September 2007

Handbook for the Undergraduate Mathematics Courses

Supplement to the Handbook

Honour School of Mathematics

Syllabus and Synopses for Part C 2007–8

for examination in 2008

Contents

1 Foreword **5**

Honour School of Mathematics 5

Honour School of Mathematics & Statistics 7

Honour School of Mathematics & Philosophy 7

Registration for Mathematics and Mathematics & Philosophy students . . . 8

"Units" and "Half-Units" and methods of examination 8

2 Mathematics Department Units **8**

C1.1: Gödel's Incompleteness Theorems and Model Theory 8

C1.1a: Gödel's Incompleteness Theorems — Dr Isaacson — 16MT . 9

C1.1b: Model Theory — Dr Koenigsmann — 16HT 9

C1.2: Analytic Topology and Axiomatic Set Theory 10

C1.2a: Analytic Topology — Prof. Haydon — 16MT 11

C1.2b: Axiomatic Set Theory — Prof. Zilber — 16HT 11

C2.1: Lie Algebras and Representation Theory of Symmetric Groups 12

C2.1a: Lie Algebras — Prof. J. Wilson — 16MT 13

C2.1b — Representation Theory of Symmetric Groups — Dr Miemietz — 16HT 14

C3.1: Lie Groups and Differentiable Manifolds 16

C3.1a: Lie Groups — Prof. Tillmann — 16MT 16

C3.1b: Differentiable Manifolds — Prof. Hitchin — 16HT 18

C3.2a — Algebraic Geometry — Dr Baldwin — 16MT 20
C4.1: Functional Analysis, Banach and C*- algebras 21
C4.1a: Functional Analysis — Prof. Batty — 16MT 21
C4.1b: Banach and C*- algebras — Dr Edwards — 16HT 22
C4.2a: Real and Harmonic Analysis — Dr Kirchheim — 16 MT 23
C5.1: PDEs for Pure and Applied Mathematicians and Calculus of Variations 24
C5.1a: PDEs for Pure and Applied Mathematicians — Dr Dyson — 16MT 24
C5.1b: Calculus of Variations — Prof. Seregin — 16HT 26
C5.2b Fixed Point Methods for Nonlinear PDEs — Prof. Niethammer — 16 HT 27
C6.1a: Solid Mechanics — Professor Ball — 16 MT 28
C6.2b: Elasticity and Plasticity — Dr J Ockendon — 16HT 30
C6.3: Perturbation Methods and Applied Complex Variables 31
C6.3a: Perturbation Methods — Dr Porter — 16MT 31
C6.3b: Applied Complex Variables — Prof. Chapman — 16HT . . . 32
C6.4a: Topics in Fluid Mechanics — Drs Fowler/Howell/Norbury/Prof. Chapman — 16MT 33
C7.1b: Quantum Theory and Quantum Computers — Dr Hannabuss — 16HT 34
C7.2b: General Relativity I — Prof. Chruściel — 16HT 36
C7.4 — Theoretical Physics 37
C7.4a Theoretical Physics I — lecturers Prof. Chalker and Dr Lukas — 24MT 37
C7.4b — Theoretical Physics II — lecturers Prof. Chalker and Dr Lukas — 16HT 38
C8.1: Mathematics and the Environment and Mathematical Physiology . . 39
C8.1a: Mathematics and the Environment — Dr Fowler/Dr Sander — 16MT 39
C8.1b: Mathematical Physiology — Prof. Maini — 16HT 40
C9.1: Analytic Number Theory and Elliptic Curves 41
C9.1a: Analytic Number Theory — Prof. Heath-Brown — 16MT . . 41
C9.1b Elliptic Curves — Prof. Flynn — 16 lectures HT 42
C10.1 Stochastic Differential Equations and Brownian Motion in Complex Analysis 44
C10.1a: Stochastic Differential Equations — Dr Qian —16MT . . . 44

C10.1b: Brownian Motion in Complex Analysis — Prof. Lyons — 16HT . . . 45

C11.1: Graph Theory and Probabilistic Combinatorics . . . 47

C11.1a: Graph Theory — Prof. Scott — 16MT . . . 47

C11.1b: Probabilistic Combinatorics — Dr Martin — 16HT . . . 48

C12.1 Numerical Linear Algebra and Continuous Optimization . . . 49

C12.1a Numerical Linear Algebra and Approximation — Dr Wathen — 16MT . . . 49

C12.1b Continuous Optimization — Prof. Gould — 16HT . . . 50

C12.2 Approximation Theory and Finite Element Methods . . . 50

C12.2a Approximation of Functions — Dr Sobey — 16MT . . . 51

C12.2b Finite Element Methods for Partial Differential Equations — Prof. Süli — 16HT . . . 52

Dissertations . . . 53

3 Extra Units — application required **53**

4 Other Units **53**

MS: Statistics Half-units . . . 53

MS1a: Graphical Models and Inference — lecturer tbc — 16MT . . . 53

MS1b: Statistical Data Mining — lecturer tbc — 12HT and 4 one-hour computer labs . . . 54

MS2a: Bioinformatics and Computational Biology — lecturer tbc — 16MT . . . 55

MS2b: Stochastic Models in Mathematical Genetics — lecturer tbc — 16HT . . . 56

MS3b: Lévy Processes and Finance — lecturer tbc — 16HT . . . 58

Computer Science: Half Units . . . 59

CCS1a: Categories, Proofs and Processes — lecturer t.b.c — 20 lectures (plus extra reading) MT . . . 60

CCS3b: Quantum Computer Science — Dr Coecke — 20 lectures (plus extra reading) HT . . . 62

CCS4b — Automata, Logics and Games — Prof. Ong — 16 lectures (plus extra reading) HT . . . 65

Philosophy . . . 67

Rise of Modern Logic — no lectures in 2007-08 but tutorials will be given for those wishing to take it. . . . 67

5 Language Classes **67**

6 Registration **72**

1 Foreword

This is the final confirmed version for Michaelmas Term 2007. It will be available in hard copy from reception in the Mathematical Institute in October 2007, and on the website at:

http://www.maths.ox.ac.uk/current-students/undergraduates/handbooks-synopses/

before the start of Michaelmas Term 2007.

Notice of misprints or errors of any kind, and suggestions for improvements in this booklet should be addressed to the Academic Assistant in the Mathematical Institute.

Honour School of Mathematics

Students staying on to take the four-year course will take the equivalent of 3 units from Part C in their fourth year. The equivalent of two units must be taken from the menu of "Mathematics department units" or offered as a Dissertation on a mathematical topic. Up to one unit may be taken from the menu of "Other units".

For the MMath in Mathematics in TT 2008 one of the classifications will be based on Part C alone. Each candidate will receive a numerical mark on each paper in each Part of the examination in the University standardised range 0-100, such that

- a First Class performance (on that paper) is indicated by a mark of 70 to 100;
- an Upper Second Class performance (on that paper) is indicated by a mark of 60 to 69;
- a Lower Second Class performance (on that paper) is indicated by a mark of 50 to 59
- a Third Class performance (on that paper) is indicated by a mark of 40 to 49;
- a Pass performance (on that paper) is indicated by a mark of 30 to 39;
- a performance at the level of a Fail (on that paper) is indicated by a mark of 0 to 29.

Note on Inter-Collegiate Classes

Where undergraduate registrations for lecture courses fall below 5, classes will not run as part of the inter-collegiate scheme but will be arranged informally by the lecturer.

The Examination Papers in Part C

Mathematics Department units and half units: Each 16 hour lecture course is examined as a half unit. Most papers are whole unit papers, that is, they contain the questions for two half units. There are four questions set for each half unit. If you are taking a full unit, then you may attempt as many questions as you like but only your two best answers on each half unit will count towards your final mark for the paper. The same applies when you take a half unit paper, or only enter for one half of a full unit paper, namely only your two best questions on that half unit count.

Marking of Papers in Part C

For the Mathematics Department papers, each question will initially be marked out of 25 and marking schemes for the questions will aim to ensure that the following qualitative criteria hold:

- 20-25 marks. A completely, or almost completely, correct answer, showing excellent understanding of the concepts and skill in carrying through the arguments and/or calculations.
- 13-19 marks. A good though not complete answer, showing understanding of the concepts and competence in handling the arguments and/or calculations. Such an answer might consist of an excellent answer to a substantial part of the question, or a good answer to the whole question which nevertheless shows some flaws in calculation or in understanding or in both.

Marks for other mathematical and non-mathematical papers will be reported to the Mathematics Examiners by the relevant assessing panel in University Standardised Mark (USM) form.

Analysis of Marks in Part C

The Board of Examiners in Part C will assign USMs for full unit and half unit papers taken in Part C and may recalibrate the raw marks to arrive at university standardised marks reported to candidates. The full unit papers are designed so that the raw marks sum to 100, however, Examiners will take into account the relative difficulty of papers when assigning USMs. In order to achieve this, Examiners may use information on candidates' performances on the earlier Parts of the examination when recalibrating the raw marks. They may also use other statistics to check that the USMs assigned fairly reflect the students' performances on a paper.

