as the torus and Klein bottle. [2 lectures]

Reading

W. A. Sutherland, *Introduction to Metric and Topological Spaces* (Oxford University Press, 1975). Chapters 2-6, 8, 9.1-9.4.
(New edition to appear shortly.)

J. R. Munkres, *Topology, A First Course* (Prentice Hall, 1974), chapters 2, 3, 7.

Further Reading

B. Mendelson, *Introduction to Topology* (Allyn and Bacon, 1975). (cheap paperback edition available).

G. Buskes, A. Van Rooij, *Topological Spaces* (Springer, 1997).

N. Bourbaki, *General Topology* (Springer, 1998).

J. Dugundji, *Topology* (Allyn and Bacon, 1966), chapters 3, 4, 5, 6, 7, 9, 11. [Although out of print, available in some libraries.]

3.2.6 Multivariable Calculus — Prof. Niethammer — 8 lectures TT

Overview

In this course, the notion of the total derivative for a function $f\colon \mathbb{R}^m \to \mathbb{R}^n$ is introduced. Roughly speaking, this is an approximation of the function near each point in $\mathbb{R}^n$ by a linear transformation. This is a key concept which pervades much of mathematics, both pure and applied. It allows us to transfer results from linear theory locally to nonlinear functions. For example, the Inverse Function Theorem tells us that if the derivative is an invertible linear mapping at a point then the function is invertible in a neighbourhood of this point. Another example is the tangent space at a point of a surface in $\mathbb{R}^3$, which is the plane that locally approximates the surface best.

Synopsis

Definition of a derivative of a function from $\mathbb{R}^m$ to $\mathbb{R}^n$; examples; elementary properties; partial derivatives; the chain rule; the gradient of a function from $\mathbb{R}^m$ to $\mathbb{R}$; Jacobian. Continuous partial derivatives imply differentiability, Mean Value Theorems. Higher order derivatives. [3 lectures]

The Inverse Function Theorem and the Implicit Function Theorem (proofs non-examinable). [2 lectures]

The definition of a submanifold of $\mathbb{R}^m$. Its tangent and normal space at a point, examples, including two-dimensional surfaces in $\mathbb{R}^3$. [2 lectures]

Lagrange multipliers. [1 lecture]

Reading

Theodore Shifrin, *Multivariable Mathematics* (Wiley, 2005). Chapters 3-6.

T. M. Apostol, *Mathematical Analysis: Modern Approach to Advanced Calculus (World Students)* (Addison Wesley, 1975). Chapters 12 and 13.

S. Dineen, *Multivariate Calculus and Geometry* (Springer, 2001). Chapters 1-4.

J. J. Duistermaat and J A C Kolk, *Multidimensional Real Analysis I, Differentiation* (Cambridge University Press, 2004).

Further Reading

William R. Wade, *An Introduction to Analysis* (Second Edition, Prentice Hall, 2000). Chapter 11.

M. P. Do Carmo, *Differential Geometry of Curves and Surfaces* (Prentice Hall, 1976).

Stephen G. Krantz and Harold R. Parks, *The Implicit Function Theorem: History, Theory and Applications* (Birkhaeuser, 2002).

3.2.7 Calculus of Variations — Prof. Tod — 8 lectures HT

Weeks 1 to 4 in Hilary Term

Overview

The calculus of variations concerns problems in which one wishes to find the minima or extrema of some quantity over a system that has functional degrees of freedom. Many important problems arise in this way across pure and applied mathematics and physics. They range from the problem in geometry of finding the shape of a soap bubble, a surface that minimizes its surface area, to finding the configuration of a piece of elastic that minimises its energy. Perhaps most importantly, the principle of least action is now the standard way to formulate the laws of mechanics and basic physics.

In this course it is shown that such variational problems give rise to a system of differential equations, the Euler-Lagrange equations. Furthermore, the minimizing principle that underlies these equations leads to direct methods for analysing the solutions to these equations. These methods have far reaching applications and will help develop students technique.

Learning Outcomes

Students will be able to formulate variational problems and analyse them to deduce key properties of system behaviour.

