Handbook for the Undergraduate Mathematics Courses
Supplement to the Handbook
Honour School of Mathematics
Syllabus and Synopses for Part C 2006–7
for examination in 2007

Contents

1 Foreword

Every effort is made to ensure that the list of courses offered is accurate at time of going online (Trinity Term 2006). However, students are advised to check the up-to-date version of this document before the start of Michaelmas Term 2006. It will be available in hard copy from reception in the Mathematical Institute in September 2006, and on the website at:

http://www.maths.ox.ac.uk/current-students/undergraduates/handbooks-synopses/

before the start of Michaelmas Term 2006.

Notice of misprints or errors of any kind, and suggestions for improvements in this booklet should be addressed to the Academic Assistant in the Mathematical Institute.

Honour School of Mathematics

Students staying on to take the four-year course will take the equivalent of 3 units from Part C in their fourth year. The equivalent of two units must be taken from the menu of "Mathematics department units" or offered as a Dissertation on a mathematical topic. Up to one unit may be taken from the menu of "Other units".

In the classification awarded at the end of the degree, unit paper marks in Part A will be given a 'weighting' of 2, and unit marks in Part B will be given a 'weighting' of 3. For those students staying on to do the fourth year, the Part C unit marks will also be taken into account with a 'weighting' of 4.

Honour School of Mathematics & Philosophy

In Part C each candidate shall offer a total of three units chosen in any combination from the lists for Mathematics and for Philosophy. Units in Mathematics are given in this booklet.

Honour School of Mathematics & Statistics

See details published by the Statistics Department.

Registration for Mathematics and Mathematics & Philosophy students

YOU MUST REGISTER BY THE END OF TRINITY TERM 2006 FOR LECTURE AND CLASS ATTENDANCE FOR ALL COURSES YOU WISH TO TAKE IN 2006–07. A REGISTRATION FORM IS ATTACHED TO THESE SYNOPSES.

SOME COMBINATIONS OF SUBJECTS ARE NOT ADVISED AND LECTURES IN THESE SUBJECTS MAY CLASH. HOWEVER, WHEN TIMETABLING LECTURES WE WILL AIM TO KEEP CLASHES TO A MINIMUM. WE WILL USE THE INFORMATION ON YOUR REGISTRATION FORMS TO PLAN CLASS TEACHING AND LECTURING. IT IS THEREFORE IMPORTANT THAT YOU RETURN YOUR REGISTRATION FORM BY MONDAY, WEEK 9, TRINITY TERM 2006. IF YOU NEED TO CHANGE YOUR REGISTRATIONS AFTER THAT DATE, PLEASE INFORM THE ACADEMIC ASSISTANT AS SOON AS POSSIBLE.

“Units” and “Half-Units” and methods of examination

All Mathematics Department lecture courses given in Michaelmas Term, other than C7.3 Advanced Quantum Mechanics are available as half-units. Most Hilary Term courses are available as half-units, the exceptions being:

1. 5.1b Calculus of Variations, where 5.1a PDEs for Pure and Applied Mathematicians is an essential pre-requisite.
2. C11.1b Probabilistic Combinatorics, where C11.1a Graph Theory is an essential pre-requisite.

Half units are examined in an examination paper of $1\frac{3}{4}$ hours, unless the synopsis states otherwise; whole units will be examined in a 3 hour exam unless the synopsis states otherwise.

All Computer Science half-units will be examined by a paper of $1\frac{1}{2}$ hours in length.

All the units and half-units described in this booklet are “M–Level”.

2 Mathematics Department Units

C1.1: Model Theory and Gödel's Incompleteness Theorems

Method of examination: 3 hour (for whole unit) or $1\frac{3}{4}$ hour (for half unit)paper.

C1.1a: Gödel's Incompleteness Theorems — Dr D Isaacson — 16MT

Recommended Prerequisites

This course presupposes basic knowledge of First Order Predicate Calculus up to and including the Soundness and Completeness Theorems.

Aims & Objectives

This course introduces important techniques and results in modern logic which go to the heart of the relationship between truth and formal proof, in particular that show how to obtain, for any consistent formal system containing basic arithmetic, a sentence in the language of that system which is true but not provable in the system.

Synopsis

Gödel numbering of a formal language; the diagonal lemma. The arithmetical undefinability of truth in arithmetic. Formal systems of arithmetic; arithmetical proof predicates. Gödel's β-function; the representation of functions and sets. ω-consistency; the first Gödel incompleteness theorem. Σ_0-completeness. Separability; the Rosser incompleteness theorem. Adequacy conditions for a provability predicate. The second Gödel incompleteness theorem; Löb's theorem. Σ_1-completeness. Abstract provability systems; the logic of provability. The undecidability of first-order logical validity. The Hilbert–Bernays arithmetized completeness theorem; a formally undecided sentence of arithmetic whose truth value is not known. The ω-rule.

Method of examination

4 examination questions.

Reading

1. R.M. Smullyan, *Gödel's Incompleteness Theorems*, OUP (1992).

Further Reading

1. G.S. Boolos and R.C. Jeffrey, *Computability and Logic*, 3rd edition, CUP (1989), Chs 15, 16, pp 170–190.

C1.1b: Model Theory — Prof B Zilber — 16HT

Recommended Prerequisites

This course presupposes basic knowledge of First Order Predicate Calculus up to and including the Soundness and Completeness Theorems. Also a familiarity with (at least the statement of) the Compactness Theorem would also be desirable.

Aims & Objectives

1. To advance the students' knowledge of the notion of an algebraic mathematical structure and of its logical theory, taking B1 Logic as a starting point.
2. To study the concepts of completeness and categoricity and introduce some more advanced technical notions.

Learning outcomes

The course deepens student's understanding of the notion of a mathematical structure and of the logical formalism that underlies every mathematical theory, taking B1 Logic as a starting point. Various examples emphasise the connection between logical notions and practical mathematics.
The concepts of completeness and categoricity will be studied and some more advanced technical notions, up to elements of modern stability theory, will be introduced.

Synopsis

Structures. The first-order language for structures. The Compactness Theorem for first-order logic. Elementary embeddings. Löwenheim-Skolem theorems. Preservation theorems for substructures.

Categoricity for first-order theories. Types and saturation. Omitting types. The Ryll-Nardzewski theorem characterizing aleph-zero categorical theories. Theories with few types.

Method of examination

4 examination questions.

Reading

1. J. Bridge, *Beginning Model Theory*, OUP (1977). (Out of print but can be found in libraries).
2. W. Hodges, *Shorter Model Theory*, CUP (1997).

Further reading

1. All topics discussed (and much more) can also be found in W. Hodges, *Model Theory*. CUP, (1993).

C1.2: Axiomatic Set Theory and Analytic Topology

Method of examination: 3 hour (for whole unit) or $1\frac{3}{4}$ hour (for half unit)paper.

C1.2a: Analytic Topology — Dr R Knight — 16MT

We find it necessary to run Analytic Topology in MT this year. Classes for Analytic Topology will be run both in MT and HT, so that students who find themselves overburdened in MT will have the opportunity to attend classes on Analytic Topology in HT.

Recommended Prerequisites

Part A Topology

Learning outcomes

The aim of the course is to present a range of major theory and theorems, both important and elegant in themselves and with important applications within topology and to mathematics as a whole. Central to the course is the general theory of compactness and Tychonoff's Theorem, one of the most important in all mathematics (with applications across mathematics and in mathematical logic) and computer science.

Synopsis

Separation axioms, Urysohn's Lemma. Separable, Lindelöf and second countable spaces. Urysohn's Metrisation Theorem. Filters and ultrafilters. Tychonoff's Theorem. Compactifications, in particular, the Alexandroff One-Point Compactification and the Stone-Čech Compactification; proper mappings. Paracompact spaces and Stone's Theorem. Connectedness and local connectedness. Components and quasi-components. Sura-Bura Lemma.

Method of examination

4 examination questions.

Reading

1. S. Willard, *General Topology*, Addison-Wesley (1970), Chs. 1–8.
2. N. Bourbaki, *General Topology*, Springer-Verlag (1989), Ch. 1.

C1.2b: Axiomatic Set Theory — Prof A Wilkie — 16HT

Recommended Prerequisites

This course presupposes basic knowledge of First Order Predicate Calculus up to and including the Soundness and Completeness Theorems, together with a course on basic set theory, including cardinals and ordinals,the Axiom of Choice and the Well Ordering Principle.

Aims & Objectives

Inner models and consistency proofs lie at the heart of modern Set Theory, historically as well as in terms of importance. In this course we shall introduce the first and most important of inner models, Gödel's constructible universe, and use it to derive some fundamental consistency results.

Synopsis

A review of the axioms of ZF set theory. The recursion theorem for the set of natural numbers and for the class of ordinals. The Cumulative Hierarchy of sets and the consistency of the Axiom of Foundation as an example of the method of inner models. Levy's Reflection Principle. Gödel's inner model of constructible sets and the consistency of the Axiom of Constructibility ($V = L$). The fact that $V = L$ implies the Axiom of Choice. Some advanced cardinal arithmetic. The fact that $V = L$ implies the Generalized Continuum Hypothesis.

Method of examination

4 examination questions.

Reading

For the review of ZF set theory:

1. D. Goldrei, *Classic Set Theory*, Chapman and Hall (1996).

For course topics (and much more):

1. K. Kunen, *Set Theory: An Introduction to Independence Proofs*, North Holland (1983) (now in paperback). Review: Chapter 1. Course topics: Chapters 3, 4, 5, 6 (excluding section 5).

Further reading

1. K. Hrbacek and T. Jech, *Introduction to Set Theory*, 3rd edition, M Dekker (1999).

C2.1: Lie Algebras and Representation Theory of Symmetric Groups

Method of examination: 3 hour (for whole unit) or $1\frac{3}{4}$ hour (for half unit)paper.

C2.1a: Lie Algebras — Prof J Wilson — 16MT

Learning outcomes

Lie Algebras are mathematical objects which, besides being of interest in their own right, elucidate problems in several areas in mathematics. The classification of the finite-dimensional complex Lie algebras is a beautiful piece of applied linear algebra. The aims of this course are to introduce Lie algebras, develop some of the techniques for studying them, and describe parts of the classification mentioned above, especially the parts concerning root systems and Dynkin diagrams. Students will learn how to utilise various techniques for working with Lie algebras, and they will gain an understanding of parts of a major classification result.

Prerequisites

A thorough knowledge of linear algebra and the second year algebra courses; in particular familiarity with group actions, quotient rings and vector spaces, with isomorphism theorems and with inner product spaces will be assumed. Some familiarity with the Jordan–Hölder theorem and the general ideas of representation theory will be an advantage.

Synopsis

Definition of Lie algebras, small dimensional examples, some classical groups and their Lie algebras (treated informally). Ideals, subalgebras, homomorphisms, modules.

Nilpotent algebras, Engel's theorem; soluble algebras, Lie's theorem. Semisimple algebras and Killing form, Cartan's criteria for solubility and semisimplicity.

The root space decomposition of a Lie algebra; root systems, Cartan matrices and Dynkin diagrams. Classification of irreducible root systems. Description (with few proofs) of the classification of complex simple Lie algebras; examples.

Method of examination

Four examination questions.

Reading

1. J.E. Humphreys. *Introduction to Lie Algebras and Representation Theory*, Graduate Texts in Mathematics 9. Springer-Verlag (1972, reprinted 1997). Chapters 1–3 are relevant and part of the course will follow Chapter 3 closely.

2. B. Hall. *Lie Groups, Lie Algebras, and Representations. An Elementary Introduction*. Graduate Texts in Mathematics, 222, Springer-Verlag (2003).

3. K. Erdmann, M. J. Wildon. *Introduction to Lie Algebras*. Springer-Verlag (2006)ISBN: 1846280400.

Additional reading

1. J.-P. Serre. *Complex Semisimple Lie Algebras*. Springer (1987). Rather condensed, assumes the basic results. Very elegant proofs.
2. N. Bourbaki. *Lie Algebras and Lie Groups*. (Masson, 1982). Chapters 1 and 4–6 are relevant; this text fills in some of the gaps in Serre's text.

C2.1b — Representation Theory of Symmetric Groups — Dr Henke — 16HT

Method of assessment

Examination questions (Part C), Weight: half-unit, Level: Part C.

Detailed course synopsis

Prerequisites: first and second year linear algebra including inner product spaces, symmetric groups, actions of (symmetric) groups; very helpful too is the basic representation theory from B2 (group algebras, simple modules, reducibility, Maschke's theorem, Wedderburn's theorem, characters).