The USMs awarded to a candidate for papers in Part C will be used to arrive at a classification for Part C of the MMath.

AvUSM-Part C = Average USM in Part C (rounded up to whole number);

- First Class:
  $AvUSM - \text{Part C} \geq 70$
- Upper Second Class:
  $70 > AvUSM - \text{Part C} \geq 60$
- Lower Second Class:
  $60 > AvUSM - \text{Part C} \geq 50$
- Third Class:
  $50 > AvUSM - \text{Part C} \geq 40$

[Note: Half unit papers count as half a paper when determining the average USM.]

Candidates leaving after four years who satisfy the Examiners will be awarded an MMath in Mathematics, with two associated classifications; for example: MMath in Mathematics: Years 2 and 3 together - First Class; Year 4 - First Class

A 'Pass' will not be awarded for Year 4. Candidates achieving:

$$AvUSM - \text{Part C} < 40,$$

may supplicate for a BA.

Descriptors

The average USM ranges used in the classification reflect the following descriptions:

- First Class: the candidate shows excellent problem-solving skills and excellent knowledge of the material, and is able to use that knowledge in unfamiliar contexts.
- Upper Second Class: the candidate shows good problem-solving skills and good knowledge of much of the material.
- Lower Second Class: the candidate shows adequate basic problem-solving skills and knowledge of much of the material.
- Third Class: the candidate shows reasonable understanding of at least part of the basic material and some problem solving skills. Threshold level.
- Pass: the candidate shows some limited grasp of basic material demonstrated by the equivalent of an average of one meaningful attempt at a question on each unit of study. A stronger performance on some papers may compensate for a weaker performance on others.
- Fail: little evidence of competence in the topics examined; the work is likely to show major misunderstanding and confusion, coupled with inaccurate calculations; the answers to questions attempted are likely to be fragmentary only.

Honour School of Mathematics & Statistics

See details published by the Statistics Department.

Honour School of Mathematics & Philosophy

In Part C each candidate shall offer a total of three units chosen in any combination from the lists for Mathematics and for Philosophy. Units in Mathematics are given in this booklet.

Registration for Mathematics and Mathematics & Philosophy students

YOU MUST REGISTER BY THE END OF TRINITY TERM 2007 FOR LECTURE AND CLASS ATTENDANCE FOR ALL COURSES YOU WISH TO TAKE IN 2007–08. A REGISTRATION FORM IS ATTACHED TO THESE SYNOPSES.

SOME COMBINATIONS OF SUBJECTS ARE NOT ADVISED AND LECTURES IN THESE SUBJECTS MAY CLASH. HOWEVER, WHEN TIMETABLING LECTURES WE WILL AIM TO KEEP CLASHES TO A MINIMUM. WE WILL USE THE INFORMATION ON YOUR REGISTRATION FORMS TO PLAN CLASS TEACHING AND LECTURING. IT IS THEREFORE IMPORTANT THAT YOU RETURN YOUR REGISTRATION FORM BY MONDAY, WEEK 9, TRINITY TERM 2007. IF YOU NEED TO CHANGE YOUR REGISTRATIONS AFTER THAT DATE, PLEASE INFORM THE ACADEMIC ASSISTANT AS SOON AS POSSIBLE.

"Units" and "Half-Units" and methods of examination

Most Mathematics Department lecture courses are available as half-units, the exceptions being:

1. C5.1b Calculus of Variations, where C5.1a PDEs for Pure and Applied Mathematicians is an essential pre-requisite.
2. C7.4 Theoretical Physics - this is available as a whole-unit only.
3. C11.1b Probabilistic Combinatorics, where C11.1a Graph Theory is an essential pre-requisite.

Half units are examined in an examination paper of $1\frac{3}{4}$ hours, unless the synopsis states otherwise; whole units will be examined in a 3 hour exam unless the synopsis states otherwise.

All Computer Science half-units will be examined by a paper of $1\frac{1}{2}$ hours in length.

All the units and half-units described in this booklet are "M–Level".

2 Mathematics Department Units

C1.1: Gödel's Incompleteness Theorems and Model Theory

Level: M-level **Method of Assessment:** Written Examination

Weight: Whole unit, or can be taken as either a Half-unit in C1.1a or a Half-unit in C1.1b

C1.1a: Gödel's Incompleteness Theorems — Dr Isaacson — 16MT

Recommended Prerequisites

This course presupposes basic knowledge of First Order Predicate Calculus up to and including the Soundness and Completeness Theorems.

Aims & Objectives

This course introduces important techniques and results in modern logic which go to the heart of the relationship between truth and formal proof, in particular that show how to obtain, for any consistent formal system containing basic arithmetic, a sentence in the language of that system which is true but not provable in the system.

Synopsis

Gödel numbering of a formal language; the diagonal lemma. The arithmetical undefinability of truth in arithmetic. Formal systems of arithmetic; arithmetical proof predicates. The representation of functions and sets. ω-consistency; the first Gödel incompleteness theorem. Σ_0 and Σ_1-completeness. Separability; the Rosser incompleteness theorem. Adequacy conditions for a provability predicate. The second Gödel incompleteness theorem; Löb's theorem. Provable Σ_1-completeness. Provability logic. The Hilbert–Bernays arithmetized completeness theorem; a formally undecided sentence of arithmetic whose truth value is not known. The ω-rule.

Reading

1. R.M. Smullyan, *Gödel's Incompleteness Theorems*, OUP (1992).

Further Reading

1. G.S. Boolos and R.C. Jeffrey, *Computability and Logic*, 3rd edition, CUP (1989), Chs 15, 16, pp 170–190.

C1.1b: Model Theory — Dr Koenigsmann — 16HT

Recommended Prerequisites

This course presupposes basic knowledge of First Order Predicate Calculus up to and including the Soundness and Completeness Theorems. Also a familiarity with (at least the statement of) the Compactness Theorem would also be desirable.

Aims & Objectives

The course deepens a student's understanding of the notion of a mathematical structure and of the logical formalism that underlies every mathematical theory, taking B1 Logic as a starting point. Various examples emphasise the connection between logical notions and practical mathematics.

The concepts of completeness and categoricity will be studied and some more advanced technical notions, up to elements of modern stability theory, will be introduced.

Learning Outcomes

1. To advance the students' knowledge of the notion of an algebraic mathematical structure and of its logical theory, taking B1 Logic as a starting point.
2. To study the concepts of completeness and categoricity and introduce some more advanced technical notions.

Synopsis

Structures. The first-order language for structures. The Compactness Theorem for first-order logic. Elementary embeddings. Löwenheim–Skolem theorems. Preservation theorems for substructures.

Categoricity for first-order theories. Types and saturation. Omitting types. The Ryll–Nardzewski theorem characterizing aleph-zero categorical theories. Theories with few types.

Reading

1. J. Bridge, *Beginning Model Theory*, OUP (1977). (Out of print but can be found in libraries).
2. W. Hodges, *Shorter Model Theory*, CUP (1997).

Further reading

1. All topics discussed (and much more) can also be found in W. Hodges, *Model Theory*. CUP, (1993).

C1.2: Analytic Topology and Axiomatic Set Theory

Level: M-level **Method of Assessment:** Written Examination

Weight: Whole unit, or can be taken as either a half-unit in C1.2a or a half-unit in C1.2b

C1.2a: Analytic Topology — Prof. Haydon — 16MT

We find it necessary to run Analytic Topology in MT. Classes for Analytic Topology will be run both in MT and HT, so that students who find themselves overburdened in MT will have the opportunity to attend classes on Analytic Topology in HT.

Recommended Prerequisites

Part A Topology; a basic knowledge of Set Theory, including ordinals and the Axiom of Choice, will also be useful.

Aims & Objectives

The aim of the course is to present a range of major theory and theorems, both important and elegant in themselves and with important applications within topology and to mathematics as a whole. Central to the course is the general theory of compactness and Tychonoff's theorem, one of the most important in all mathematics (with applications across mathematics and in mathematical logic) and computer science.

Synopsis

Separation axioms, Urysohn's lemma. Separable, Lindelöf and second countable spaces. Urysohn's metrization theorem. Filters and ultrafilters. Tychonoff's theorem. Compactifications, in particular, the Alexandroff One-Point Compactification and the Stone–Čech Compactification. Paracompact spaces and A.H. Stone's theorem. Connectedness and local connectedness. Components and quasi-components. Totally disconnected compact spaces, Boolean algebras and Stone spaces.

Reading

1. S. Willard, *General Topology*, Addison–Wesley (1970), Chs. 1–8.
2. N. Bourbaki, *General Topology*, Springer–Verlag (1989), Ch. 1.