Synopsis

The basic variational problem and Euler's equation. Examples, including axi-symmetric soap films. Extension to several dependent variables. Hamilton's principle for free particles and particles subject to holonomic constraints. Equivalence with Newton's second law. Geodesics on surfaces. Extension to several independent variables. Examples including Laplace's equation. Lagrange multipliers and variations subject to constraint. The Rayleigh-Ritz method and eigenvalue problems for Sturm-Liouville equations.

Reading

Arfken Weber, *Mathematical Methods for Physicists* (5th edition, Academic Press, 2005). Chapter 17.

Further Reading

N. M. J. Woodhouse, *Introduction to Analytical Dynamics* (1987). Chapter 2 (in particular 2.6). (This is out of print, but still available in most College libraries.)

M. Lunn, *A First Course in Mechanics* (OUP, 1991). Chapters 8.1, 8.2.

P. J. Collins, *Differential and Integral Equations* (O.U.P., 2006). Chapters 11, 12.

3.2.8 Classical Mechanics — Prof. Tod — 8 lectures HT

Weeks 5 to 8 in Hilary Term

Overview

This course extends the study of the dynamics of point particles in the first year to the study of extended rigid bodies moving in three dimensions.

The course provides powerful applications of the Lagrangian theory to a range of systems, in particular to the study of small oscillations near equilibrium, and it introduces some key classical ideas that also play an important role in modern physical theory, notably angular momentum and its connection with rotations.

Synopsis

Lagrangian equations of motion with and without holonomic constraints. Oscillations near equilibrium; normal frequencies, normal modes.

Angular momentum of a system of particles about a fixed point and about the centre of mass. The description of the motion of a rigid body with one fixed point in terms of a time-dependent rotation matrix. Definition of angular velocity. Moments of inertia, kinetic energy, and angular momentum of a rigid body with axial symmetry. Gyroscopes and the classical integrable cases of rigid body motion.

Reading

N. M. J. Woodhouse, *Introduction to Analytical Mechanics* (1987). Chapters 3 and 6. (This is out of print, but still available in most College libraries.)

Further Reading

M. Lunn, *A First Course in Mechanics* (OUP, 1991). Chapters 6, 7.2, 7.3, 8.3 and 8.4.

3.2.9 Quantum Theory — Dr Sparks — 8 lectures HT

Lectures are one hour each week in HT

Overview

Quantum theory was born out of the attempt to understand the interactions between radiation, described by Maxwell's theory of electromagnetism, and matter, described by Newton's mechanics.

Although there remain deep mathematical and physical questions at the frontiers of the subject, the resulting theory encompasses not just the mechanical but also the electrical and chemical properties of matter. Many of the key components of modern technology such as transistors and lasers were developed using quantum theory.

In quantum theory particles also have some wave-like properties. This introductory course explores some of the consequences of this culminating in an elementary treatment of the hydrogen atom.

Synopsis

The Schrödinger equation; stationary states, quantum states of a particle in a box; interpretation of the wave function, probability density and current. Boundary conditions; conservation of current, tunnelling, parity.

Expectation values of observable eigenvalues and eigenfunctions.

The one-dimensional harmonic oscillator, higher-dimensional oscillators and normal modes.

The rotationally symmetric and general radial states of the hydrogen atom with fixed nucleus.

Reading

K C Hannabuss, *Introduction to Quantum Mechanics* (OUP, 1997). Chapters 1-5.

Further Reading

B J Bransden and C J Joachain, *Introduction to Quantum Mechanics* (Longman, 1995).

I P Grant, *Mathematical Institute Notes* (1991).

A I M Rae, *Quantum Mechanics* (3rd edition, Institute of Physics, 1993).

L I Schiff, *Quantum Mechanics* (3rd edition, Mc Graw Hill, 1968).

3.2.10 Fluid Dynamics and Waves — Dr Howell — 16 lectures HT

Overview

This course introduces students to the mathematical theory of inviscid fluids. The theory provides insight into physical phenomena such as flight, vortex motion, and water waves. The course also explains important concepts such as conservation laws and dispersive waves and, thus, serves as an introduction to the mathematical modelling of continuous media.