Learning outcomes

The representation theory of the symmetric groups is a special case of the representation theory of finite groups. While representations over a field of characteristic zero are well-understood, fundamental questions over a field of prime characteristic remain widely open. The course will be algebraic and combinatorial in flavour. It will follow the approach taken by G. James. The final goal is to construct and parametrise the simple modules of the symmetric groups over an arbitrary field. On the way the theory over a field of characteristic zero will be developed. Other highlights of the course include combinatorial algorithms like the Robinson-Schensted-Knuth correspondence.

Detailed Synopsis

Reminder on the background needed. Partitions, conjugacy classes of symmetric groups and parametrisations of simple modules. Group algebras, modules, simple modules and characters.

Counting standard tableaux of fixed shape: Young diagrams, Young tableaux, standard tableau, Young-Frobenius formula, hook numbers, hook formula.

Robinson-Schensted-Knuth algorithm, its inverse and the Robinson-Schensted-Knuth correspondence.

Construction of fundamental modules for the symmetric groups: Action of symmetric groups on tableaux, tabloids, permutation modules, polytabloids, Specht modules, standard basis for Specht modules. Young's rule.

S_n-modules over a field of characteristic zero: Basic combinatorial lemma, simplicity of Specht modules in characteristic zero, classification of simple S_n-modules in characteristic zero. Character tables of symmetric groups, Murnaghan-Nakayama rule to calculate character values. Applications.

S_n-modules over a field of prime characteristic: Submodule Theorem, construction of simple S_n-modules, classification of simple S_n-modules. Applications.

Reading

1. W Fulton. *Young Tableaux*, London Mathematical Society Student Texts 35. Cambridge University Press (1997). From Part I and II.

2. D Knuth. *The Art of Computer Programming*, Volume 3. Addison-Wesley (1998). From Chapter 5.

3. B E Sagan. *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*, Graduate Texts in Mathematics 203. Springer-Verlag (2000). Chapters 1 - 2.

Further Reading

1. W Fulton, J Harris. *Representation Theory. A first course*, Graduate Texts in Mathematics. Readings in Mathematics 129. Springer Verlag (1991). From Part I.

2. G James. *The Representation Theory of the Symmetric Groups*, Lecture Notes in Mathematics 682, Springer Verlag (1978).

3. G James, A Kerber. *The Representation Theory of the Symmetric Groups*, Encyclopaedia of Mathematics and its Applications 16. Addison-Wesley (1981). From Chapter 7.

4. R Stanley. *Enumerative Combinatorics. Volume 2*, Cambridge Studies in Advanced Mathematics 62. Cambridge University Press (1999).

C3.1 Topology and Groups and Algebraic Topology

Method of examination: 3hour (for whole unit) or $1\frac{3}{4}$ hour (for half unit) paper.

C3.1a Topology and Groups – Dr Lackenby – 16 MT

Prerequisites

2nd year Groups in action 2nd year Topology

Learning outcomes

This course introduces the important link between topology and group theory. On the one hand, associated to each space, there is a group, known as its fundamental group. This can be used to solve topological problems using algebraic methods. On the other hand, many results about groups are best proved and understood using topology. For example, presentations of groups, where the group is defined using generators and relations, have a topological interpretation. The endpoint of the course is the Nielsen-Shreier theorem, an important, purely algebraic result, which is proved using topological techniques.

Synopsis

Homotopic mappings, homotopy equivalence. Simplicial complexes. Simplicial approximation theorem.

The fundamental group of a space. The fundamental group of a circle. Application: the fundamental theorem of algebra. The fundamental groups of spheres.

Free groups. Existence and uniqueness of reduced representatives of group elements. The fundamental group of a bouquet of circles.

Groups defined by generators and relations (with examples). Tietze transformations.

The Seifert van Kampen theorem.

Cell complexes. The fundamental group of a cell complex (with examples).

The realization of any finitely presented group as the fundamental group of a finite cell complex. Van Kampen diagrams.

The free product of two groups. Amalgamated free products. HNN extensions.

Covering spaces. Liftings of paths and homotopies. A covering map induces an injection between fundamental groups. The use of covering spaces to determine fundamental groups: the circle again, and real projective n-space. The correspondence between covering spaces and subgroups of the fundamental group. Regular covering spaces and normal subgroups.

Cayley graphs of a group. The relationship between the universal cover of a cell complex, and the Cayley graph of its fundamental group. The Cayley 2-complex of a group.

The Nielsen-Schreier theorem (every subgroup of a free group is free) proved using covering spaces.

Reading

1. John Stillwell, *Classical Topology and Combinatorial Group Theory*, Springer-Verlag, 1993.

Additional Reading

1. D. Cohen, *Combinatorial group theory: a topological approach*, LMS Student Texts 14, 1989, Chs 1-7.
2. A. Hatcher, *Algebraic Topology*, CUP 2001, Ch. 1.
3. D.L. Johnson, *Presentations of groups, London Mathematical Society, Student Texts 15*, Second Edition, Cambridge University Press, 1997. Chs. 1-5, 10,13
4. W. Magnus, A. Karrass, and D. Solitar, *Combinatorial Group Theory*, Dover Publications, 1976. Chs. 1-4
5. M. Hall, Jr, *The Theory of Groups*, Macmillan 1959, Chs. 1-7, 12, 17

C3.1b Algebraic Topology — Prof Tillmann — 16 HT

Prerequisites

Ideally students will also take the Topology and Groups course in Michaelmas, though this is not absolutely necessary.

Aims and Objectives

Homology theory is a subject that pervades much of modern mathematics. Its basic ideas are used in nearly every branch, pure and applied. In this course, the homology groups of topological spaces are studied. These powerful invariants have many attractive applications. For example we will prove that the dimension of a vector space is a topological invariant and the fact that a 'hairy ball can not be combed'.

Learning outcomes

At the end of the course, students are expected to understand the basic algebraic and geometric ideas that underpin homology and cohomology theory. These include the cup product and Poincaré Duality for manifolds. They should be able to choose between the different homology theories and to use calculational tools such as the Mayer-Vietoris sequence to effectively compute the homology and cohomology of easy examples, including projective spaces, surfaces, certain simplicial spaces and cell complexes. At the end of the course, students should also have developed a sense of how the ideas of homology and cohomology may be applied to problems from other branches of mathematics.

Synopsis

Chain complexes of abelian groups and their homology. Short exact sequences. Delta (and simplicial) complexes and their homology. Euler characteristic.

Singular homology of topological spaces. Relation of the first homology group to the fundamental group. Relative homology and the Five Lemma. Homotopy invariance and excision (details of proofs not examinable). Mayer-Vietoris Sequence. Equivalence of simplicial and singular homology. Axioms of homology.

Degree of a self-map of a sphere. Cell complexes and cellular homology. Application: the hairy ball theorem.

Cohomology of spaces and the Universal Coefficient Theorem (proof not examinable). Cup products. Künneth Theorem (without proof). Topological manifolds and orientability. The fundamental class of an orientable, closed manifold and the degree of a map between manifolds of the same dimension. Poincaré Duality (without proof).

Reading

1. A. Hatcher, *Algebraic Topology*, CUP (2001). Chapter 3 and 4.
2. G. Bredon, *Topology and Geometry*, Springer (1997). Chapter 4 and 5.
3. J. Vick, *Homology Theory*, GTM 145 Springer (1973).

C4.1: Functional Analysis and Banach and C* Algebras

Method of examination: 3 hour (for whole unit) or $1\frac{3}{4}$ hour (for half unit)paper.

C4.1a: Functional Analysis — Prof C Batty — 16MT

Recommended Prerequisites

Part A Topology, B4 Analysis

Learning outcomes

This course builds on B4, by extending the theory of Banach spaces and operators. The course exhibits the benefits of completeness (through the Baire category theorems) and compactness, giving the course a topological flavour. On the other hand, many of the topics studied have applications in differential equations and other areas of applied mathematics, some of which will appear in the form of examples.

Synopsis

Normed spaces and Banach spaces; subspaces and quotient spaces. Direct sums.

Compactness in Banach spaces; totally bounded sets. Ascoli's Theorem, Riesz criterion.

Dual spaces; completion; reflexive spaces. Hahn-Banach theorems. Weak and weak* topologies. The Banach-Alaoglu theorem.

The Baire category theorem. The open-mapping theorem. The closed-graph theorem. The principle of uniform boundedness.

Compact operators; spectral theory. Compact self-adjoint operators.

Closed (unbounded) operators. Operators with compact resolvent. Unbounded self-adjoint operators.

Method of examination

4 examination questions.

Reading

1. B. Bollobas, *Linear Analysis*, Second Edition, Cambridge University Press (1999).
2. M. Reed and B. Simon, *Methods of Modern Mathematical Physics I: Functional Analysis*, Academic Press (1972).

Alternative Reading

1. P.D. Lax, *Functional Analysis*, Wiley (2002).

C4.1b: Banach and C* Algebras — Dr C M Edwards — 16HT

Recommended Prerequisites

B4 Analysis

Learning outcomes

The suggestion that the observables of quantum mechanics should be represented by linear operators on Hilbert spaces led Murray and von Neumann into a detailed study of algebras of bounded linear operators on Hilbert space. The abstract version of such algebras are known as C*-algebras and investigations into their structure continue today. Their importance now extends beyond functional analysis and physics into geometry and number theory. The outcomes of this course should be the appreciation and understanding of the main problems in Banach algebras and C* algebras. Details are included in the synopsis below.

Synopsis

The course begins with an introduction to the general theory of Banach algebras and C*-algebras culminating in the Gelfand Representation Theorem for commutative C*-algebras. The second part of the course concentrates on the relationship that exists between the algebraic and geometric properties of non-commutative C*-algebras. The final part concerns itself with representations of C*-algebras on Hilbert spaces culminating with the Gelfand-Naimark Theorem.

Method of examination

4 examination questions.

Reading

1. G.I. Murphy, *C*-algebras and Operator Theory*, Academic (1990), Chs. 2, 3 and parts of Ch. 5.

Alternative (more advanced) sources:

1. G.K. Pedersen, *C*-algebras and their Automorphism Groups*, Academic (1979).
2. S. Sakai, *C*-algebras and W*-algebra*, Springer (1971).

C4.2a: Real and Harmonic Analysis — Dr B Kirchheim — 16 MT

Method of examination: $1\frac{3}{4}$ hour paper - 4 questions.

Weight: half-unit.

Aims and Objectives

The aim of the course is to introduce basic analytical tools for the study of fine properties (local structure) of modern (i.e. weakly differentiable) solutions to variational problems and PDEs, together with rigorous convergence results for Fourier analysis in this general setting.

Learning Outcomes

One objective of the course is to refine the integration theory taught in Part A, dealing with questions like "Does the Fundamental Theorem of Calculus hold also for the Lebesgue integral?"

The second part of the lectures utilizes these results to build a modern theory of Fourier series and transforms – answering natural questions about convergence for all integrable functions, touching upon recent concepts like wavelets and being used in both pure and applied mathematics (from number theory to PDE).

Synopsis

The Hardy-Littlewood maximal operator and Lebesgue's differentiation theorem, absolutely continuous functions, interpolation of linear operators.

Fourier series: Summablity methods, Gibb's Phenomenon, sets of divergence.

Fourier transform on the real line: definition in the space of integrable functions and basic properties, Plancherel's theorem and the space of square integrable functions, Fourier transform of measures and Bochner's theorem.

Prerequisites

Familiarity with the Lebesgue integral (to be very briefly revised) and B4a Banach spaces is expected, knowledge of B4b Hilbert spaces would be useful.

Reading

1. W. Rudin, *Real and Complex Analysis*, Mc Graw-Hill, 1966 (and new editions, 1987).
2. Y. Katznelson, *An Introduction to Harmonic Analysis*, John Wiley 1968 (1st ed).

Further Reading

1. T.W. Koerner; *Fourier Analysis*, Cambridge University Press 1986.

2. E.M. Stein, G. Weiss; *Introduction to Fourier Analysis in Euclidean spaces*, Princeton UP, 1971.

3. E.M. Stein, R. Shakarchi; *Fourier Analysis : an introduction*, Princeton UP, 2003.

C5.1: Partial Differential Equations for Pure and Applied Mathematicians and Calculus of Variations

C5.1a PDEs for Pure and Applied Mathematicians can be taken as a half-unit, but C5.1b Calculus of Variations cannot.

Method of examination: 3 hour (for whole unit) or $1\frac{3}{4}$ hour (for half unit)paper.