C1.2b: Axiomatic Set Theory — Prof. Zilber — 16HT

Recommended Prerequisites

This course presupposes basic knowledge of First Order Predicate Calculus up to and including the Soundness and Completeness Theorems, together with a course on basic set theory, including cardinals and ordinals,the Axiom of Choice and the Well Ordering Principle.

Aims & Objectives

Inner models and consistency proofs lie at the heart of modern Set Theory, historically as well as in terms of importance. In this course we shall introduce the first and most important of inner models, Gödel's constructible universe, and use it to derive some fundamental consistency results.

Synopsis

A review of the axioms of ZF set theory. The recursion theorem for the set of natural numbers and for the class of ordinals. The Cumulative Hierarchy of sets and the consistency of the Axiom of Foundation as an example of the method of inner models. Levy's Reflection Principle. Gödel's inner model of constructible sets and the consistency of the Axiom of Constructibility ($V = L$). The fact that $V = L$ implies the Axiom of Choice. Some advanced cardinal arithmetic. The fact that $V = L$ implies the Generalized Continuum Hypothesis.

Reading

For the review of ZF set theory:

1. D. Goldrei, *Classic Set Theory*, Chapman and Hall (1996).

For course topics (and much more):

1. K. Kunen, *Set Theory: An Introduction to Independence Proofs*, North Holland (1983) (now in paperback). Review: Chapter 1. Course topics: Chapters 3, 4, 5, 6 (excluding section 5).

Further reading

1. K. Hrbacek and T. Jech, *Introduction to Set Theory*, 3rd edition, M Dekker (1999).

C2.1: Lie Algebras and Representation Theory of Symmetric Groups

Level: M-level **Method of Assessment:** Written Examination

Weight: Whole-unit, or can be taken as either a Half-unit in C2.1a or a Half-unit in C2.1b.

C2.1a: Lie Algebras — Prof. J. Wilson — 16MT

Recommended Prerequisites

Part B course B2a. A thorough knowledge of linear algebra and the second year algebra courses; in particular familiarity with group actions, quotient rings and vector spaces, with isomorphism theorems and with inner product spaces will be assumed. Some familiarity with the Jordan–Hölder theorem and the general ideas of representation theory will be an advantage.

Aims & Objectives

Lie Algebras are mathematical objects which, besides being of interest in their own right, elucidate problems in several areas in mathematics. The classification of the finite-dimensional complex Lie algebras is a beautiful piece of applied linear algebra. The aims of this course are to introduce Lie algebras, develop some of the techniques for studying them, and describe parts of the classification mentioned above, especially the parts concerning root systems and Dynkin diagrams.

Learning Outcomes

Students will learn how to utilise various techniques for working with Lie algebras, and they will gain an understanding of parts of a major classification result.

Synopsis

Definition of Lie algebras, small-dimensional examples, some classical groups and their Lie algebras (treated informally). Ideals, subalgebras, homomorphisms, modules.

Nilpotent algebras, Engel's theorem; soluble algebras, Lie's theorem. Semisimple algebras and Killing form, Cartan's criteria for solubility and semisimplicity.

The root space decomposition of a Lie algebra; root systems, Cartan matrices and Dynkin diagrams. Classification of irreducible root systems. Description (with few proofs) of the classification of complex simple Lie algebras; examples.

Reading

1. J.E. Humphreys. *Introduction to Lie Algebras and Representation Theory*, Graduate Texts in Mathematics 9. Springer–Verlag (1972, reprinted 1997). Chapters 1–3 are relevant and part of the course will follow Chapter 3 closely.

2. B. Hall. *Lie Groups, Lie Algebras, and Representations. An Elementary Introduction.* Graduate Texts in Mathematics, 222, Springer–Verlag (2003).

3. K. Erdmann, M. J. Wildon. *Introduction to Lie Algebras.* Springer–Verlag (2006)ISBN: 1846280400.

Additional reading

1. J.-P. Serre. *Complex Semisimple Lie Algebras.* Springer (1987). Rather condensed, assumes the basic results. Very elegant proofs.

2. N. Bourbaki. *Lie Algebras and Lie Groups.* (Masson, 1982). Chapters 1 and 4–6 are relevant; this text fills in some of the gaps in Serre's text.

C2.1b — Representation Theory of Symmetric Groups — Dr Miemietz — 16HT

Recommended Prerequisites

A thorough knowledge of linear algebra and the second year algebra courses; in particular familiarity with the symmetric groups, (symmetric) group actions, quotient vector spaces, with isomorphism theorems and with inner product spaces will be assumed. Some familiarity with basic representation theory from B2 (group algebras, simple modules, reducibility, Maschke's theorem, Wedderburn's theorem, characters) will be an advantage.

Aims & Objectives

The representation theory of the symmetric groups is a special case of the representation theory of finite groups. While representations over a field of characteristic zero are well-understood, fundamental questions over a field of prime characteristic remain widely open. The course will be algebraic and combinatorial in flavour. It will follow the approach taken by G. James. The final goal is to construct and parametrise the simple modules of the symmetric groups over an arbitrary field. On the way the theory over a field of characteristic zero will be developed. Other highlights of the course include combinatorial algorithms like the Robinson–Schensted–Knuth correspondence.

Synopsis

Reminder: conjugacy classes of symmetric groups and parametrisations of simple modules.

Counting standard tableaux of fixed shape: Young diagrams and tableaux, standard tableaux, Young–Frobenius formula, hook formula. Robinson–Schensted–Knuth algorithm and correspondence.

Construction of fundamental modules for symmetric groups: Action of symmetric groups on tableaux, tabloids and polytabloids; permutation modules on Young subgroups, Specht modules. Standard basis for Specht modules. Homomorphisms between permutation modules, Young's rule. Examples and applications.

Basic combinatorial lemma, simplicity of Specht modules in characteristic zero, classification of simple S_n-modules in characteristic zero. Character tables of symmetric groups, Murnaghan–Nakayama rule.

Submodule Theorem, construction and classification of simple S_n-modules over a field of prime characteristic. Properties of simple modules over a field of prime characteristic. Decomposition matrices. Examples and applications.

Reading

1. W Fulton. *Young Tableaux*, London Mathematical Society Student Texts 35. Cambridge University Press (1997). From Part I and II.
2. D Knuth. *The Art of Computer Programming*, Volume 3. Addison–Wesley (1998). From Chapter 5.
3. B E Sagan. *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*, Graduate Texts in Mathematics 203. Springer–Verlag (2000). Chapters 1 – 2.

Additional Reading

1. W Fulton, J Harris. *Representation Theory. A first course*, Graduate Texts in Mathematics. Readings in Mathematics 129. Springer Verlag (1991). From Part I.
2. G James. *The Representation Theory of the Symmetric Groups*, Lecture Notes in Mathematics 682, Springer Verlag (1978).
3. G James, A Kerber. *The Representation Theory of the Symmetric Groups*, Encyclopaedia of Mathematics and its Applications 16. Addison–Wesley (1981). From Chapter 7.
4. R Stanley. *Enumerative Combinatorics. Volume 2*, Cambridge Studies in Advanced Mathematics 62. Cambridge University Press (1999).

C3.1: Lie Groups and Differentiable Manifolds

Level: M-level. **Method of Assessment:** Written Examination.

Weight: Whole unit, or can be taken either as a half-unit in Lie Groups or a half-unit in Differentiable Manifolds.

C3.1a: Lie Groups — Prof. Tillmann — 16MT

Recommended Prerequisites:

2nd year Groups in Action, Topology, Multivariable Calculus.

Aims & Objectives

The theory of Lie Groups is one of the most beautiful developments of pure mathematics in the twentieth century, with many applications to geometry, theoretical physics and mechanics, and links to both algebra and analysis. Lie groups are groups which are simultaneously manifolds, so that the notion of differentiability makes sense, and the group multiplication and inverse maps are differentiable. However this course introduces the theory in a more concrete way via groups of matrices, in order to minimise the prerequisites.

Learning Outcomes

Students will have learned the basic theory of topological matrix groups and their representations. This will include a firm understanding of root systems and their role for representations.

Synopsis

The exponential map for matrices, Ad and ad, the Campbell–Baker–Hausdorff series.

Linear Groups, their Lie algebras and the Lie correspondance. Homomorphisms and coverings of linear groups. Examples including $SU(2)$, $SO(3)$ and $SL(2;\mathbb{R}) \cong SU(1,1)$.

The compact and complex classical Lie groups. Cartan subgroups, Weyl groups, weights, roots, reflections.

Informal discussion of Lie groups as manifolds with differentiable group structures; quotients of Lie groups by closed subgroups.

Bi-invariant integration on a compact group (statement of existence and basic properties only). Representations of compact Lie groups. Tensor products of representations. Complete reducibility, Schur's lemma. Characters, orthogonality relations.