Synopsis

Incompressible flow. Convective derivative, streamlines and particle paths. Euler's equations of motion for an inviscid fluid. Bernoulli's theorem. Vorticity, circulation and Kelvin's Theorem.

Irrotational incompressible flow; velocity potential. Two-dimensional flow, stream function and complex potential. Line sources and vortices. Method of images, circle theorem and Blasius's Theorem.

Uniform flow past a circular cylinder. Circulation, lift. Use of conformal mapping to determine flow past a flat wing. Water waves, including effects of finite depth and surface tension. Dispersion, simple introduction to group velocity. The vorticity equation and vortex motion.

Reading

D. J. Acheson, *Elementary Fluid Dynamics* (OUP, 1997). Chapters 1, 3.1-3.5, 4.1-4.8, 4.10-4.12, 5.1, 5.2, 5.6, 5.7.

3.2.11 Probability — Dr Laws — 16 lectures HT

Overview

The first half of the course takes further the probability theory that was developed in the first year. The aim is to build up a range of techniques that will be useful in dealing with mathematical models involving uncertainty. The second half of the course is concerned with Markov chains in discrete time and Poisson processes in one dimension, both with developing the relevant theory and giving examples of applications.

Synopsis

Continuous random variables. Jointly continuous random variables, independence, conditioning, bivariate distributions, functions of one or more random variables. Moment generating functions and applications. Characteristic functions, definition only. Examples to include some of those which may have later applications in Statistics.

Basic ideas of what it means for a sequence of random variables to converge in probability, in distribution and in mean square. Chebychev and Markov inequalities. The weak law of large numbers and central limit theorem for independent identically distributed variables with a second moment. Statements of the continuity and uniqueness theorems for moment generating functions.

Discrete-time Markov chains: definition, transition matrix, n-step transition probabilities, communicating classes, absorption, irreducibility, calculation of hitting probabilities and mean hitting times, recurrence and transience. Invariant distributions, mean return time, positive recurrence, convergence to equilibrium (proof not examinable). Examples of applications in areas such as: genetics, branching processes, Markov chain Monte Carlo. Poisson processes in one dimension: exponential spacings, Poisson counts, thinning and superposition.

Reading

G. R. Grimmett and D. R. Stirzaker, *Probability and Random Processes* (3rd edition, OUP, 2001). Chapters 4, 6.1-6.5, 6.8.

R. Grimmett and D. R. Stirzaker, *One Thousand Exercises in Probability* (OUP, 2001).

G. R. Grimmett and D J A Welsh, *Probability: An Introduction* (OUP, 1986). Chapters 6, 7.4, 8, 11.1-11.3.

J. R. Norris, *Markov Chains* (CUP, 1997). Chapter 1.

D. R. Stirzaker, *Elementary Probability* (Second edition, CUP, 2003). Chapters 7-9 excluding 9.9.

3.2.12 Statistics — Dr Myers — 16 lectures HT

Overview

Building on the first year course, this course develops statistics for mathematicians, emphasising both its underlying mathematical structure and its application to the logical interpretation of scientific data. Advances in theoretical statistics are generally driven by the need to analyse new and interesting data which come from all walks of life.

Synopsis

Estimation: observed and expected information, statement of large sample properties of maximum likelihood estimators in the regular case, methods for calculating maximum likelihood estimates, large sample distribution of sample estimators using the delta method.

Hypothesis testing: simple and composite hypotheses, size, power and p-values, Neyman-Pearson Lemma, distribution theory for testing means and variances in the normal model, generalized likelihood ratio, statement of its large sample distribution under the null hypothesis, analysis of count data.

Confidence intervals: exact intervals, approximate intervals using large sample theory, relationship to hypothesis testing.

Regression: correlation, least squares and maximum likelihood estimation, use of matrices, distribution theory for the normal model, hypothesis tests and confidence intervals for linear regression problems, examining assumptions by plotting residuals.

Examples: statistical techniques will be illustrated with relevant data sets in the lectures.

Reading

F. Daly, D. J. Hand, M. C. Jones, A. D. Lunn and K. J. McConway, *Elements of Statistics* (Addison Wesley, 1995). Chapters 7-10 (and Chapters 1-6 for background).