C5.1a: Partial Differential Equations for Pure and Applied Mathematicians — Dr Dyson — 16MT

Weight: Half-unit.

Prerequisites: Lebesgue integration would be useful but is not essential.

Aims & Objectives

The course will introduce some of the modern techniques in partial differential equations that are central to the theoretical and numerical treatments of linear and nonlinear partial differential equations arising in science, geometry and other fields.

It provides valuable background for the Part C courses on Calculus of Variations and Finite Element Methods.

Learning outcomes

Students will learn techniques and results, such as Sobolev spaces, weak convergence, weak solutions, the direct method of calculus of variations, embedding theorems, the Lax-Milgram theorem, the Fredholm Alternative and the Hilbert Schmidt theorem and how to apply these to obtain existence and uniqueness results for linear and nonlinear elliptic partial differential equations.

Synopsis

Part I Function Spaces:

Why are function spaces important for partial differential equations?

User's guide to the Lebesgue integral. Definition of Banach spaces, separability and dual spaces. Definition of Hilbert space. The spaces $L^p(\Omega)$, $1 \leq p \leq \infty$, where $\Omega \subset \mathbb{R}^n$ is open. Minkowski and Hölder inequalities. Statement that $L^p(\Omega)$ is a Banach space, and that the dual of L^p is $L^{p'}$, for $1 \leq p < \infty$. Statement that L^2 is a Hilbert space.

Weak and weak* convergence in L^p spaces. Examples. A bounded sequence in a separable Hilbert space has a weakly convergent subsequence.

Mollifiers and the density of smooth functions in L^p for $1 \leq p < \infty$.

Definition of weak derivatives and their uniqueness. Definition of Sobolev space $W^{m,p}(\Omega)$, $1 \leq p \leq \infty$. $H^m(\Omega) = W^{m,2}(\Omega)$. Definition of $W_0^{1,p}(\Omega)$, $1 \leq p < \infty$.

Part II Elliptic Problems:

The direct method of calculus of variations: The Poincaré inequality. Proof of the existence and uniqueness of a weak solution to Poisson's equation $-\Delta u = f$, with zero Dirichlet boundary conditions and $f \in L^2(\Omega)$, with Ω bounded. Discussion of regularity of solutions.

The Lax Milgram lemma and Gårding's inequality. Existence and uniqueness of weak solutions to general linear uniformly elliptic equations.

Embedding theorems (proofs omitted except $W^{1,1}(a,b) \hookrightarrow C[a,b]$).

Compact operators and self adjoint operators. Fredholm Alternative and Hilbert Schmidt Theorem. Examples including $-\Delta$ with zero Dirichlet boundary conditions.

A nonlinear elliptic problem treated by the direct method.

Reading

1. Lawrence C. Evans, *Partial differential equations*, (Graduate Studies in Mathematics), 2004, American Mathematical Society

2. M. Renardy and R.C. Rogers *An introduction to partial differential equations*, 2004, Springer-Verlag, New York.

Additional Reading

1. E. Kreyszig, *Introductory Functional Analysis with Applications*, Wiley (revised edition, 1989)

2. J. Rauch, *Partial differential equations*, 1992, Springer-Verlag, New York.

C5.1b: Calculus of Variations — Dr Crooks — 16HT

Prerequisites:

The Section M course 'C5.1a: Partial Differential Equations for Pure and Applied Mathematicians'. Some familiarity with the Lebesgue integral is essential, and some knowledge of elementary functional analysis (e.g. Banach spaces and their duals, weak convergence) an advantage.

Aims & Objectives

The aim of the course is to give a modern treatment of the calculus of variations from a rigorous perspective, blending classical and modern approaches and applications.

Learning outcomes

Students will learn rigorous results in the classical and modern one-dimensional calculus of variations and see possible behaviour and application of these results in examples. They will see some examples of multi-dimensional problems.

Synopsis

Classical and modern examples of variational problems (e.g. brachistochrone, models of phase transformations).

One-dimensional problems, function spaces and definitions of weak and strong relative minimizers. Necessary conditions; the Euler-Lagrange and Du Bois-Reymond equations, theory of the second variation, the Weierstrass condition. Sufficient conditions; field theory and sufficiency theorems for weak and strong relative minimizers. The direct method of the calculus of variations and Tonelli's existence theorem. Regularity of minimizers. Examples of singular minimizers and the Lavrentiev phenomenon. Problems whose infimum is not attained. Relaxation and generalized solutions. Isoperimetric problems and Lagrange multipliers.

Multi-dimensional problems, done via some examples.

Reading

G. Buttazzo, M. Giaquinta, S. Hildebrandt, *One-dimensional Variational Problems, Oxford Lecture Series in Mathematics*, Vol. 15, OUP (1998). Ch 1, Sections 1.1, 1.2 (treated differently in course), 1.3, Ch 2 (background), Ch 3, Sections 3.1, 3.2, Ch 4, Sections 4.1, 4.3.

Additional Reading

1. U. Brechtken-Manderscheid, *Introduction to the Calculus of Variations*, Chapman & Hall (1991).
2. H. Sagan, *Introduction to the Calculus of Variations*, Dover (1992).
3. J. Troutman, *Variational Calculus and Optimal Control*, Springer-Verlag (1995).

C6.1: Solid Mechanics

Method of examination: 3 hour (for whole unit) or $1\frac{3}{4}$ hour (for half unit)paper.

C6.1 Solid Mechanics – Professor J.M. Ball – 16 MT

Part C course.

Method of assessment

Four examination questions. $1\frac{3}{4}$ hour examination. Weight: Half-unit.

Prerequisites

There are no formal prerequisites. In particular it is not necessary to have taken any courses in fluid mechanics, though having done so provides some background in the use of similar concepts. Use is made of some elementary linear algebra in $\mathbf{R}^3$ (for example, eigenvalues, eigenvectors and diagonalization of symmetric 3×3 matrices), and revision of this material, for example from the Mods Linear Algebra course 3.2.1, is useful preparation. The necessary material is summarized in the course.

The first module of 6 lectures is useful preparation for C6.2 Elasticity and Plasticity. Taken together the two courses will provide a broad overview of modern solid mechanics, with a variety of approaches.

Aims and Objectives

Solid mechanics is a vital ingredient of materials science and engineering, and is playing an increasing role in biology. It has a rich mathematical structure. The aim of the course is to derive the basic equations of elasticity theory, the central model of solid mechanics, and give some interesting applications to the behaviour of materials.

Learning outcomes

Students will learn basic techniques of modern continuum mechanics, such as kinematics of deformation, stress, constitutive equations and the relation between nonlinear and linearized models. They will also gain an insight into some recent developments in applications of mathematics to a variety of different materials.

Synopsis

(1) Nonlinear and linear elasticity (6 lectures)

Lagrangian and Eulerian descriptions of motion, analysis of strain. Balance laws of continuum mechanics. Frame-indifference. Cauchy and Piola-Kirchhoff stress. Constitutive equations for a nonlinear elastic material. Material symmetry, isotropy. Linear elasticity as a linearization of nonlinear elasticity.

(2) Exact solutions in elastostatics. (6 lectures)

Universal deformations for compressible materials. Incompressibility and models of rubber. Exact solutions for incompressible materials, including the Rivlin cube, simple shear, torsion of a cylinder, inflation of a balloon. Cavitation in polymers.

(3) Phase transformations in solids (4 lectures)

Martensitic phase transformations, twins and microstructure.

Austenite-martensite interfaces. The shape-memory effect.

Reading

1. R. J. Atkin & N. Fox, *An introduction to the theory of elasticity*, Longman, 1980.

2. M.E. Gurtin, *An introduction to continuum mechanics*, Academic Press, 1981.

Further Reading

1. Stuart S. Antman, *Nonlinear problems of elasticity*, Applied mathematical sciences v. 107, Springer-Verlag, 1995.

2. Jerrold E. Marsden, Thomas J.R. Hughes, *Mathematical foundations of elasticity*, Prentice-Hall, 1983.

3. Philippe G. Ciarlet, *Mathematical elasticity*, Studies in mathematics and its applications ; v. 20, 27, 29, North-Holland 1988-

4. Kaushik Bhattacharya, *Microstructure of Martensite - Why it forms and how it gives rise to the shape-memory effect*, Oxford University Press 2003.

C6.2: Elasticity and Plasticity

C6.2: Elasticity and Plasticity — Lecturer TBA — 16HT

Method of examination: $1\frac{3}{4}$ hour paper - 4 questions.

Recommended Prerequisites

There are no prerequisites but familiarity with classical and fluid mechanics (Part A) and simple perturbation theory (Part C course) will be useful. The complementary Part C course, Solid Mechanics, will be especially useful.

Learning outcomes

The course gives a rapid review of mathematical models for basic solid mechanics. Benchmark solutions are reviewed for static problems and wave propagation in linear-elastic materials. It is then shown how these results can be used as a basis for practically useful problems involving rods, plates, and shells. Also simple geometrically nonlinear models will be introduced to explain buckling, contact and fracture at the most basic level. Yield and plasticity will be discussed at a similar level, both microscopically and macroscopically and there will be a brief introduction to composite fields (composite materials, thermo- and visco-elasticity).
Learning outcomes are an appreciation and understanding of the topic listed in the syllabus below.

Synopsis

Review of tensors, conservation laws, Navier equations. Antiplane strain, plain strain, torsion. Elastic wave propagation, Rayleigh waves. Ad hoc approximations for thin materials; simple bifurcation theory and buckling; simple mixed boundary value problems, brittle fracture and smooth contact; simple ideas about homogenization, composite materials, thermo- and visco-elasticity.

Reading

1. R.M. Hill, *Mathematical Theory of Plasticity*, Oxford Clarendon Press, 1998.
2. A.E.H. Love, *Treatise on the Mathematical Theory of Elasticity*, Dover, 1944.
3. L.D. Landau and E.M. Lifshitz, *Theory of Elasticity*, Pergamon Press, 1986.

C6.3: Perturbation Methods and Applied Complex Variables

Method of examination: 3 hour (for whole unit) or $1\frac{3}{4}$ hour (for half unit)paper.

C6.3a: Perturbation Methods — Prof Chapman and Dr J. Ockendon — 16MT

Recommended Prerequisites

Part A Differential Equations and Complex Analysis (B5 and B6 are relevant but not essential)

Learning outcomes

Perturbation methods underlie almost all applications of physical applied mathematics: for example, boundary layers in viscous flow, celestial mechanics, optics, shock waves, reaction-diffusion equations and nonlinear oscillations. The aims of the course are to give a clear and systematic account of modern perturbation theory and to show how it can be applied to differential equations.
Learning outcomes are an appreciation and understanding of the topic listed in the syllabus below.

Synopsis

Asymptotic expansions. Asymptotic evaluation of integrals: Laplace's method, method of steepest descent; Stokes phenomenon, exponential asymptotics. Regular and singular perturbation methods. Methods of multiple scales, WKB method , boundary layers, transition layers. Applications to partial differential equations.

Method of examination

4 examination questions.

Reading

1. E.J. Hinch, *Perturbation Methods*, CUP (1991), Chs. 1–3, 5–7.
2. C.M. Bender and S.A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*, McGraw-Hill (1978), Chs. 6, 7, 9–11.
3. J. Kevorkian and J.D. Cole, *Perturbation Methods in Applied Mathematics*, Springer-Verlag, (1981), Chs. 1, 2.1–2.5, 3.1, 3.2, 3.6, 4.1, 5.2.

C6.3b: Applied Complex Variables — Lecturer TBA — 16HT

Recommended Prerequisites

The course requires second year core analysis (complex analysis). It continues the study of complex variables in the directions suggested by contour integration and conformal mapping. Part A Fluid Dynamics and Waves is desirable as it provides motivation for some of the topics studied.

Aims & Objectives

The course begins where core second-year complex analysis leaves off, and is devoted to extensions and applications of that material. It is assumed that students will be familiar with inviscid two-dimensional hydrodynamics (Part A Fluid Dynamics and Waves) to the extent of the existence of a harmonic streamfunction and velocity potential in irrotational incompressible flow, and Bernoulli's equation.

Synopsis

1-2 Review of core real and complex analysis, especially contour integration, Fourier transforms.

2-4 Conformal mapping. Riemann mapping theorem (statement only). Schwarz-Christoffel formula. Solution of Laplace's equation by conformal mapping onto a canonical domain.

5-6 Applications to inviscid hydrodynamics: flow past an aerofoil and other obstacles by conformed mapping; free streamline flows, hodograph plane.

7-8 Flow with free boundaries in porous media. Construction of solutions using conformal mapping. The Schwarz function.