Statements of Weyl's character formula, the theorem of the highest weight and the Borel–Weil theorem, with proofs for $SU(2)$ only.

Reading

W. Rossmann, *Lie Groups: An Introduction through Linear Groups*, (Oxford, 2002), Chapters 1–3 and 6.

A. Baker, *Matrix Groups: An Introduction to Lie Group Theory*, (Springer Undergraduate Mathematics Series).

Further Reading

J. F. Adams, *Lectures on Lie Groups* (University of Chicago Press, 1982).

R. Carter, G. Segal and I. MacDonald, *Lectures on Lie Groups and Lie Algebras* (LMS Student Texts, Cambridge, 1995).

J. F. Price, *Lie Groups and Compact Groups* (LMS Lecture Notes 25, Cambridge, 1977).

C3.1b: Differentiable Manifolds — Prof. Hitchin — 16HT

Recommended Prerequisites

2nd year core algebra, topology, multivariate calculus. Useful but not essential: groups in action, geometry of surfaces.

Aims & Objectives

A manifold is a space such that small pieces of it look like small pieces of Euclidean space. Thus a smooth surface, the topic of the B3 course, is an example of a (2-dimensional) manifold.

Manifolds are the natural setting for parts of classical applied mathematics such as mechanics, as well as general relativity. They are also central to areas of pure mathematics such as topology and certain aspects of analysis.

In this course we introduce the tools needed to do analysis on manifolds. We prove a very general form of Stokes' Theorem which includes as special cases the classical theorems of Gauss, Green and Stokes. We also introduce the theory of de Rham cohomology, which is central to many arguments in topology.

Learning Outcomes

The candidate will be able to manipulate with ease the basic operations on tangent vectors, differential forms and tensors both in a local coordinate description and a global coordinate-free one; have a knowledge of the basic theorems of de Rham cohomology and some simple examples of their use; know what a Riemannian manifold is and what geodesics and harmonic forms are.

Synopsis

Smooth manifolds and smooth maps. Tangent vectors, the tangent bundle, induced maps. Vector fields and flows, the Lie bracket and Lie derivative.

Exterior algebra, differential forms, exterior derivative, Cartan formula in terms of Lie derivative. Orientability. Partitions of unity, integration on oriented manifolds.

Stokes' theorem. De Rham chomology and discussion of de Rham's theorem. Applications of de Rham theory including degree.

Riemannian metrics.

Reading

1. M. Spivak, *Calculus on Manifolds*, (W. A. Benjamin, 1965).

2. M. Spivak, *A Comprehensive Introduction to Differential Geometry*, Vol. 1, (1970).

3. W. Boothby, *An Introduction to Differentiable Manifolds and Riemannian Geometry*, 2nd edition, (Academic Press, 1986).
4. M. Berger and B. Gostiaux, *Differential Geometry: Manifolds, Curves and Surfaces.* Translated from the French by S. Levy, (Springer Graduate Texts in Mathematics, 115, Springer–Verlag (1988)) Chapters 0–3, 5–7.
5. F. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, (Springer Graduate Texts in Mathematics, 1994).
6. D. Barden and C. Thomas, *An Introduction to Differential Manifolds.* (Imperial College Press, London, 2003.)

C3.2a — Algebraic Geometry — Dr Baldwin — 16MT

Level: M-level **Method of Assessment:** Written Examination.

Weight: Half-unit.

Recommended Prerequisites

B3b Algebraic Curves is a prerequisite. (B9a Polynomial Rings and Galois theory are useful but not essential.)

Aims & Objectives:

Algebraic geometry is the study of algebraic varieties: an algebraic variety is, roughly speaking, a locus defined by polynomial equations. One of the advantages of algebraic geometry is that it is purely algebraically defined and applies to any field, including fields of finite characteristic. It is geometry based on algebra rather than on calculus, but over the real or complex numbers it provides a rich source of examples and inspiration to other areas of geometry.

Learning Outcomes:

1. A familiarity with what quasi-projective varieties are.
2. Awareness of the interplay of algebra and geometry via Hilbert's Nullstellensatz.
3. An appreciation of simple examples of birational equivalence, and of resolution of singularities.

Synopsis:

Affine algebraic varieties, the Zariski topology, morphisms of affine varieties, coordinate rings, regular functions, irreducible varieties.

Projective spaces, projective varieties. The projective closure of an affine variety. Morphisms of projective varieties. Weighted projective spaces.

Veronese maps, Segre maps and products of varieties, Grassmannians. Degree and the Hilbert polynomial.

Abstract (general) algebraic varieties. Morphisms of algebraic varieties.

Tangent spaces, smooth points.

Rational maps and birational equivalence. Blowing up along an ideal. Resolution of singularities (statement only).

Reading

KE Smith et al, *An Invitation to Algebraic Geometry*, Springer (2000), Chapters 1–8.

Supplementary reading

1. M Reid,*Undergraduate Algebraic Geometry*, LMS Student Texts 12, Cambridge (1988).
2. K Hulek,*Elementary Algebraic Geometry*, Student Mathematical Library, 20. American Mathematical Society, 2003.

C4.1: Functional Analysis, Banach and C*- algebras

Level: M-level **Method of Assessment:** Written Examination

Weight: Whole unit, or can be taken as either a half-unit in C4.1a or a half-unit in C4.1b

C4.1a: Functional Analysis — Prof. Batty — 16MT

Recommended Prerequisites

Part A Topology, B4 Analysis

Aims & Objectives

This course builds on B4, by extending the theory of Banach spaces and operators. The course exhibits the benefits of completeness (through the Baire category theorems) and compactness, giving the course a topological flavour. On the other hand, many of the topics studied have applications in differential equations and other areas of applied mathematics, some of which will appear in the form of examples.

Synopsis

Normed spaces and Banach spaces; subspaces and quotient spaces. Direct sums.

Compactness in Banach spaces; totally bounded sets. Ascoli's Theorem, Riesz criterion.

Dual spaces; completion; reflexive spaces. Hahn–Banach theorems. Weak and weak* topologies. The Banach–Alaoglu theorem.

The Baire category theorem. The open-mapping theorem. The closed-graph theorem. The principle of uniform boundedness.

Compact operators; spectral theory. Compact self-adjoint operators.

Closed (unbounded) operators. Operators with compact resolvent. Unbounded self-adjoint operators.

Reading

1. B. Bollobas, *Linear Analysis*, Second Edition, Cambridge University Press (1999).
2. M. Reed and B. Simon, *Methods of Modern Mathematical Physics I: Functional Analysis*, Academic Press (1972).

Alternative Reading

1. P.D. Lax, *Functional Analysis*, Wiley (2002).

C4.1b: Banach and C*- algebras — Dr Edwards — 16HT

Recommended Prerequisites

B4 Analysis

Aims & Objectives

The suggestion that the observables of quantum mechanics should be represented by linear operators on Hilbert spaces led Murray and von Neumann into a detailed study of algebras of bounded linear operators on Hilbert space. The abstract version of such algebras are known as C*-algebras and investigations into their structure continue today. Their importance now extends beyond functional analysis and physics into geometry and number theory.

Learning Outcome

The outcome of this course should be the appreciation and understanding of the main problems in Banach algebras and C*- algebras. Details are included in the synopsis below.

Synopsis

The course begins with an introduction to the general theory of Banach algebras and C*-algebras culminating in the Gelfand Representation Theorem for commutative C*-algebras. The second part of the course concentrates on the relationship that exists between the algebraic and geometric properties of non-commutative C*-algebras. The final part concerns itself with representations of C*-algebras on Hilbert spaces culminating with the Gelfand–Naimark Theorem.

Reading

1. G.I. Murphy, *C*-algebras and Operator Theory*, Academic (1990), Chs. 2, 3 and parts of Ch. 5.

Alternative (more advanced) sources:

1. G.K. Pedersen, *C*-algebras and their Automorphism Groups*, Academic (1979).
2. S. Sakai, *C*-algebras and W*-algebra*, Springer (1971).

C4.2a: Real and Harmonic Analysis — Dr Kirchheim — 16 MT

Level: M-level **Method of Assessment:** Written Examination

Weight: Half-unit.

Recommended Prerequisites

Familiarity with the Lebesgue integral (to be very briefly revised) and B4a Banach spaces is expected, knowledge of B4b Hilbert spaces would be useful.

Aims and Objectives

The aim of the course is to introduce basic analytical tools for the study of fine properties (local structure) of modern (i.e. weakly differentiable) solutions to variational problems and PDEs, together with rigorous convergence results for Fourier analysis in this general setting.

Learning Outcome

One objective of the course is to refine the integration theory taught in Part A, dealing with questions like "Does the Fundamental Theorem of Calculus hold also for the Lebesgue integral?"