J. A. Rice, *Mathematical Statistics and Data Analysis* (2nd edition, Wadsworth, 1995). Sections 8.5, 8.6, 9.1-9.7, 9.9, 10.3-10.6, 11.2, 11.3, 12.2.1, 13.3, 13.4.

Further Reading

G. Casella and R. L. Berger, *Statistical Inference* (2nd edition, Wadsworth, 2001).

3.2.13 Numerical Analysis — Prof. Wendland — 16 lectures HT

Overview

Scientific computing pervades our lives: modern buildings and structures are designed using it, medical images are reconstructed for doctors using it, the cars and planes we travel on are designed with it, the pricing of "Instruments" in the financial market is done using it, tomorrow's weather is predicted with it. The derivation and study of the core, underpinning algorithm for this vast range of applications defines the subject of Numerical Analysis. This course gives an introduction to that subject.

Through studying the material of this course students should gain an understanding of numerical methods, their derivation, analysis and applicability. They should be able to solve certain mathematically posed problems using numerical algorithms. This course is designed to introduce numerical methods - i.e. techniques which lead to the (approximate) solution of mathematical problems which are usually implemented on computers. The course covers derivation of useful methods and analysis of their accuracy and applicability.

The course begins with a study of methods and errors associated with computation of functions which are described by data values (interpolation or data fitting). Following this we turn to numerical methods of linear algebra, which form the basis of a large part of computational mathematics, science, and engineering. Key ideas here include algorithms for linear equations, least squares, and eigenvalues built on LU and QR matrix factorizations. The course will also include the simple and computationally convenient approximation of curves: this includes the use of splines to provide a smooth representation of complicated curves, such as arise in computer aided design. Use of such representations leads to approximate methods of integration. Techniques for improving accuracy through extrapolation will also be described. The course requires elementary knowledge of functions and calculus and of linear algebra.

Although there are no assessed practicals for this course, the classwork will involve a mix of written work and Matlab programming. No previous knowledge of Matlab is required. Specifically, like Numerical Solution of Differential Equations, Numerical Analysis has 16 lectures, no practicals, and 7 classes per term. There will be some simple use of Matlab which will be demonstrated both in lectures and in problem classes.

Learning Outcomes

At the end of the course the student will know how to:

- Find the solution of linear systems of equations.
- Compute eigenvalues and eigenvectors of matrices.
- Approximate functions of one variable by polynomials and piecewise polynomials (splines).
- Compute good approximations to one-dimensional integrals.
- Increase the accuracy of numerical approximations by extrapolation.
- Use Matlab to achieve these goals.

Synopsis

Lagrange interpolation [1 lecture],

Newton-Cotes quadrature [2 lectures],

Gaussian elimination and LU factorization [2 lectures],

QR factorization [1 lecture],

Eigenvalues: Gershgorin's Theorem, symmetric QR algorithm [3 lectures],

Best approximation in inner product spaces, least squares, orthogonal polynomials [4 lectures],

Piecewise polynomials, splines [2 lectures],

Richardson Extrapolation. [1 lecture].

Reading

You can find the material for this course in many introductory books on Numerical Analysis such as

A. Quarteroni, R Sacco and F Saleri, *Numerical Mathematics* (Springer, 2000).

K. E. Atkinson, *An Introduction to Numerical Analysis* (2nd Edition, Wiley, 1989).

S. D. Conte and C. de Boor, *Elementary Numerical Analysis* (3rd Edition, Graw-Hill, 1980).

G. M. Phillips and P. J. Taylor, *Theory and Applications of Numerical Analysis* (2nd Edition, Academic Press, 1996).

W. Gautschi, *Numerical Analysis: An Introduction* (Birkhauser, 1977).

H. R. Schwarz, *Numerical Analysis: A Comprehensive Introduction* (Wiley, 1989).

But the main recommended book for this course is:

E. Suli and D. F. Mayers, *An Introduction to Numerical Analysis* (CUP, 2003). Of which the relevant chapters are: 6, 7, 2, 5, 9, 11.