9-15 Transform methods, complex Fourier transform. Solution of mixed boundary value problems motivated by thin aerofoil theory and the theory of cracks in elastic solids. Index. Riemann Hilbert problems, Wiener-Hopf method.

16 Stokes phenomenon.

Method of examination

4 examination questions.

Reading

1. M. J. Ablowitz and A. S. Fokas, *Complex Variables: Introduction and Applications*, 2nd edition, C.U.P., Cambridge (2003). ISBN 0521534291.

2. J. Ockendon, Howison, Lacey and Movichan, *Applied Partial Differential Equations*, Oxford 1999, Pages 195-212.

3. G. F. Carrier, M. Krook and C. E. Pearson, *Functions of a Complex Variable*, McGraw-Hill, New York, (1966). Reprinted by Hod Books, 1983. ISBN 0962197300 (Out of print).

C7: Mathematical Physics

Method of examination: 3 hour (for whole unit) or $1\frac{3}{4}$ hour (for half unit)paper.

C7.1b: Quantum Theory and Quantum Computers — Dr Hannabuss — 16HT

Prerequisites:

Quantum Mechanics.

Aims and Objectives

This course builds directly on the first course in quantum mechanics and covers a series of important topics, particularly features of systems containing several particles. The behaviour of identical particles in quantum theory is more subtle than in classical mechanics, and an understanding of these features allows one to understand the periodic table of elements and the rigidity of matter. It also introduces a new property of entanglement linking particles which can be quite widely dispersed.

There are rarely neat solutions to problems involving several particles, so usually one needs some approximation methods. In very complicated systems, such as the molecules of gas in a container, quantum mechanical uncertainty is compounded by ignorance about other details of the system and requires tools of quantum statistical mechanics.

Two state quantum systems enable one to encode binary information in a new way which permits superpositions. This leads to a quantum theory of information processing, and by exploiting entanglement to other ideas such as quantum teleportation.

Synopsis

Identical particles, symmetric and anti-symmetric states, Fermi–Dirac and Bose–Einstein statistics and atomic structure.

Heisenberg representation, interaction representation, time dependent perturbation theory and Feynman–Dyson expansion. Approximation methods, Rayleigh–Schrödinger time-independent perturbation theory and variation principles. The virial theorem. Helium.

Mixed states, density operators. The example of spin systems. Purification. Gibbs states and the KMS condition.

Entanglement. The EPR paradox, Bell's inequalities, Aspect's experiment. GHZ states

Quantum information processing, qubits and quantum computing. The no-cloning theorem, quantum teleportation. Quantum logic gates. Quantum operations. The quantum Fourier transform. Grover's search algorithm.

Reading

1. Hannabuss, *Introduction to quantum mechanics* OUP (1997). Chapters 10-12 and 14, 16, supplemented by lecture notes on quantum computers on the web

Further reading:

A popular non-technical account of the subject:

A Hey and P Walters, *The new Quantum Universe*, Cambridge (2003).

Also designed for an Oxford course, though only covering some material:

I P Grant, *Classical and Quantum Mechanics*, Mathematical Institute Notes (1991).

An encyclopaedic account of quantum computing:

M A Nielsen and I L Chuang: *Quantum Computation*, Cambridge University Press, (2000).

Even more paradoxes can be found in:

Y Aharanov and D Rohrlich: *Quantum Paradoxes*, Wiley-VCH (2005).

Those who read German can find further material on entanglement in:

J Audretsch: *Verschränkte Systeme*, Wiley-VCH (2005).

Other accounts of the first part of the course:

L I Schiff, *Quantum Mechanics*, 3rd edition, Mc Graw Hill (1968).

B J Bransden and C J Joachain, *Introduction to quantum mechanics*, Longman (1995)

A I M Rae, *Quantum Mechanics*, 4th edition, Institute of Physics (1993)

C7.2b: General Relativity — Dr A Hodges — 16HT

Recommended Prerequisites

B7 Relativity and Electromagnetism taken in 2005-6.

Aims & Objectives

The course is intended as an elementary introduction to general relativity, its basic physical concepts of its observational implications, and the new insights that it provides into the nature of space time, and the structure of the universe. Some familiarity with special relativity and electromagnetism covered in the B7 course will be assumed. The lectures will review Newtonian gravitation, tensor calculus and continuum physics in special relativity, tensor calculus in Minkowski space, physics in curved space time and the Einstein field equations. This will suffice for an account of simple applications to planetary motion, the bending of light and the existence of black holes.

Learning outcomes

This course starts by asking how the theory of gravitation can be made consistent with the special-relativistic framework. Physical considerations (the principle of equivalence, general covariance) are used to motivate and illustrate the mathematical machinery of tensor calculus. The technical development is kept as elementary as possible, emphasising the use of local inertial frames. A similar elementary motivation is given for Einstein's equations and the Schwarzschild solution. Orbits in the Schwarzschild solution are given a unified treatment which allows a simple account of the three classical tests of Einstein's theory. Finally, use of Eddington-Finkelstein coordinates shows students how the theory of black holes emerges and exposes the radical consequences of Einstein's theory for space-time structure. Cosmological solutions are not discussed.

The learning outcomes are an understanding and appreciation of the ideas and concepts described above.

Synopsis

Review of Newtonian gravitation theory and problems of constructing a relativistic generalisation. Review of Special Relativity. The equivalence principle. Tensor formulation of special relativity (including general particle motion, tensor form of Maxwell's equations and the energy momentum-tensor of dust). Curved space time. Local inertial coordinates. General coordinate transformations, elements of Riemannian geometry (including connections, curvature and geodesic deviation). Mathematical formulation of General Relativity, Einstein's equations (properties of the energy-momentum tensor will be needed in the case of dust only). The Schwarzschild solution; planetary motion, the bending of light, and black holes.

Method of examination

4 examination questions.

Reading

1. L.P. Hughston and K.P. Tod, *An Introduction to General Relativity*, LMS Student Text 5, CUP (1990), Chs 1-18.

2. N.M.J. Woodhouse, *Notes on Special Relativity*, (Mathematical Institute Notes. Revised edition; published in a revised form as *Special Relativity, Lecture notes in Physics m6*, Springer-Verlag, (1992), Chs 1-7

Further Reading

1. B. Schutz, *A First Course in General Relativity*, CUP (1990).

2. R.M. Wald, *General Relativity*, Chicago (1984).

3. W. Rindler, *Essential Relativity*, Springer-Verlag, 2nd edition (1990).

C7.3 Advanced Quantum Mechanics — Prof Candelas and Prof Ross —16MT, plus a reading course — HT

Assessment

The course will be examined with an examination paper set by Physics and an additional mini project making up 50% of the assessment.
Copies of the mini project must be submitted to the Chairman of Examiners Honour School of Mathematics, c/o the Clerk of the Schools, Examination Schools, High Street, Oxford by midday on Friday of Week 9 of Hilary Term. Copies should be in a folder or envelope giving the cadidate's examination number only.

This is a course on advanced quantum mechanics for both mathematicians and physicists that will be taught jointly between Mathematics and Physics. To allow for the differences in syllabus between prior Quantum Mechanics courses in the two departments the 16 lectures are divided into a first part consisting of 5 lectures aimed at mathematicians that will be given by Philip Candelas and a further 11 lectures for both mathematicians and physicists given by Graham Ross. The first 5 lectures review cover topics that are new for mathematicians but are review for physicists. The course will be supported by structured reading together with weekly problem sets and classes.

Aim

This course provides a thorough introduction to perturbation theory, scattering theory and relativistic quantum mechanics. The topics chosen are closely linked with effects in the hydrogen atom providing many examples of the application of the perturbation theory techniques. Expansion in spherical harmonics, first introduced in relation to the hydrogen atom, has repeated application in respect to scattering by a central potential. Scattering theory in turn, both through the correspondence between waves and particles and through consideration of high energy processes, leads naturally to a search for relativistic generalizations of the Schrödinger equation, a search which turns out to have far-reaching consequences. The topics that are covered are interesting in themselves and have many important applications. They also lead to, and provide a very good preparation for, a further course on relativistic quantum field theory.

The outcomes will be an appreciation and understanding of the topics mentioned above.

Part II: (6 lectures) Nonrelativistic Scattering:

Introduction to Scattering Theory
Scattering by a potential, the differential cross section, stationary states and the scattering amplitude, calculation of the cross section using probability currents.

Integral Scattering Equation
Definition of the Green Function, the Lipmann Schwinger equation, determination of the Green function ($\mathbf{x}$ and $\mathbf{k}$ space), the Born Series, calculation of the Born approximation for a Yukawa potential.

The operator formulation of the Lippmann-Schwinger equation
Introduction to the operator formalism, determination of the Green function (regularisation in the complex plane), the Born Series in operator notation.

Scattering by a central potential : the method of partial waves
Angular momentum stationary states, expansion of a plane wave in terms of free spherical waves, partial waves in a central potential, the definition of a phase shift, expression of the cross section in terms of the phase shifts, unitarity and the optical theorem.

Part III: Relativistic Quantum Mechanics

The construction of a relativistic wave equation
identification of probability density, 4-vector formulation, historical perspective the problem of the negative energy and probability states, the Pauli Weisskopf reinterpretation of the probability density, and the Feynman Stuckelberg interpretation of the negative energy states.

The relativistic treatment of scattering
Lorentz invariant form of the electromagnetic potential, the scattering amplitude and the current density, propagator of the Klein Gordon equation, determination of the scattering amplitude.

The Dirac equation
Dirac matrices and the relativistic generalisation of the Schrodinger equation, Diracs derivation the 'square root' of the Schrodinger equation, hole theory interpretation, free particle solutions.

The non-relativistic correspondence
introduction of electromagnetism, the coupled equations for the upper and lower spinor components, Non-relativistic limit, the gyromagnetic ratio.

Symmetries
Angular momentum, spin and helicity, parity.

Reading for Part I

Separation of the Laplacian in spherical coordinates, spherical harmonics
J D Jackson, *Classical Electrodynamics*, John Wiley and Sons (WIE), 1999,ISBN: 047130932X,SS3.1–3.5.

The hydrogen atom
Many references, for example, K C Hannabuss *Introduction to Quantum Theory*, Publisher: Clarendon Press,ISBN: 0198537948, 1997, SS4.
L I Schiff *Quantum Mechanics*, Publisher: McGraw-Hill Education, 1968,ISBN: 0070856435, SS10.

Stationary perturbation theory
Hannabuss SS12,13, Schiff SS31.

The variational principle
Hannabuss SS14, Schiff SS33.

Reading for Parts II and III:

J J Sakurai *Modern Quantum Mechanics*, Publisher: Addison Wesley, ISBN: 0201539292, 1993, contains an excellent treatment of the scattering theory topics of the course.

C Cohen-Tanoudgi, B Diu and F Laloe *Quantum Mechanics vol II*, Publisher: John Wiley & Sons Inc, ISBN: 0471164356, 1977, a useful introduction to non-relativistic scattering theory.

Schiff remains a good reference for these parts of the course.

I J R Aitchison *Relativistic Quantum Mechanics*, Publisher: Macmillan, ISBN: 0333126947, 1972, is an excellent introduction to the relativistic aspects of the course.

J D Bjorken and S D Drell *Relativistic Quantum Mechanics*, Publisher: McGraw-Hill Education, ISBN: 0070054940, 1965,is a 'classic' text .

C8.1: Mathematics and the Environment and Mathematical Physiology

Method of examination: 3 hour (for whole unit) or $1\frac{3}{4}$ hour (for half unit)paper.

C8.1a: Mathematics and the Environment — Dr A Fowler — 16MT

Recommended Prerequisites

B6 highly recommended.

Learning outcomes

The aim of the course is to illustrate the techniques of mathematical modelling in their particular application to environmental problems. The mathematical techniques used are drawn from the theory of ordinary differential equations and partial differential equations. However, the course does require the willingness to become familiar with a range of different scientific disciplines. In particular, familiarity with the concepts of fluid mechanics will be useful.
Learning outcomes are an appreciation and understanding of the topic listed in the syllabus below.

Synopsis

Applications of mathematics to environmental or geophysical problems involving the use of models with ordinary and partial differential equations. Examples to be considered are: Climate dynamics. River flow and sediment transport. Glacier dynamics.

Method of examination

4 examination questions.

Reading

1. A. C. Fowler, *Mathematics and the Environment*, Mathematical Institute lecture notes. (Revised edition, September 2004.)