The second part of the lectures utilizes these results to build a modern theory of Fourier series and transforms — answering natural questions about convergence for all integrable functions, touching upon recent concepts like wavelets and being used in both pure and applied mathematics (from number theory to PDE).

Students at the end of this course will have advanced their knowledge in these topics and be able to apply them.

Synopsis

The Hardy–Littlewood maximal operator and Lebesgue's differentiation theorem, absolutely continuous functions, interpolation of linear operators.

Fourier series: Summablity methods, Gibb's Phenomenon, sets of divergence.

Fourier transform on the real line: definition in the space of integrable functions and basic properties, Plancherel's theorem and the space of square integrable functions, Fourier transform of measures and Bochner's theorem.

Reading

1. W. Rudin, *Real and Complex Analysis*, Mc Graw–Hill, 1966 (and new editions, 1987).
2. Y. Katznelson, *An Introduction to Harmonic Analysis*, John Wiley 1968 (1st ed).

Further Reading

1. T.W. Körner; *Fourier Analysis*, Cambridge University Press 1986.
2. E.M. Stein, G. Weiss; *Introduction to Fourier Analysis in Euclidean Spaces*, Princeton UP, 1971.
3. E.M. Stein, R. Shakarchi; *Fourier Analysis : an introduction*, Princeton UP, 2003.

C5.1: PDEs for Pure and Applied Mathematicians and Calculus of Variations

Level: M-level **Method of Assessment:** Written Examination

Weight: Whole-unit, or can be taken as a half-unit in C5.1a. C51b cannot be taken on its own.

C5.1a: PDEs for Pure and Applied Mathematicians — Dr Dyson — 16MT

Recommended Prerequisites: Lebesgue integration would be useful but is not essential.

Aims & Objectives

The course will introduce some of the modern techniques in partial differential equations that are central to the theoretical and numerical treatments of linear and nonlinear partial differential equations arising in science, geometry and other fields.

It provides valuable background for the Part C courses on Calculus of Variations, Fixed Point Methods for Nonlinear PDEs, and Finite Element Methods.

Learning Outcomes

Students will learn techniques and results, such as Sobolev spaces, weak convergence, weak solutions, the direct method of calculus of variations, embedding theorems, the Lax–Milgram theorem, the Fredholm Alternative and the Hilbert–Schmidt theorem and how to apply these to obtain existence and uniqueness results for linear and nonlinear elliptic partial differential equations.

Synopsis

Part I Function Spaces:

Why are function spaces important for partial differential equations?

User's guide to the Lebesgue integral. Definition of Banach spaces, separability and dual spaces. Definition of Hilbert space. The spaces $L^p(\Omega)$, $1 \leq p \leq \infty$, where $\Omega \subset \mathbb{R}^n$ is open. Minkowski and Hölder inequalities. Statement that $L^p(\Omega)$ is a Banach space, and that the dual of L^p is $L^{p'}$, for $1 \leq p < \infty$ where $\frac{1}{p} = \frac{1}{p'} = 1$. Statement that L^2 is a Hilbert space.

Weak and weak* convergence in L^p spaces. Examples. A bounded sequence in a separable Hilbert space has a weakly convergent subsequence.

Mollifiers and the density of smooth functions in L^p for $1 \leq p < \infty$.

Definition of weak derivatives and their uniqueness. Definition of Sobolev space $W^{m,p}(\Omega)$, $1 \leq p \leq \infty$. $H^m(\Omega) = W^{m,2}(\Omega)$. Definition of $W_0^{1,p}(\Omega)$, $1 \leq p < \infty$.

Part II Elliptic Problems:

The direct method of calculus of variations: The Poincaré inequality. Proof of the existence and uniqueness of a weak solution to Poisson's equation $-\Delta u = f$, with zero Dirichlet boundary conditions and $f \in L^2(\Omega)$, with Ω bounded. Discussion of regularity of solutions.

The Lax–Milgram theorem and Gårding's inequality. Existence and uniqueness of weak solutions to general linear uniformly elliptic equations.

Embedding theorems (proofs omitted except $W^{1,1}(a,b) \hookrightarrow C[a,b]$).

Compact operators and self adjoint operators. Fredholm Alternative and Hilbert–Schmidt Theorem. Examples including $-\Delta$ with zero Dirichlet boundary conditions.

A nonlinear elliptic problem treated by the direct method.

Reading

1. Lawrence C. Evans, *Partial Differential Equations*, (Graduate Studies in Mathematics), 2004, American Mathematical Society
2. M. Renardy and R.C. Rogers *An Introduction to Partial Differential Equations*, 2004, Springer–Verlag, New York.

Additional Reading

1. E. Kreyszig, *Introductory Functional Analysis with Applications*, Wiley (revised edition, 1989)
2. J. Rauch, *Partial Differential Equations*, 1992, Springer–Verlag, New York.

C5.1b: Calculus of Variations — Prof. Seregin — 16HT

Recommended Prerequisites:

See also "Weight" above. Some familiarity with the Lebesgue integral is essential, and some knowledge of elementary functional analysis (e.g. Banach spaces and their duals, weak convergence) an advantage.

Aims & Objectives

The aim of the course is to give a modern treatment of the calculus of variations from a rigorous perspective, blending classical and modern approaches and applications.

Learning Outcomes

Students will learn rigorous results in the classical and modern one-dimensional calculus of variations and see possible behaviour and application of these results in examples. They will see some examples of multi-dimensional problems.

Synopsis

Classical and modern examples of variational problems (e.g. brachistochrone, models of phase transformations).

One-dimensional problems, function spaces and definitions of weak and strong relative minimizers. Necessary conditions; the Euler–Lagrange and Du Bois–Reymond equations, theory of the second variation, the Weierstrass condition. Sufficient conditions; field theory and sufficiency theorems for weak and strong relative minimizers. The direct method of the calculus of variations and Tonelli's existence theorem. Regularity of minimizers. Examples of singular minimizers and the Lavrentiev phenomenon. Problems whose infimum is not attained. Relaxation and generalized solutions. Isoperimetric problems and Lagrange multipliers.

Multi-dimensional problems, done via some examples.

Reading

1. G. Buttazzo, M. Giaquinta, S. Hildebrandt, *One-dimensional Variational Problems, Oxford Lecture Series in Mathematics*, Vol. 15, OUP (1998). Ch 1, Sections 1.1, 1.2 (treated differently in course), 1.3, Ch 2 (background), Ch 3, Sections 3.1, 3.2, Ch 4, Sections 4.1, 4.3.

Additional Reading

1. U. Brechtken-Manderscheid, *Introduction to the Calculus of Variations*, Chapman & Hall (1991).

2. H. Sagan, *Introduction to the Calculus of Variations*, Dover (1992).

3. J. Troutman, *Variational Calculus and Optimal Control*, Springer–Verlag (1995).

C5.2b Fixed Point Methods for Nonlinear PDEs — Prof. Niethammer — 16 HT

Level: M-level. **Method of Assessment:** Written Examination.

Weight: Half-Unit.

Recommended Prerequisites:

C51.a: Partial Differential Equations for Pure and Applied Mathematics. Some knowledge of functional analysis, in particular Banach spaces (as in B4) and compactness (as in Part A Topology), is useful.

Aims and Objectives

This course gives an introduction to the techniques of nonlinear functional analysis with emphasis on the major fixed point theorems and their applications to nonlinear differential equations and variational inequalities, which abound in applications such as fluid and solid mechanics, population dynamics and geometry.

Learning Outcomes

Besides becoming acquainted with the fixed point theorems of Banach, Brouwer and Schauder, students will see the abstract principles in a concrete context. Hereby they also reinforce techniques from elementary topology, functional analysis, Banach spaces, compactness methods, calculus of variations and Sobolev spaces.

Synopsis

Examples of nonlinear differential equations and variational inequalities. Banach's fixed point theorem and applications. Brouwer's fixed point theorem, proof via Calculus of Variations and Null–Lagrangians. Compact operators and Schauder's fixed point theorem. Sobolev spaces, Poincaré inequality, embedding theorems (without proof). Applications of Schauder's fixed point theorem to nonlinear elliptic equations. Variational inequalities and monotone operators. Applications of monotone operator theory to nonlinear elliptic equations (p-Laplace, stationary Navier–Stokes)

Reading

E. Zeidler, *Nonlinear Functional Analysis I + II*, Springer–Verlag, 1986/89

Further Reading

M.S. Berger, *Nonlinearity and Functional Analysis*, Academic Press, 1977.

K. Deimling, *Nonlinear Functional Analysis*, Springer–Verlag, 1985.

L.C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics 19, AMS, 1998.

L. Nirenberg, *Topics in Nonlinear Functional Analysis*, Courant Institute Lecture Notes, AMS, 2001.

R.E. Showalter, *Monotone Operators in Banach Spaces and Nonlinear Partial Differential Equations*, Mathematical Surveys and Monographs, vol. 49, AMS, 1997.