2. K. Richards, *Rivers*, Methuen 1982.

3. G. B. Whitham, *Linear and Nonlinear Waves.* Wiley, New York. 1974.

4. W. S. B. Paterson, *The Physics of Glaciers*, 3rd edition, Pergamon Press 1994.

5. J. T. Houghton, *The Physics of Atmospheres*, 3rd ed. C.U.P., Cambridge 2002.

C8.1b: Mathematical Physiology — Prof Maini — 16HT

Recommended Prerequisites

B8a highly recommended.

Learning outcomes

The course aims to provide an introduction which can bring students within reach of current research topics in physiology, by synthesising a coherent description of the physiological background with realistic mathematical models and their analysis. The concepts and treatment of oscillations, waves and stability are central to the course, which develops ideas introduced in the more elementary B8a course. In addition, the lecture sequence aims to build understanding of the workings of the human body by treating in sequence problems at the intracellular, intercellular, whole organ and systemic levels.
Learning outcomes are an appreciation and understanding of the topic listed in the syllabus below.

Synopsis

Review of enzyme reactions and Michaelis-Menten theory. Trans-membrane ion transport: Hodgkin-Huxley and Fitzhugh-Nagumo models.

Excitable media; wave propagation in neurons.

Calcium dynamics: calcium-induced calcium release. Intracellular oscillations and wave propagation.

The electrochemical action of the heart. Spiral waves, tachycardia and fibrillation. The heart as a pump. Regulation of blood flow.

Respiration and CO_2 control. Mackey and Grodins models.

Regulation of stem cell and blood cell production. Dynamical diseases.

Method of examination

4 examination questions.

Reading

The principal text is:

1. J. Keener and J. Sneyd, *Mathematical Physiology*, Springer-Verlag (1998). Chs. 1, 4, 5, 9-12, 14-17.

Subsidiary mathematical texts are:

1. J. D. Murray, *Mathematical Biology*, Springer-Verlag, 2nd ed., 1993.

2. L. A. Segel, *Modeling Dynamic Phenomena in Molecular and Cellular Biology*, CUP (1984).

3. L. Glass and M. C. Mackey, *From Clocks to Chaos*, Princeton University Press (1988).

4. P. Grindrod, *Patterns and Waves*, Oxford University Press (1991).

General physiology texts are:

1. R. M. Berne and M. N. Levy, *Principles of Physiology*, 2nd ed. Mosby, St. Louis (1996).

2. J. R. Levick, *An Introduction to Cardiovascular Physiology*, 3rd ed. Butterworth-Heinemann, Oxford (2000).

3. A. C. Guyton and J. E. Hall, *Textbook of Medical Physiology*, 10th ed. W. B. Saunders Co., Philadelphia (2000).

C9.1: Analytic Number Theory and Elliptic Curves

Method of examination: 3 hour (for the whole unit) or $1\frac{3}{4}$ hour (for half unit) paper.

C9.1a: Analytic Number Theory — Prof Heath-Brown — 16MT

Prerequisites

Complex analysis (holomorphic and meromorphic functions, Cauchy's Residue Theorem, Evaluation of integrals by contour integration, Uniformly convergent sums of holomorphic functions). Elementary number theory (Unique Factorization Theorem).

Aims and Objectives

The course aims to introduce students to the theory of prime numbers, showing how the irregularities in this elusive sequence can be tamed by the power of complex analysis. The course builds up to the Prime Number Theorem which is the corner-stone of prime number theory, and culminates in a description of the Riemann Hypothesis, which is arguably the most important unsolved problem in modern mathematics.

Learning outcomes

The course will teach students to handle multiplicative functions, to deal with Dirichlet series as functions of a complex variable, and to prove the Prime Number Theorem and simple variants.

Synopsis

Introductory material on primes

Arithmetic functions - Mobius function, Euler function, Divisor function, Sigma function - multiplicativity

Dirichlet series - Euler products - von Mangoldt function

Riemann Zeta-function - analytic continuation to $Re(s) > 0$

Non-vanishing of zeta(s) on $Re(s) = 1$

Proof of the prime number theorem

The Riemann hypothesis and its significance.

The Gamma function, the functional equation for zeta(s), the value of zeta(s) at negative integers.

Reading

1. T.M. Apostol, *Introduction to analytic number theory. Undergraduate Texts in Mathematics*, Springer-Verlag, 1976. chapters 2,3,11,12 and 13.

2. M. Ram Murty, *Problems in analytic number theory*, (Springer, 2001). Chapters 1 - 5
3. G.H. Hardy and E.M. Wright, *An introduction to the theory of numbers*, Fifth edition. OUP 1979. Chapters 16 ,17 and 18
4. G.J.O. Jameson, *The Prime Number Theorem* LMS Student Texts, 53 CUP 2003

C9.1b Elliptic Curves — Dr V Flynn — 16 lectures HT

Recommended Prerequisites

It is helpful, but not essential, if students have already taken a standard introduction to algebraic curves and algebraic number theory. For those students who may have gaps in their background, I have placed the file "Preliminary Reading" permanently on the Elliptic Curves webpage, which gives in detail (about 30 pages) the main prerequisite knowledge for the course. Go first to: www.maths.ox.ac.uk/current-students/undergraduates/lecture-material/ then click on "C9 Elliptic Curves" and then click on the pdf file "Preliminary Reading".

Aims & Objectives

Elliptic curves give the simplest examples of many of the most interesting phenomena which can occur in algebraic curves; they have an incredibly rich structure and have been the testing ground for many developments in algebraic geometry whilst the theory is still full of deep unsolved conjectures, some of which are amongst the oldest unsolved problems in Mathematics. The course will concentrate on arithmetic aspects of elliptic curves defined over the rationals, with the study of the group of rational points, and explicit determination of the rank, being the primary focus. Using elliptic curves over the rationals as an example, we will be able to introduce many of the basic tools for studying arithmetic properties of algebraic varieties.

Learning outcomes

On completing the course, students should be able to understand and use properties of elliptic curves, such as the group law, the torsion group of rational points, and 2-isogenies between elliptic curves. They should be able to understand and apply the theory of fields with valuations, emphasising the p-adic numbers, and be able to prove and apply Hensel's Lemma in problem solving. They should be able to understand the proof of the Mordell-Weil Theorem for the case when an elliptic curve has a rational point of order 2, and compute ranks in such cases, for examples where all homogeneous spaces for descent-via-2-isogeny satisfy the Hasse principle. They should also be able to apply the elliptic curve method for the factorisation of integers.

Synopsis

Non-singular cubics and the group law; Weierstrass equations.
Elliptic curves over finite fields; Hasse estimate (stated without proof).

p-adic fields (basic definitions and properties).
1-dimensional formal groups (basic definitions and properties).
Curves over p-adic fields and reduction mod p.
Computation of torsion groups over $\mathbb{Q}$; the Nagell-Lutz theorem.
2-isogenies on elliptic curves defined over $\mathbb{Q}$, with a $\mathbb{Q}$-rational point of order 2.
Weak Mordell-Weil Theorem for elliptic curves defined over $\mathbb{Q}$, with a $\mathbb{Q}$-rational point of order 2.
Height functions on abelian groups and basic properties.
Heights of points on elliptic curves defined over $\mathbb{Q}$; statement (without proof) that this gives a height function on the Mordell-Weil group.
Mordell-Weil Theorem for elliptic curves defined over $\mathbb{Q}$, with a $\mathbb{Q}$-rational point of order 2.
Explicit computation of rank using descent via 2-isogeny.
Public keys in cryptography; Pollard's $(p-1)$ method and the elliptic curve method of factorisation.

Reading

1. J.W.S. Cassels, *Lectures on Elliptic Curves*, LMS Student Texts 24, Cambridge University Press, 1991.
2. N. Koblitz, *A Course in Number Theory and Cryptography*, Graduate Texts in Mathematics 114, Springer, 1987.
3. J.H. Silverman and J. Tate, *Rational Points on Elliptic Curves*, Undergraduate Texts in Mathematics, Springer, 1992.
4. J.H. Silverman, *The Arithmetic of Elliptic Curves*, Graduate Texts in Mathematics 106, Springer, 1986.

Further Reading

1. A. Knapp, *Elliptic Curves. Mathematical Notes 40*, Princeton University Press, 1992.
2. G, Cornell, J.H. Silverman and G. Stevans (editors), *Modular Forms and Fermat's Last Theorem*, Springer, 1997.
3. J.H. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*, Graduate Texts in Mathematics 151, Springer, 199

C10.1 Stochastic Differential Equations and Brownian Motion in Complex Analysis

Method of examination: 3 hour (for the whole unit) or $1\frac{3}{4}$ hour (for half unit) paper.

C10.1a: Stochastic Differential Equations — Dr Z. Qian —16MT

Stochastic differential equations have been used extensively in many areas of application, including finance and social science as well as Chemistry . This course develops the basic theory of Ito's calculus and stochastic differential equations, and gives a few applications.

Learning outcomes

To develop an appreciation of stochastic calculus as a tool that can be used for defining and understanding diffusive systems.

Synopsis

Ito's calculus: stochastic integrals with respect to martingales, Ito's lemma, Levy's theorem on characteristic of Brownian motion, exponential martingales, exponential inequality, Girsanov's theorem, The Martingale Representation Theorem. Stochastic differential equations: strong solutions, questions of existence and uniqueness, diffusion processes, Cameron-Martin formula, weak solution and martingale problem. Some selected applications chosen from option pricing, stochastic filtering etc.

Reading — Main texts

1. Dr Qian's online notes.
2. B. Oksendal: *Stochastic Differential Equations: An introduction with applications*, Universitext, Springer (6th edition). Chapters II, III, IV, V, part of VI, Chapter VIII (F).
3. F. C. Klebaner: *Introduction to Stochastic Calculus with Applications*, Imperial College Press. Sections 3.1 - 3.5, 3.9, 3.12. Chapters 4, 5, 11.

Alternative reading

1. H. P. McKean: *Stochastic Integrals*, Academic Press, New York and London (1969).

Further reading

1. N. Ikeda & S. Watanabe: *Stochastic Differential Equations and Diffusion Processes*, North-Holland Publishing Company.

2. I. Karatzas and S. E. Shreve: *Brownian Motion and Stochastic Calculus (GTM 113)*, Springer-Verlag.

3. L. C. G. Rogers & D. Williams: *Diffusions, Markov Processes and Martingales Vol1 (Foundations) and Vol 2 (Ito Calculus)*, Cambridge University Press.

C10.1b: Brownian Motion in Complex Analysis — Prof T Lyons — 16HT

Recommended Prerequisites

The second year course on Complex Variable. At least one of (B10a) Martingales through Measure and (C10.1) Stochastic Differential Equations. To have attended both would be desirable.

Learning outcomes

To develop an appreciation of the role that Brownian motion and martingales can play in pure mathematics.

Aims & Objectives

Randomness plays a key feature in the behaviour of many high dimensional systems and so is intimately connected with applications. However, it also plays a key role in our understanding of many aspects of pure mathematics. This course will look at the deep interaction between 2 dimensional Brownian motion and complex analysis. At the core of these interactions is the conformal invariance of Brownian motion observed by Levy and the relationship with Harmonic Functions (based on Martingales) first observed by Kakutani and Doob.

Since that time there have been many developments and connections. The Hardy spaces of Fefferman and Stein, Value Distribution Theory, and most recently the stochastic Lowener equation (a topic of current and very exciting research).

We will use the conformal properties of Brownian motion to examine and prove some deep theorems about value distributions for complex functions.

Synopsis

Brownian motion. Continuous martingales and Levy's characterisation in terms of Brownian motion. Conformal invariance of Brownian motion. Brownian motion tangles about two points and a proof of Picard's theorems. Harmonic functions on the disk and the solution of the Dirichlet problem. Burkholders Inequalities Hardy spaces. Harmonic and superharmonic functions - via probability. [Nevanlinna's Theorems].

Method of examination

4 examination questions.

Reading

There isn't a perfect book for this course and we will refer to research papers to a limited extent.

1. The Notes of Prof Lyons

2. D. Burkholder, *Distribution function inequalities for martingales*, Ann. Probability, 1 (1973) 19–42.

Further Reading

1. McKean, *Stochastic Integrals*, (1969). Hard, short, with much relevant material and some mistakes! Excellent for the able!

2. K. E. Petersen, *Brownian Motion, Hardy Spaces and Bounded Mean Oscillation*, LMS Lecture Note Series, 28, Cambridge University Press (1977).