C6.1a: Solid Mechanics — Professor Ball — 16 MT

Level: M-level **Method of Assessment:** Written Examination

Weight: Half-unit.

Recommended Prerequisites

There are no formal prerequisites. In particular it is not necessary to have taken any courses in fluid mechanics, though having done so provides some background in the use of similar concepts. Use is made of some elementary linear algebra in $\mathbb{R}^3$ (for example, eigenvalues, eigenvectors and diagonalization of symmetric 3×3 matrices), and revision of this material, for example from the Mods Linear Algebra course, is useful preparation. The necessary material is summarized in the course.

The first module of 6 lectures is useful preparation for C6.2b Elasticity and Plasticity. Taken together the two courses will provide a broad overview of modern solid mechanics, with a variety of approaches.

Aims & Objectives

Solid mechanics is a vital ingredient of materials science and engineering, and is playing an increasing role in biology. It has a rich mathematical structure. The aim of the course is to derive the basic equations of elasticity theory, the central model of solid mechanics, and give some interesting applications to the behaviour of materials.

Learning Outcomes

Students will learn basic techniques of modern continuum mechanics, such as kinematics of deformation, stress, constitutive equations and the relation between nonlinear and linearized models. They will also gain an insight into some recent developments in applications of mathematics to a variety of different materials.

Synopsis

(1) Nonlinear and linear elasticity (6 lectures)

Lagrangian and Eulerian descriptions of motion, analysis of strain. Balance laws of continuum mechanics. Frame-indifference. Cauchy and Piola–Kirchhoff stress. Constitutive equations for a nonlinear elastic material. Material symmetry, isotropy. Linear elasticity as a linearization of nonlinear elasticity.

(2) Exact solutions in elastostatics. (6 lectures)

Universal deformations for compressible materials. Incompressibility and models of rubber. Exact solutions for incompressible materials, including the Rivlin cube, simple shear, torsion of a cylinder, inflation of a balloon. Cavitation in polymers.

(3) Phase transformations in solids (4 lectures)

Martensitic phase transformations, twins and microstructure.

Austenite-martensite interfaces. The shape-memory effect.

Reading

1. R. J. Atkin & N. Fox, *An Introduction to the Theory of Elasticity*, Longman, 1980.
2. M.E. Gurtin, *An Introduction to Continuum Mechanics*, Academic Press, 1981.

Further Reading

1. Stuart S. Antman, *Nonlinear Problems of Elasticity*, Applied mathematical sciences v. 107, Springer–Verlag, 1995.

2. Jerrold E. Marsden, Thomas J.R. Hughes, *Mathematical Foundations of Elasticity*, Prentice–Hall, 1983.

3. Philippe G. Ciarlet, *Mathematical Elasticity*, Studies in mathematics and its applications ; v. 20, 27, 29, North–Holland 1988–

4. Kaushik Bhattacharya, *Microstructure of Martensite — Why it forms and how it gives rise to the shape-memory effect*, Oxford University Press 2003.

C6.2b: Elasticity and Plasticity — Dr J Ockendon — 16HT

Level: M-level **Method of Assessment:** Written Examination

Weight: Half-unit.

Recommended Prerequisites: Familiarity with classical and fluid mechanics (Part A) and simple perturbation theory (Part C course) will be useful. The complementary Part C course, Solid Mechanics, will be especially useful.

Aims & Objectives

The course gives a rapid review of mathematical models for basic solid mechanics. Benchmark solutions are reviewed for static problems and wave propagation in linear-elastic materials. It is then shown how these results can be used as a basis for practically useful problems involving rods, plates, and shells. Also simple geometrically nonlinear models will be introduced to explain buckling, contact and fracture at the most basic level. Yield and plasticity will be discussed at a similar level, both microscopically and macroscopically and there will be a brief introduction to composite fields (composite materials, thermo- and visco-elasticity).

Learning Outcomes

Learning outcomes are an appreciation and understanding of the topic listed in the syllabus below.

Synopsis

Review of tensors, conservation laws, Navier equations. Antiplane strain, plain strain, torsion. Elastic wave propagation, Rayleigh waves. Ad hoc approximations for thin materials; simple bifurcation theory and buckling; simple mixed boundary value problems, brittle fracture and smooth contact; simple ideas about homogenization, composite materials, thermo- and visco-elasticity.

Reading

1. R.M. Hill, *Mathematical Theory of Plasticity*, Oxford Clarendon Press, 1998.

2. A.E.H. Love, *Treatise on the Mathematical Theory of Elasticity*, Dover, 1944.
3. L.D. Landau and E.M. Lifshitz, *Theory of Elasticity*, Pergamon Press, 1986.

C6.3: Perturbation Methods and Applied Complex Variables

Level: M-level **Weight:** Whole-unit, or can be taken as either a half-unit in C6.3a or a half-unit in C6.3b.

Method of Assessment: Written Examination

C6.3a: Perturbation Methods — Dr Porter — 16MT

Recommended Prerequisites

Part A Differential Equations and Core Analysis (Complex Analysis). B5 and B6 are relevant but not essential.

Aims & Objectives

Perturbation methods underlie almost all applications of physical applied mathematics: for example, boundary layers in viscous flow, celestial mechanics, optics, shock waves, reaction-diffusion equations and nonlinear oscillations. The aims of the course are to give a clear and systematic account of modern perturbation theory and to show how it can be applied to differential equations.

Learning Outcomes

Learning outcomes are an appreciation and understanding of the topics listed in the syllabus below.

Synopsis

Asymptotic expansions. Asymptotic evaluation of integrals: Laplace's method, method of steepest descent; Stokes phenomenon, exponential asymptotics. Regular and singular perturbation methods. Methods of multiple scales, WKB method , boundary layers, transition layers. Applications to partial differential equations.

Reading

1. E.J. Hinch, *Perturbation Methods*, CUP (1991), Chs. 1–3, 5–7.
2. C.M. Bender and S.A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*, McGraw–Hill (1978), Chs. 6, 7, 9–11.

3. J. Kevorkian and J.D. Cole, *Perturbation Methods in Applied Mathematics*, Springer–Verlag, (1981), Chs. 1, 2.1–2.5, 3.1, 3.2, 3.6, 4.1, 5.2.

C6.3b: Applied Complex Variables — Prof. Chapman — 16HT

Recommended Prerequisites

The course requires second year core analysis (complex analysis). It continues the study of complex variables in the directions suggested by contour integration and conformal mapping. Part A Fluid Dynamics and Waves is desirable as it provides motivation for some of the topics studied.

Aims & Objectives

The course begins where core second-year complex analysis leaves off, and is devoted to extensions and applications of that material. It is assumed that students will be familiar with inviscid two-dimensional hydrodynamics (Part A Fluid Dynamics and Waves) to the extent of the existence of a harmonic streamfunction and velocity potential in irrotational incompressible flow, and Bernoulli's equation.

Synopsis

1–2 Review of core real and complex analysis, especially contour integration, Fourier transforms.

2–4 Conformal mapping. Riemann mapping theorem (statement only). Schwarz–Christoffel formula. Solution of Laplace's equation by conformal mapping onto a canonical domain.

5–6 Applications to inviscid hydrodynamics: flow past an aerofoil and other obstacles by conformed mapping; free streamline flows, hodograph plane.

7–8 Flow with free boundaries in porous media. Construction of solutions using conformal mapping. The Schwarz function.

9–15 Transform methods, complex Fourier transform. Solution of mixed boundary value problems motivated by thin aerofoil theory and the theory of cracks in elastic solids. Index. Riemann Hilbert problems, Wiener–Hopf method.

16 Stokes phenomenon.

Reading

1. M. J. Ablowitz and A. S. Fokas, *Complex Variables: Introduction and Applications*, 2nd edition, C.U.P., Cambridge (2003). ISBN 0521534291.

2. J. Ockendon, Howison, Lacey and Movichan, *Applied Partial Differential Equations*, Oxford 1999, Pages 195–212.

3. G. F. Carrier, M. Krook and C. E. Pearson, *Functions of a Complex Variable*, McGraw–Hill, New York, (1966). Reprinted by Hod Books, 1983. ISBN 0962197300 (Out of print).

C6.4a: Topics in Fluid Mechanics — Drs Fowler/Howell/Norbury/Prof. Chapman — 16MT

Level: M-level **Method of Assessment:** Written Examination

Weight: Half-unit.

Recommended Prerequisites: B6 fluid mechanics.

Aims and Objectives

The course will expand and illuminate the 'classical' fluid mechanics taught in the third year course B6, and illustrate its modern application in a number of different areas in industry and geoscience.

Learning Outcomes

The course will develop an understanding of how fluid mechanics arises in modern applications, and will show how it can be used to develop an understanding in these.

Synopsis

Convection: Earth's mantle and core; magma chambers. Stability, boundary layers, parameterised convection.

Rotating flows: atmosphere and oceans. Waves, geostrophy, quasi-geostrophy, baroclinic instability.

Two-phase flows: boilers, condensers, fluidised beds. Flow régimes. Homogeneous, drift-flux, two-fluid models. Ill-posedness, waves, density wave oscillations.

Thin film flows: coatings and foams. Lubrication theory: gravity flows, Marangoni effects. Droplet dynamics, contact lines, menisci. Drying and wetting. Foam drainage.

Reading

J. S. Turner, *Buoyancy Effects in Fluids*, C. U. P., Cambridge (1973).

A. E. Gill, *Atmosphere-Ocean Dynamics*, Academic Press, San Diego (1982).

J. Pedlosky, *Geophysical Fluid Dynamics*, Springer–Verlag, Berlin (1979).

D. A. Drew and S. L. Passman, *Theory of Multicomponent Fluids*, Springer–Verlag, Berlin (1999).

P. B. Whalley, *Boiling, Condensation and Gas-Liquid Flow*, O. U. P., Oxford (1987).

D. Weaire and S. Hutzler, *The Physics of Foams*, O. U. P., Oxford (1999).

Further Reading

G. K. Batchelor, H. K. Moffatt and M. G. Worster (eds.), *Perspectives in Fluid Dynamics*, C. U. P., Cambridge (2000).

C7.1b: Quantum Theory and Quantum Computers — Dr Hannabuss — 16HT

Level: M-level **Method of Assessment:** Written Examination

Weight: Half-unit

Recommended Prerequisites

B7.1a Quantum Mechanics.

Aims and Objectives

This course builds directly on the first course in quantum mechanics and covers a series of important topics, particularly features of systems containing several particles. The behaviour of identical particles in quantum theory is more subtle than in classical mechanics, and an understanding of these features allows one to understand the periodic table of elements and the rigidity of matter. It also introduces a new property of entanglement linking particles which can be quite widely dispersed.

There are rarely neat solutions to problems involving several particles, so usually one needs some approximation methods. In very complicated systems, such as the molecules of gas in a container, quantum mechanical uncertainty is compounded by ignorance about other details of the system and requires tools of quantum statistical mechanics.

Two state quantum systems enable one to encode binary information in a new way which permits superpositions. This leads to a quantum theory of information processing, and by exploiting entanglement to other ideas such as quantum teleportation.

Learning Outcomes

Know about quantum mechanics of many particle systems, statistics, entanglement, and applications to quantum computing.

Synopsis

Identical particles, symmetric and anti-symmetric states, Fermi–Dirac and Bose–Einstein statistics and atomic structure.

Heisenberg representation, interaction representation, time dependent perturbation theory and Feynman–Dyson expansion. Approximation methods, Rayleigh–Schrödinger time-independent perturbation theory and variation principles. The virial theorem. The ground state of helium.

Entanglement. The EPR paradox, Bell's inequalities, Aspect's experiment. GHZ states

Mixed states, density operators. The example of spin systems. Purification. Gibbs states and the KMS condition.

Quantum information processing, qubits and quantum computing. The no-cloning theorem, quantum teleportation. Quantum logic gates. Schmidt decompostion. Positive operator-valued measures. The quantum Fourier transform. Shor's factorisation algorithm.

Reading

1. Hannabuss, *Introduction to Quantum Mechanics* OUP (1997). Chapters 10–12 and 14, 16, supplemented by lecture notes on quantum computers on the web

Further reading:

A popular non-technical account of the subject:

A Hey and P Walters, *The New Quantum Universe*, Cambridge (2003).

Also designed for an Oxford course, though only covering some material:

I P Grant, *Classical and Quantum Mechanics*, Mathematical Institute Notes (1991).

A concise account of quantum information theory:

S Stenholm and K-A Suominen *Quantum Approach to Informatics*, Wiley (2005).

An encyclopaedic account of quantum computing:

M A Nielsen and I L Chuang: *Quantum Computation*, Cambridge University Press, (2000).

Even more paradoxes can be found in:

Y Aharanov and D Rohrlich: *Quantum Paradoxes*, Wiley–VCH (2005).

Those who read German can find further material on entanglement in:

J Audretsch: *Verschränkte Systeme*, Wiley–VCH (2005).

Other accounts of the first part of the course:

L I Schiff, *Quantum Mechanics*, 3rd edition, Mc Graw Hill (1968).

B J Bransden and C J Joachain, *Introduction to Quantum Mechanics*, Longman (1995)

A I M Rae, *Quantum Mechanics*, 4th edition, Institute of Physics (1993)

C7.2b: General Relativity I — Prof. Chruściel — 16HT

Level: M-level **Method of Assessment:** Written Examination.

Weight: Half-unit

Recommended Prerequisites

B7.2a Relativity and Electromagnetism.

Aims & Objectives

The course is intended as an elementary introduction to general relativity, its basic physical concepts of its observational implications, and the new insights that it provides into the nature of space time, and the structure of the universe. Familiarity with special relativity and electromagnetism as covered in the B7 course will be assumed. The lectures will review Newtonian gravitation, tensor calculus and continuum physics in special relativity, physics in curved space time and the Einstein field equations. This will suffice for an account of simple applications to planetary motion, the bending of light and the existence of black holes.

Learning Outcomes

This course starts by asking how the theory of gravitation can be made consistent with the special-relativistic framework. Physical considerations (the principle of equivalence, general covariance) are used to motivate and illustrate the mathematical machinery of tensor calculus. The technical development is kept as elementary as possible, emphasising the use of local inertial frames. A similar elementary motivation is given for Einstein's equations and the Schwarzschild solution. Orbits in the Schwarzschild solution are given a unified treatment which allows a simple account of the three classical tests of Einstein's theory. Finally, the analysis of extensions of the Schwarzschild solution show how the theory of black holes emerges and exposes the radical consequences of Einstein's theory for space-time structure. Cosmological solutions are not discussed.

The learning outcomes are an understanding and appreciation of the ideas and concepts described above.

Synopsis

Review of Newtonian gravitation theory and problems of constructing a relativistic generalisation. Review of Special Relativity. The equivalence principle. Tensor formulation of special relativity (including general particle motion, tensor form of Maxwell's equations and the energy momentum-tensor of dust). Curved space time. Local inertial coordinates.

General coordinate transformations, elements of Riemannian geometry (including connections, curvature and geodesic deviation). Mathematical formulation of General Relativity, Einstein's equations (properties of the energy-momentum tensor will be needed in the case of dust only). The Schwarzschild solution; planetary motion, the bending of light, and black holes.

Reading

1. L.P. Hughston and K.P. Tod, *An Introduction to General Relativity*, LMS Student Text 5, CUP (1990), Chs 1–18.
2. N.M.J. Woodhouse, *Notes on Special Relativity*, (Mathematical Institute Notes. Revised edition; published in a revised form as *Special Relativity, Lecture notes in Physics m6*, Springer–Verlag, (1992), Chs 1–7

Further Reading

1. B. Schutz, *A First Course in General Relativity*, CUP (1990).
2. R.M. Wald, *General Relativity*, Chicago (1984).
3. W. Rindler, *Essential Relativity*, Springer–Verlag, 2nd edition (1990).

C7.4 — Theoretical Physics

Note: This unit is offered by the Physics Department.

Level: M-level **Method of Assessment:** Written Examination.

Weight: Whole-unit only.

Recommended Prerequisites: Part A Electromagnetism, Part A Classical Mechanics, B7.1a: Quantum Mechanics, C7.1b: Quantum Theory and Quantum Computers, B7.2a: Special Relativity and Electromagnetism.

C7.4a Theoretical Physics I — lecturers Prof. Chalker and Dr Lukas — 24MT

Aims and Objectives

This course is intended to give an introduction to some aspects of field theory and related ideas. These are important in particular for treating systems with an infinite number of degrees of freedom. An aim is to present some core ideas and important applications in a unified way. These applications include the classical mechanics of continuum systems, the quantum mechanics and statistical mechanics of many-particle systems, and some basic aspects of relativistic quantum field theory.

Learning Outcomes

Learning outcomes are an understanding of the individual topics listed in the syllabus below, and of the connections between them.