3. T. K. Carne, *Brownian Motion and Nevanlinna Theory*. Proc. London. Math. Soc. 52 (1986), 349-68

4. Richard F. Bass, *Probabilistic Techniques in Analysis*. Springer-Verlag New York Inc, ISBN: 0387943870 (1995).

5. Lars Ahlfors, *Complex Analysis*, McGraw-Hill, ISBN: 0070006571, (1979).

6. Jean-Claude Gruet, *Nevanlinna Theory, Fuchsian Functions and Brownian Motion Windings*, Source: Rev. Mat. Iberoamericana 18 (2002), no. 2, 301–324.

C11.1: Graph Theory and Probabilistic Combinatorics

C11.1a Graph Theory can be taken as a half unit, but C11.1b Probabilistic Combinatorics cannot.

Method of examination: 3 hour (for the whole unit) or $1\frac{3}{4}$ hour (for C11.a Graph Theory half unit) paper.

C11.1a: Graph Theory — Dr A Scott — 16MT

Recommended Prerequisites

None beyond elementary probability theory.

Learning outcomes

To develop an appreciation of extremal methods in the analysis and understanding of graphical structures.

Aims & Objectives

Graphs are among the simplest mathematical structures, but nevertheless have a very rich and well-developed structural theory. Graph Theory is an important area of mathematics, and also has many applications in other fields such as computer science.

The main aims of the course are to discuss the use of extremal methods for the study of graph structure, and to give an introduction to probabilistic techniques.

Synopsis

Introduction. Trees. Euler circuits. Planar graphs.
Matchings and Hall's Theorem. Connectivity.
Extremal problems. Long paths and cycles. Turán's Theorem.
Graph colouring. The Theorem of Brooks. The chromatic polynomial.
Ramsey's Theorem.
Introduction to probabilistic techniques. Random graphs. Graphs with large girth and chromatic number.
Szemerédi's Regularity Lemma.

Method of examination

4 examination questions.

Reading

1. B. Bollobás, *Modern Graph Theory*, GTM 184, Springer-Verlag (1998).

Further Reading

1. J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, Elsevier (1976); available online at http://www.ecp6.jussieu.fr/pageperso/bondy/books/gtwa/gtwa.html.
2. R. Diestel, *Graph Theory*, second edition, GTM 173, Springer-Verlag (2000).
3. D. West, *Introduction to Graph Theory*, second edition, Prentice-Hall (2001).
4. N. Alon and J. Spencer, *The Probabilistic Method*, second edition, Wiley (2000).

C11.1b: Probabilistic Combinatorics — Dr Martin — 16HT

Recommended Prerequisites

C11.1a Graph Theory and Part A Probability. Some knowledge of discrete-time martingales would be helpful but is not essential.

Learning outcomes

To develop an appreciation of probabilistic methods in discrete mathematics.

Aims and objectives

Probabilistic combinatorics is a very active field of mathematics, with connections to other areas such as computer science and statistical physics. Probabilistic methods are essential for the study of random discrete structures and for the analysis of algorithms, but they can also provide a powerful and beautiful approach for answering deterministic questions. The aim of this course is to introduce some fundamental probabilistic tools and present a few applications.

Synopsis

Spaces of random graphs. Threshold functions.
First and second moment methods. Chernoff bounds. Applications to Ramsey numbers and random graphs.
Lovasz Local Lemma. Two-colourings of hypergraphs (property B).
Poisson approximation, and application to the distribution of small subgraphs. Janson's inequality.
Concentration of measure. Martingales and the Azuma-Hoeffding inequality.
Chromatic number of random graphs.
Talagrand's inequality.

Method of examination

4 examination questions.

Reading

1. N. Alon and J.H. Spencer. *The Probabilistic Method*, second edition, Wiley, 2000.

Further reading:

1. B. Bollobás, *Random Graphs*, second edition, CUP, 2001.
2. S.Janson, T. Luczak and A.Rucinski, *Random Graphs*, John Wiley and Sons, 2000.
3. M. Mitzenmacher and E. Upfal. *Probability and Computing : Randomized Algorithms and Probabilistic Analysis*, Cambridge University Press, New York (NY), 2005.

Numerical Linear Algebra and Analysis

C12.1 Numerical Linear Algebra and Continuous Optimization

Method of examination: 3 hour (for the whole unit) or $1\frac{3}{4}$ hour (for half unit) paper.

C12.1a Numerical Linear Algebra and Approximation — Dr Wathen — 16MT

Synopsis

Common problems in linear algebra. Matrix structure, singular value decomposition. QR factorization, the QR algorithm for eigenvalues. Direct solution methods for linear systems, Gaussian elimination and its variants. Iterative solution methods for linear systems.

Approximation theory, Chebyshev semi-iterative methods, conjugate gradients, convergence analysis using approximation theory. Preconditioning.

Reading List

1. L N Trefethen and D Bau III, *Numerical Linear Algebra*, SIAM, 1997
2. J W Demmel, *Applied Numerial Linear Algebra*, SIAM, 1997
3. A Greenbaum, *Iterative Solution Methods for Linear Systems*, SIAM, 1997
4. G H Golub and C F van Loan, *Matrix Computations*, John Hopkins University Press, 3rd edition, 1996
5. M J D Powell, *Approximation Theory and Methods*, CUP, 1981

C12.1b Continuous Optimization — Dr Gould — 16HT

Aims

Optimisation deals with the problem of minimising or maximising a mathematical model of an objective function such as cost, fuel consumption etc. under a set of side constraints on the domain of definition of this function. Optimisation theory is the study of the mathematical properties of optimisation problems and the analysis of algorithms for their solution. The aim of this course is to provide an introduction to nonlinear continuous optimisation specifically tailored to the background of mathematics students.

Synopsis

[1] Preliminaries: convex sets and functions, Cholesky and QR factorisations, Sherman–Morrison–Woodbury formula, implicit function theorem, global versus local optimisation, convergence rates, optimality conditions for unconstrained optimisation.

[2–4] Line-search methods for unconstrained optimisation: steepest descent, congujate gradients, Fletcher–Reeves method, Newton–Raphson method, symmetric rank one method, Broyden–Fletcher–Goldfarb–Shanno method, practical line searches.

[5–7] Trust region methods for unconstrained optimisation: Cauchy point, dogleg method, two dimensional subspace minimisation, Steihaug's method, characterisation of exact solutions.

[8–11] Optimality conditions for constrained optimisation: convex separation, Farkas' lemma, linear programming duality, constraint qualification, Lagrangian function, Karush–Kuhn–Tucker conditions, second order optimality conditions, Lagrangian duality.

[12–16] Nonlinearly constrained optimisation: merit functions and homotopy idea, penalty function method, augmented Lagrangian method, barrier method, sequential quadratic programming.

Reading List

Lecture notes will be made available for downloading from the course webpage.
To complement the notes, reading assignments will be given from the book of J.Nocedal and S.J.Wright, *Numerical Optimisation*, Springer 1999.

C12.2 Approximation Theory and Finite Element Methods

Method of examination: 3 hour (for the whole unit) or $1\frac{3}{4}$ hour (for half unit) paper.

C12.2a Approximation of Functions — Dr Sobey — 16MT

Overview

The central idea in approximation of functions can be illustrated by the question: Given a set of functions A and an element u ? A, if we select a subset B ? A, can we choose an element U ? B so that U approximates u in some way? The course focusses on this question in the context of functions when the way we measure 'goodness' of approximation is either with an integral least square norm or with an infinity norm of the difference u-U. The choice of measure leads to further questions: is there a best approximation; if a best approximation exists, is it unique, how accurate is a best approximation and can we develop algorithms to generate good approximations? This course aims to give a grounding in the advanced theory of such ideas, the analytic methods used and important theorems for real functions. As well as being a beautiful subject in its own right, approximation theory is the foundation for many of the algorithms of computational mathematics and numerical analysis.

Synopsis

Introduction to approximation. Approximation in L2. Approximation in L?: Oscillation Theorem, Exchange Algorithm. Approximation with splines. Rational approximation. Approximation of perodic functions.

Syllabus

Introduction to approximation. Approximation in L2. Approximation in L?: Oscillation Theorem, Exchange Algorithm. Approximation with splines. Rational approximation. Approximation of perodic functions.

Reading

1. Powell, M.J.D. *Approximation theory and methods* (CUP).
2. Davis, P.J. *Interpolation & Approximation* (Dover).

C12.2b Finite Element Methods for Partial Differential Equations — Prof Süli — 16HT

Synopsis

Finite element methods represent a powerful and general class of techniques for the approximate solution of partial differential equations; the aim of this course is to provide an introduction to their mathematical theory, with special emphasis on theoretical questions such as accuracy, reliability and adaptivity; practical issues concerning the development of efficient finite element algorithms will also be discussed.

Syllabus

Elements of function spaces. Elliptic boundary value problems: existence, uniqueness and regularity of weak solutions.

Finite element methods: Galerkin orthogonality and Cea's lemma. Piecewise polynomial approximation in Sobolev spaces. Optimal error bounds in the energy norm. Variational crimes.

The Aubin-Nitsche duality argument. Superapproximation properties in mesh-dependent norms. A posteriori error analysis by duality: reliability, efficiency and adaptivity.

Finite element approximation of initial boundary value problems: Stability and error analysis.

Prerequisites

While no formal prerequisites are assumed, students who take this course will find it helpful to attend the Michaelmas Term lecture course Partial Differential Equations for Pure and Applied Mathematicians.

Reading List

1. S. Brenner & R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer-Verlag, Second Edition 2002 [Chapters 0,1,2,3; Chapter 4: Secs. 4.1–4.4, Chapter 5: Secs. 5.1–5.7].

2. K. Eriksson, D. Estep, P. Hansbo, & C. Johnson, *Computational Differential Equations*, CUP, 1996. [Chapters 5, 6, 8, 14 – 17].

3. C. Johnson, *Numerical Solution of Partial Differential Equations by the Finite Element Method*, CUP, 1990. [Chapters 1–4; Chapter 8: Secs. 8.1–8.4.2; Chapter 9: Secs. 9.1–9.5].

4. E. Suli, *Finite Element Methods for Partial Differential Equations*, Oxford University Computing Laboratory, 2001.

Dissertations

The guidance notes by the Projects Committee and application form are available at:

http://www.maths.ox.ac.uk/current-students/undergraduates/projects/.

YOu may offer either a whole-unit or a half-unit Dissertation in a Mathematical topic. You must apply to the Mathematics Project Committe in advance for approval. Proposals should be addressed to The Secretary to the Projects Committee, Room F1, The Mathematical Institute, and must be received before 12noon on Friday of Week 3 of Michaelmas Full Term.

3 Extra Units - application required

There will be no extra units in the academic year 2006-07.

4 Other Units

MS: Statistics

Please see the Statistics Website for up-to-date course information.

MS1a: Graphical Models and Inference — Prof Lauritzen — 16MT

Recommended Prerequisites

OBS1 Applied statistics and OBS2 Statistical Inference would be helpful but not essential.

Aims & Objectives

Graphical models have become increasingly important in many areas where statistics play a role. They enable the description and analysis of complex stochastic systems via their natural modularity, expressed in terms of (mathematical) graphs which encode conditional independence structure. The modules correspond typically to well-understood, classical models. This course builds upon develops the specific theory and computational tools needed in the analysis of graphical models for categorical and multivariate Gaussian data as well as Bayesian graphical models for complex stochastic systems.

Synopsis

Topics in MT06 include:

1. Conditional independence and Markov properties.
2. Log-linear graphical models for categorical data.
3. Gaussian graphical models.
4. Graphical models for complex stochastic systems

Method of examination

4 examination questions, $1\frac{3}{4}$ hour paper.

Reading

1. D. Edwards, *Introduction to Graphical Models* (2nd ed.), Springer-Verlag, New York (2002).
2. S. L. Lauritzen, *Graphical Models*, Oxford University Press, Oxford (1996).
3. P. J. Green, N. L. Hjort and S. Richardson, eds. *Highly Structured Stochastic Systems*, Oxford University Press, Oxford (2003).

MS1b: Statistical Data Mining — Lecturer tba — 12HT

Recommended Prerequisites

Part A Probability and Statistics. OBS1 Applied Statistics would be an advantage.

Aims & Objectives

'Data mining' is now widely used to find interesting patterns in large databases, for example in insurance, in marketing and in many scientific fields. With large amounts of data we can search for quite subtle patterns.

This course concentrates on the statistical tools used to identify patterns, and then to identify those which are interesting not just the result of chance associations.

Synopsis

Fundamentals of pattern recognition, machine learning and data mining.

Exploratory methods: principal components analysis, biplots, independent component analysis, multidimensional scaling.

Cluster Analysis: K-means, hierarchical methods, vector quantisation, self-organising maps.

Linear discriminant analysis, logistic discrimination, linear separation and perceptrons.

Classification trees. Splitting criteria, existence of pruning sequences. V-fold cross-validation.

Feed-forward neural networks. Universal approximation properties, back-propagation, training algorithms, assessment of fit.

Method of examination

3 examination questions and 2 assessed practicals.

Reading

1. C. Bishop, *Neural Networks for Pattern Recognition*, Oxford UP (1995).
2. D. Hand, H. Mannila, P. Smyth, *Principles of Data Mining*, MIT Press (2001).
3. I. H. Witten and E. Franke, *Data Mining. Practical Machine Learning Tools and Techniques with Java Implementations*, Morgan Kaufmann (2000).

Further Reading

1. B. D. Ripley, *Pattern Recognition and Neural Networks*, Cambridge UP (1996).

MS2a: Bioinformatics and Computational Biology — Dr Nicholls and Prof Hein — 16MT

Recommended Prerequisites

None. In particular, no previous knowledge of Genetics will be necessary.

Aims & Objectives

Modern molecular biology generates large amounts of data, such as sequences, structures and expression data, that needs different forms of statistical analysis and modelling to be properly interpreted. The fields of Bioinformatics and Computational Biology have this as their subject matter and there is no sharp boundary between them. Bioinformatics has an applied flavour while Computational Biology is viewed as the study of the models, statistical methodology and algorithms needed to do bioinformatics analysis. This course aims to present core topics of these fields with and emphasis on modelling and computation.

Draft Synopsis

Fundamental Data Structures in Biology: Sequences, Genes and RNA secondary structure.

Stochastic Models of Sequence and Genome Evolution including models of single nucleotide/amino acid/codon evolution.

Phylogenies: enumerating phylogenies, the probability of sequences related by a specified phylogeny, the minimal number of events needed to explain a data set (Parsimony).

Likelihood and algorithms (Markov Chain Monte Carlo) for inference based on the likelihood. Software packages for sample-based inference.

Alignment Algorithms. Comparing 2 strings, an arbitrary number of strings, find segments of high similarity in 2 strings.

Common Patterns in a set of Sequences.

Method of examination

4 examination questions.

Reading

1. C. Semple and M. Steel, *Phylogenetics*, Oxford University Press (2003).
2. Durbin et al., *Biological Sequence Analysis*, Cambrigde University Press (1998).
3. T. Jiang et al., (editors) *Current Topics in Computational Biology*, MIT Press, (2003).
4. M. S. Waterman et al., *Computational Genome Analysis: An Introduction*, Springer (2004).

MS2b: Stochastic Models in Mathematical Genetics — Prof B Griffiths and Prof A Etheridge — 16HT

Aims & Objectives

The aim of the lectures is to introduce modern Stochastic models in Mathematical Population Genetics that describe the distribution of gene frequencies and ancestry in a population or sample of genes. Stochastic and Graph theoretic properties of coalescent and gene trees are studied in the first eight lectures. Diffusion process models of gene frequencies and their applications are studied in the second eight lectures.

Synopsis

Evolutionary models in Mathematical Genetics:

The Wright-Fisher model. The Genealogical Markov chain describing the number ancestors back in time of a collection of genes.

The Coalescent process describing the stochastic behaviour of the ancestral tree of a collection of genes. Mutations on ancestral lineages in a coalescent tree. Inferring the time to the most recent common ancestor in a sample of genes from the number of mutations occurring to the genes. Models with a variable population size.

The frequency spectrum and age of a mutation. Ewens' sampling formula for the probability distribution of the allele configuration of genes in a sample in the infinitely-many-alleles model. Hoppe's urn model for the infinitely-many-alleles model.

The infinitely-many-sites model of mutations on DNA sequences. Gene trees as perfect phylogenies describing the mutation history of a sample of DNA sequences. Graph theoretic constructions and characterizations of gene trees from DNA sequence variation. Gusfield's construction algorithm of a tree from DNA sequences. Examples of gene trees from data. The probability distribution of a gene tree.

Diffusion process models in Mathematical Genetics:

Introduction to diffusion processes. The stochastic process describing the distribution of the gene frequency of an allele forward in time for a two-allele model. The Moran model. The diffusion process limit from the Moran model. The generator of a diffusion process with two allele types.

Heuristic introduction to Stochastic differential equations. Examples in using the diffusion process generator and Stochastic differential equations. The mean time to absorption or fixation of an allele.

The genealogy of the diffusion process describing the gene frequency of an allele. The underlying infinite-particle coalescent.

Two allele models with mutation and selection. Stationary distributions of diffusion process. Sampling from the stationary distribution.

A brief introduction to diffusion process models with more than two types. The Dirichlet distribution describing the stationary distribution of allele frequencies. The Poisson-Dirichlet process.

Method of examination

4 examination questions.

Reading

1. R. Durrett, *Probability Models for DNA Sequence Evolution*, Springer (2002).

2. W. J. Ewens, *Mathematical Population Genetics*, 2nd ed, Springer (2004).

3. J. R. Norris, *Markov Chains*, Cambridge University Press (1999).

4. M. Slatkin and M. Veuille, *Modern Developments in Theoretical Population Genetics*, Oxford Biology (2002).

5. S. Tavaré and O. Zeitouni, *Lectures on Probability Theory and Statistics, Ecole d'Eté de Probabilités de Saint-Flour XXXI - 2001*, Lecture Notes in Mathematics 1837. Springer (2004).

MS3b: Lévy Processes and Finance — Dr Winkel — 16HT

Method of examination

4 examination questions.

Prerequisites

Part A Probability is a prerequisite. BS3a/OBS3a Applied Probability or B10 Martingales and Financial Mathematics would be useful, but are by no means essential; some material from these courses will be reviewed without proof.

Aims

Lévy processes form a central class of stochastic processes, contain both Brownian motion and the Poisson process, and are prototypes of Markov processes and semimartingales. Like Brownian motion, they are used in a multitude of applications ranging from biology and physics to insurance and finance. Like the Poisson process, they allow to model abrupt moves by jumps, which is an important feature for many applications. In the last ten years Lévy processes have seen a hugely increased attention as is reflected on the academic side by a number of excellent graduate texts and on the industrial side realising that they provide versatile stochastic models of financial markets. This continues to stimulate further research in both theoretical and applied directions. This course will give a solid introduction to some of the theory of Lévy processes as needed for financial and other applications.

Synopsis

Review of (compound) Poisson processes, Brownian motion (informal), Markov property. Connection with random walks, [Donsker's theorem], Poisson limit theorem. Spatial Poisson processes, construction of Lévy processes.
Special cases of increasing Lévy processes (subordinators) and processes with only positive jumps. Subordination. Examples and applications. Financial models driven by Lévy processes. Stochastic volatility. Level passage problems. Applications: option pricing, insurance ruin, dams.
Simulation: via increments, via simulation of jumps, via subordination. Applications: option pricing, branching processes.

Reading

1. J.F.C. Kingman: *Poisson processes*, Oxford University Press (1993), Ch.1-5, 8.
2. A.E. Kyprianou: *Introductory lectures on fluctuations of Lévy processes with Applications*, Springer (2006), Ch. 1-3, 8-9.
3. W. Schoutens: *Lévy processes in finance: pricing financial derivatives*, Wiley (2003).

Further reading

1. J. Bertoin: *Lévy processes*, Cambridge University Press (1996), Sect. 0.1-0.6, I.1, III.1-2, VII.1.
2. K. Sato: *Lévy processes and infinite divisibility*,Cambridge University Press (1999), Ch. 1-2, 4, 6, 9.

Computer Science: Half Units

Please see the Computing Laboratory website for full, up-to-date course information.

CCS1: Categories, Proofs and Programs — Prof Abramsky — 16 MT

Overview

Category Theory is a powerful mathematical formalism which has become an important tool in modern mathematics, logic and computer science. One main idea of Category Theory is to study mathematical 'universes', collections of mathematical structures and their structure-preserving transformations, as mathematical structures in their own right, i.e. categories - which have their own structure-preserving transformations (functors). This is a very powerful perspective, which allows many important structural concepts of mathematics to be studied at the appropriate level of generality, and brings many common underlying structures to light, yielding new connections between apparently different situations.

Another important aspect is that set-theoretic reasoning with elements is replaced by reasoning in terms of arrows. This is more general, more robust, and reveals more about the intrinsic structure underlying particular set-theoretic representations.

Category theory has many important connections to logic. We shall in particular show how it illuminates the study of formal proofs as mathematical objects in their own right. This will involve looking at the Curry-Howard isomorphism between proofs and programs, and at Linear Logic, a resource-sensitive logic. Both of these topics have many important applications in Computer Science.

Category theory has also deeply influenced the design of modern (especially functional) programming languages, and the study of program transformations. One exciting recent development we will look at will be the development of the idea of coalgebra, which allows the formulation of a notion of coinduction, dual to that of mathematical induction, which provides powerful principles for defining and reasoning about infinite objects.

This course will develop the basic ideas of Category Theory, and explore its applications to the study of roofs in logic, and to the algebraic structure of programs and programming languages.

Synopsis

- Introduction to category theory. Categories, functors, natural transformations. Isomorphisms. monics and epics. Products and coproducts. Universal constructions. Cartesian closed categories. Symmetric monoidal closed categories. The ideas will be illustrated with many examples, from both mathematics and Computer Science.

- Introduction to structural proof theory. Natural deduction, simply typed lambda calculus, the Curry-Howard correspondence. Introduction to Linear Logic. The connection between logic and categories.

- Further topics in category theory. Algebras, coalgebras and monads. Connections to programming (structural recursion and corecursion), and to programming languages (monads as types for computational effects).

Reading List

Lecture notes will be provided.

The following books provide useful background reading.

1. B. C. Pierce, *Basic Category Theory for Computer Science*, MIT Press, 1991, ISBN: 0262660717.
2. F. W. Lawvere and S. H. Schanuel, *Conceptual Mathematics*, Cambridge University Press, 1997, ISBN: 0521478170.
3. Saunders Mac Lane *Categories for the Working Mathematician*, Springer; 2 edition, 1998, ISBN: 0387984038.
4. R. Crole, *Categories for Types*, Cambridge University Press, 1994), ISBN: 0521457017.
5. J.-Y. Girard, Y. Lafont and P. Taylor, *Proofs and Types*, Cambridge University Press, 1989), ISBN: 0521371813.
6. R. Bird and O. de Moor, *Algebra of Programming*, Prentice Hall PTR; 1st edition, 1996, ISBN: 013507245X.

Of these, the book by Pierce provides a very accessible and user-friendly first introduction to the subject.

Required Background

Some familiarity with basic discrete mathematics: sets, functions, relations, mathematical induction. Basic familiarity with logic: propositional and predicate calculus. Some first acquaintance with abstract algebra: vector spaces and linear maps, and/or groups and group homomorphisms. Some familiarity with programming, particularly functional programming, would be useful but is not essential.

CCS2: Domain Theory — Prof Abramsky — 16 HT

Overview

Domain theory is a mathematical theory of information and computation. It is based on the idea of states of (in general) partial information, ordered by how much information they contain. On this basis, a beautiful mathematical theory has been developed, with deep applications to many topics in Computer Science, in particular to the semantics of programming languages. In this course, we shall develop both the mathematical theory, and the applications. Particular themes will: the ideas of continuity and approximation supported by domain theory, which has important connections with topology, and gives a basis for computation with infinite objects; the development of a rich theory of fixpoints, as a foundation for recursive definitions; developing a rich set of data type constructions, and recursive definitions of domains themselves; and powerdomains, to support ideas of non-deterministic and probabilistic computation.

Synopsis

1. Basic concepts of partial orders. Mathematical models of syntax.
2. Fixpoints and recursive definitions, illustrated by a range of examples.
3. Data types: functions, sums and products.
4. Recursive types. Applications: streams, trees, untyped lambda calculus.
5. Algebraicity and continuity. Approximating infinite objects.
6. Powerdomains for non-determinism and probability. Applications to transition systems and processes.

Reading List

Notes will be provided.

Additional Reading

1. S. Abramsky and A. Jung, *Domain Theory*, in Handbook of Logic in Computer Science
2. B. Davey and H. Priestley, *Introduction to Lattices and Order*, Cambridge University Press; 2 edition, 2002, ISBN: 0521784514.
3. G. Plotkin, *Notes on Domains*, available from web.

Required Background

Discrete mathematics: sets, functions, relations, order relations. Domain theory would make a good combination with the course on Lambda calculus, but neither requires the other. Similar remarks apply to the course on Categories, Proofs and Programs.

CCS3: Quantum Computer Science — Dr Coecke — 16 HT

Pre-requisites

We do not assume any prior knowledge of quantum mechanics. Some knowledge of basic linear algebraic notions such as vector spaces and matrices is however a pre-requisite. The course notes do comprise an overview of this material so we advise students with a limited background in linear algebra to consult the course notes before the course starts.

Overview

Both physics and computer science have been very dominant scientific and technological disciplines in the previous century. Quantum Computer Science aims at combining both and hence promises to play an important similar role in this century. Combining the existing expertise in bothfields proves to be a non-trivial but very exciting interdisciplinary journey. Besides the actual issue of building a quantum computer or realizing quantum protocols it involves a fascinating encounter of concepts and formal tools which arose in distinct disciplines.

This course provides an interdisciplinary introduction to the emerging field of quantum computer science, explaining (very) basic quantum mechanics (including finite dimensional Hilbert spaces and the tensor product thereof), quantum entanglement, its structure and its physical consequences (e.g. non-locality, no-cloning principle), and introduces qubits. We give detailed discussions of some key algorithms and protocols such as Shor's factorization algorithm, quantum teleportation and quantum key exchange, and analyze the challenges these pose for computer science, mathematics etc. We also provide a more conceptual semantic analysis of some of the above. Other important issues such as quantum information theory (including mixed states) will also be covered (although not in great detail). We also discuss alternative computational paradigms and models for the circuit model, we argue the need for high-level methods, provide some recent results concerning categorical semantics and delineate the remaining scientific challenges for the future.

Learning Outcomes

The student will know by the end of the course what quantum computing and quantum protocols are about, why they matter, and what the scientific prospects concerning are. This includes a structural understanding of some basic quantum mechanics, knowledge of important algorithms such as Shor's algorithm and important protocols such as quantum teleportation. He/she will also know where to find more details and will be able to access these. Hence this course also offers computer science and mathematics students a first stepping-stone for research in the field, with a particular focus on the newly developing field of quantum computer science semantics, to which Oxford University Computing Laboratory has provided pioneering contributions.

Synopsis

1 Historical and physical context
1.1 The birth of quantum mechanics
1.2 The status of quantum mechanics
1.3 The birth of quantum informatics
1.4 The status of quantum informatics
2 Qubits vs. bits
2.1 Acting on qubits
2.2 Describing a qubit with complex numbers
2.3 Describing two qubits
3 von Neumanns pure state formalism
3.1 Hilbert space
3.2 Matrices
3.3 Tensor structure
3.4 Dirac notation
4 Protocols from entanglement
4.1 Bell-base and Bell-matrices
4.2 Teleportation and entanglement swapping
5 The structure of entanglement
5.1 Map-state duality and compositionality
5.2 The logic of bipartite entanglement
5.3 Quantifying entanglement
5.4 Trace
6 Algorithms and gates
6.1 The Deutch-Jozsa algorithm
6.2 Shor's factoring algorithm
6.2.1 Period finding
6.2.2 Factoring and code-breaking
6.3 Grover's search algorithm
6.4 Quantum key distribution
6.5 Special gates
7 Semantics for quantum informatics
7.1 What is semantics?
7.2 Semantics for systems
7.3 Semantics for quantum systems
7.4 Classical uncertainty and open systems

Syllabus

Finite dimensional vector space, inner-product, complex numbers, linear adjoints, unitary maps, projectors, trace, tensor product of Hilbert spaces, Dirac notation, bit, qubit, entanglement, map-state duality, no-cloning, quantum circuits, quantum gates, Shor's algorithm, Grover's algorithm, quantum teleportation, quantum key-exchange, teleportation and measurement based quantum computing, decoherence, mixed states, quantum information, quantum logic, quantum categorical semantics.

Reading List

Lecture notes which cover the whole course and which provide detailed pointers to additional reading will be made available.

Standard books on the subject that might be of use are:

1. Gruska, J. (1999) *Quantum Computing.* McGraw-Hill.
2. Nielsen, M. and Chuang, I. L. (2000) *Quantum Computation and Quantum Information.* Cambridge University Press.
3. Kitaev, A. Yu., Shen, A. H. and Vyalyi, M. N. (2001) *Classical and Quantum Computing.* Graduate Studies in Mathematics 47, American Mathematical Society.

On-line available courses elsewhere which can be consulted are:

1. http://www.weizmann.ac.il/chemphys/schmuel/comp/comp.html
2. http://www.theory.caltech.edu/people/preskill/ph229/

A different angle which is also very much reflected in the course is available at:

- http://arxiv.org/abs/quant-ph/0510032

Philosophy

Rise of Modern Logic — Dr V. Halbach — 16 MT

Weight: one unit.

Essential Prerequisites

Philosophy of Mathematics, B1 Foundations

Synopsis

The eight lectures shall concentrate on Frege's and Russell's logicism, the origins of set theory (Cantor and Zermelo), Hilbert's programme and the impact of Gödel's theorems on it. Further topics will include the rise of first-order logic and intuitionism.

Further details can be found in the Mathematics & Philosophy Handbook.

Method of examination

A three hour paper in Trinity Term and a 5000 word essay to be submitted by Week 0 in Trinity Term.

Reading

Much of the relevant material is contained in

Jean van Heijenoort (ed.), *From Frege to Godel. A Source Book in Mathematical Logic, 1879-1931*, (Harvard University Press).

A reading list will be available shortly at my website.

5 Language Classes

Mathematics and Mathematics & Statistics students may apply to take language classes. In 2006-7 French Language classes will be run in MT and HT. The courses are run by the University Language Centre.

Students wishing to take language classes should attend the presentation given in Trinity Term at 3.00pm, Friday 12th May (Week 3), at the Language Centre, 12 Woodstock Road. Preliminary registration forms will be available at this presentations but the meeting is a briefing at which students may ask questions. Inability to attend will not affect students' later registration.

If there is sufficient demand, two class will be offered, one for students with good GCSE-level (or equivalent) French and an additional class for those with AS-level/A-level (or equivalent) French. Allocation of a place in a class will depend on performance in the preliminary test which will be held in Week 1 of MT.

Successful completion of the French for Maths course will not contribute to the class of degree awarded in Mathematics. However, upon successful completion of the course, students will be issued with a certificate of achievement which will be sent to their college, and the details may appear on their transcript.

Places at these classes are limited, so students are advised that they should indicate their interest at this stage. If you are interested but were unable to attend this presentation for some reason please contact the Academic Administrator in the Mathematical Institute (academic.administrator@maths.ox.ac.uk; (2)73530) as soon as possible.

Aims and rationale

The general aim of the language courses is to develop the student's ability to communicate (in both speech and writing) in French to the point where he or she can function in an academic or working environment in a French-speaking country.

The courses have been designed with general linguistic competence in the four skills of reading, writing, speaking and listening in mind. There will be opportunities for participants to develop their own particular interests through presentations and assignments.

Form and Content

Each course will consist of a thirty-two contact hour course, together with associated work. Classes will be held in the Language Centre at two hours per week in Michaelmas and Hilary Terms.

The content of the courses is based on course books together with a substantial amount of supplementary material prepared by the language tutors. Participants should expect to spend an additional two hours per week on preparation and follow-up work.

Each course aims to cover similar ground in terms of grammar, spoken and written language and topics. Areas covered will include:

Grammar:

1. all major tenses will be presented and/or revised, including the subjunctive
2. passive voice
3. pronouns
4. formation of adjectives, adverbs, comparatives
5. use of prepositions
6. time expressions

Speaking

1. guided spoken expression for academic, work and leisure contact (e.g. giving presentations, informal interviews, applying for jobs)
2. expressing opinions, tastes and preferences
3. expressing cause, consequence and purpose

Writing

1. guided letter writing for academic and work contact
2. summaries and short essays

Listening

1. listening practice of recorded materials (e.g. news broadcasts, telephone messages, interviews)
2. developing listening comprehension strategies

Topics (with related readings and vocabulary, from among the following)

1. life, work and culture
2. the media in each country
3. social and political systems
4. film, theatre and music
5. research and innovation
6. sports and related topics
7. student-selected topics

Teaching staff

The courses are taught by Language Centre tutors or Modern Languages Faculty instructors.

Teaching and learning approaches

Each course uses a communicative methodology which demands active participation from students. In addition, there will be some formal grammar instruction. Students will be expected to prepare work for classes and homework will be set. Students will also be given training in strategies for independent language study, including computer-based language learning, which will enable them to maintain and develop their language skills after the course.

Entry

Two classes will be formed according to level of French at entry (in 2004-2005 these were at about post-GCSE and post-A level standard). The minimum entry standard is a good GCSE Grade or equivalent. Registration for each option will take place at the start of Michaelmas Term. A preliminary qualifying test is then taken which candidates must pass in order to be allowed to join the course.

Learning outcomes

The learning outcomes will be based on internationally agreed criteria for specific levels (known as the ALTE levels), which are as follows:

Basic Level (corresponds to ALTE Level 2 "Can-do" statements)

1. Can express opinions on professional, cultural or abstract matters in a limited way, offer advice and understand instructions.
2. Can give a short presentation on a limited range of topics.
3. Can understand routine information and articles, including factual articles in newspapers, routine letters from hotels and letters expressing personal opinions.
4. Can write letters or make notes on familiar or predictable matters.

Threshold Level (corresponds to ALTE Level 3 "Can-do" statements)

1. Can follow or give a talk on a familiar topic or keep up a conversation on a fairly wide range of topics, such as personal and professional experiences, events currently in the news.
2. Can give a clear presentation on a familiar topic, and answer predictable or factual questions.
3. Can read texts for relevant information, and understand detailed instructions or advice.
4. Can make notes that will be of reasonable use for essay or revision purposes.
5. Can make notes while someone is talking or write a letter including non-standard requests.

Higher Level (corresponds to ALTE Level 4 "Can-do" statements)

1. Can contribute effectively to meetings and seminars within own area of work or keep up a casual conversation with a good degree of fluency, coping with abstract expressions.
2. Can follow abstract argumentation, for example, balancing alternatives and drawing a conclusion.
3. Can read quickly enough to cope with an academic course, to read the media for information or to understand non-standard correspondence.
4. Can prepare/draft professional correspondence, take reasonably accurate notes in meetings or write an essay which shows an ability to communicate, giving few difficulties for the reader.

Assessment

There will a preliminary qualifying test in Michaelmas Term. There are three parts to this test: Reading Comprehension, Listening Comprehension, and Grammar. Candidates who have not studied or had contact with French for some time are advised to revise thoroughly, making use of the Language Centre's French resources.

Students' achievement will be assessed through a variety of means, including continuous assessment of oral performance, a written final examination, and a project or assignment chosen by individual students in consultation with their language tutor.

Reading comprehension, writing, listening comprehension and speaking are all examined. The oral component may consist of an interview, a presentation or a candidate's performance in a formal debate or discussion.

Feedback Oral and written feedback from students are encouraged. A Feedback Form will be distributed at the end of each course.

6 Registration

LECTURES AND CLASSES: Students will have to register in advance for the lectures and classes they wish to take. Students will have to register by Monday of Week 9 of Trinity Term 2006 using the form found at https://www.maths.ox.ac.uk/current-students/undergraduates/forms/. In the case of unexpected excessive demand for inter-collegiate class places for an option, the lecturer may apply for a quota to be imposed. Students should check their e-mail regularly over summer, as those affected by the introduction of such a quota will be contacted by email, and presented with an opportunity to make a case for a place on a quota and to name an alternative option.

LECTURES: Some combinations of subjects are not advised and lectures may clash. Details are given below. We will use the information on your registration forms to aim to keep clashes to a minimum. However, because of the large number of options available in Part C some clashes are inevitable, and we must aim to accommodate the maximum number of student preferences.

Lecture Timetabling in Part C, 2006-7

The Teaching Committee has agreed that the following clashes be allowed.

C1.1 Model Theory & Godel's Incompleteness Theorems	may clash with	C6.1 Solid Mechanics
C1.2 Axiomatic Set Theory & Analytic Topology		C6.2 Elasticity and Plasticity
C2.1 Group Theory & Lie Algebras		C6.3 Perturbation Methods & Applied Complex Variables
C3.1 Topology and Groups and Algebraic Topology		C8.1 Mathematics and the Environment & Mathematical Physiology
C9.1 Analytic Number Theory & Elliptic Curves		All MS options
		All CNA options