Synopsis

1. The mathematical description of systems with an infinite number of degrees of freedom: functionals, functional differentiation, and functional integrals. Multi-dimensional Gaussian integrals. Random fields: properties of a Gaussian field. Perturbation theory for non-Gaussian functional integrals. Path integrals and quantum mechanics. Treatment of free particle and of harmonic oscillator. [5 lectures]

2. Classical field theory: fields, Lagrangians and Hamiltonians. The least action principle and field equations. Space-time and internal symmetries: $U(1)$ example, Noether current. The idea of an irreducible representation of a group. Irreducible representations of $SU(2)$ and application to global internal symmetry. Simple representations of the Lorentz group via $SU(2) \times SU(2)$ without proof. $U(1)$ gauge symmetry, action of scalar QED and derivation of Maxwell's eqns in covariant form. [5 lectures]

3. Landau theory and phase transitions: phase diagrams, first-order and continuous phase transitions. Landau–Ginsburg–Wilson free energy functionals. Examples including liquid crystals. Critical phenomena and scaling theory. [5 lectures]

4. The link between quantum mechanics and the statistical mechanics of one-dimensional systems via Wick rotation. Transfer matrices for one-dimensional systems in statistical mechanics. [4 lectures]

5. Stochastic processes and path integrals: the Langevin and Fokker–Planck equation. Brownian motion of single particle. Rouse model of polymer dynamics. [4 lectures]

C7.4b — Theoretical Physics II — lecturers Prof. Chalker and Dr Lukas — 16HT

1 Canonical quantisation and connection to many body theory: quantised elastic waves; quantisation of free scalar field theory; many-particle quantum systems. [4 lectures]

2 Path integrals and quantum field theory: generating functional and free particle propagator for scalar and $U(1)$ gauge fields (in Lorentz gauge). [5 lectures]

3 Perturbation theory at tree level for decay and scattering processes. Examples from pure scalar theories and scalar QED. Goldstone theorem. [4 lectures]

4 Canonical transformations in quantum field theory: Bogoliubov transformations applied to bose condensates, magnons in antiferromagnets, and to BCS theory. [4 lectures]

[Total: 40 lectures]

Reading

The lecturers are aware of no book that presents all parts of this course in a unified way and at an appropriate level. For this reason, detailed lecture notes will be made available.

Some books that cover parts of the course are:

D. Bailin and A. Love, *Introduction to Gauge Field Theory*, mainly chapters 1 – 6 for sections 1, 2, 6, 7 and 8.

R. P. Feynman, *Statistical Mechanics*, mainly chapters 3, 4 and 6 for sections 1, 6 and 9.

F. Reif, *Statistical and Thermal Physics*, chapter 15 for section 5.

J. M. Yeomans, *Statistical Mechanics of Phase Transitions*, chapters 1 – 5 for sections 3 and 4.

C8.1: Mathematics and the Environment and Mathematical Physiology

Level: M-level **Method of Assessment:** Written Examination

Weight: Whole unit, or can be taken as either a Half-unit in C8.1a or a Half-unit in C8.1b.

C8.1a: Mathematics and the Environment — Dr Fowler/Dr Sander — 16MT

Recommended Prerequisites

B6 highly recommended.

Aims & Objectives

The aim of the course is to illustrate the techniques of mathematical modelling in their particular application to environmental problems. The mathematical techniques used are drawn from the theory of ordinary differential equations and partial differential equations. However, the course does require the willingness to become familiar with a range of different scientific disciplines. In particular, familiarity with the concepts of fluid mechanics will be useful.

Learning Outcomes

Learning outcomes are an appreciation and understanding of the topic listed in the syllabus below.

Synopsis

Applications of mathematics to environmental or geophysical problems involving the use of models with ordinary and partial differential equations. Examples to be considered are: Climate dynamics. River flow and sediment transport. Glacier dynamics.

Reading

1. A. C. Fowler, *Mathematics and the Environment*, Mathematical Institute lecture notes. (Revised edition, September 2004.)
2. K. Richards, *Rivers*, Methuen 1982.
3. G. B. Whitham, *Linear and Nonlinear Waves.* Wiley, New York. 1974.
4. W. S. B. Paterson, *The Physics of Glaciers*, 3rd edition, Pergamon Press 1994.
5. J. T. Houghton, *The Physics of Atmospheres*, 3rd ed. C.U.P., Cambridge 2002.

C8.1b: Mathematical Physiology — Prof. Maini — 16HT

Recommended Prerequisites

B8a highly recommended.

Aims & Objectives

The course aims to provide an introduction which can bring students within reach of current research topics in physiology, by synthesising a coherent description of the physiological background with realistic mathematical models and their analysis. The concepts and treatment of oscillations, waves and stability are central to the course, which develops ideas introduced in the more elementary B8a course. In addition, the lecture sequence aims to build understanding of the workings of the human body by treating in sequence problems at the intracellular, intercellular, whole organ and systemic levels.

Learning Outcomes

Learning outcomes are an appreciation and understanding of the topic listed in the syllabus below.

Synopsis

Review of enzyme reactions and Michaelis–Menten theory. Trans-membrane ion transport: Hodgkin–Huxley and Fitzhugh–Nagumo models.

Excitable media; wave propagation in neurons.

Calcium dynamics: calcium-induced calcium release. Intracellular oscillations and wave propagation.

The electrochemical action of the heart. Spiral waves, tachycardia and fibrillation. The heart as a pump. Regulation of blood flow.

Respiration and CO_2 control. Mackey and Grodins models.

Regulation of stem cell and blood cell production. Dynamical diseases.

Reading

The principal text is:

1. J. Keener and J. Sneyd, *Mathematical Physiology*, Springer–Verlag (1998). Chs. 1, 4, 5, 9–12, 14–17.

Subsidiary mathematical texts are:

1. J. D. Murray, *Mathematical Biology*, Springer–Verlag, 2nd ed., 1993.
2. L. A. Segel, *Modeling Dynamic Phenomena in Molecular and Cellular Biology*, CUP (1984).
3. L. Glass and M. C. Mackey, *From Clocks to Chaos*, Princeton University Press (1988).
4. P. Grindrod, *Patterns and Waves*, Oxford University Press (1991).

General physiology texts are:

1. R. M. Berne and M. N. Levy, *Principles of Physiology*, 2nd ed. Mosby, St. Louis (1996).
2. J. R. Levick, *An Introduction to Cardiovascular Physiology*, 3rd ed. Butterworth–Heinemann, Oxford (2000).
3. A. C. Guyton and J. E. Hall, *Textbook of Medical Physiology*, 10th ed. W. B. Saunders Co., Philadelphia (2000).

C9.1: Analytic Number Theory and Elliptic Curves

Level: M-Level. **Method of Assessment:**Written Examination.

Weight: Whole unit, or can be taken as either a half-unit in C9.1a or a half-unit in C9.1b.

C9.1a: Analytic Number Theory — Prof. Heath-Brown — 16MT

Prerequisites

Complex analysis (holomorphic and meromorphic functions, Cauchy's Residue Theorem, Evaluation of integrals by contour integration, Uniformly convergent sums of holomorphic functions). Elementary number theory (Unique Factorization Theorem).

Aims and Objectives

The course aims to introduce students to the theory of prime numbers, showing how the irregularities in this elusive sequence can be tamed by the power of complex analysis. The course builds up to the Prime Number Theorem which is the corner-stone of prime number theory, and culminates in a description of the Riemann Hypothesis, which is arguably the most important unsolved problem in modern mathematics.

Learning Outcomes

Students will learn to handle multiplicative functions, to deal with Dirichlet series as functions of a complex variable, and to prove the Prime Number Theorem and simple variants.

Synopsis

Introductory material on primes.

Arithmetic functions — Möbius function, Euler function, Divisor function, Sigma function — multiplicativity.

Dirichlet series — Euler products — von Mangoldt function.

Riemann Zeta-function — analytic continuation to $Re(s) > 0$.

Non-vanishing of zeta(s) on $Re(s) = 1$.

Proof of the prime number theorem.

The Riemann hypothesis and its significance.

The Gamma function, the functional equation for $\zeta(s)$, the value of $\zeta(s)$ at negative integers.

Reading

1. T.M. Apostol, *Introduction to Analytic Number Theory. Undergraduate Texts in Mathematics*, Springer–Verlag, 1976. chapters 2,3,11,12 and 13.

2. M. Ram Murty, *Problems in Analytic Number Theory*, (Springer, 2001). Chapters 1 – 5

3. G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers*, Fifth edition. OUP 1979. Chapters 16 ,17 and 18

4. G.J.O. Jameson, *The Prime Number Theorem*, LMS Student Texts, 53 CUP 2003

C9.1b Elliptic Curves — Prof. Flynn — 16 lectures HT

Recommended Prerequisites

It is helpful, but not essential, if students have already taken a standard introduction to algebraic curves and algebraic number theory. For those students who may have gaps in their background, I have placed the file "Preliminary Reading" permanently on the Elliptic Curves webpage, which gives in detail (about 30 pages) the main prerequisite knowledge for the course. Go first to: www.maths.ox.ac.uk/current-students/undergraduates/lecture-material/ then click on "C9 Elliptic Curves" and then click on the pdf file "Preliminary Reading".

Aims & Objectives

Elliptic curves give the simplest examples of many of the most interesting phenomena which can occur in algebraic curves; they have an incredibly rich structure and have been the testing ground for many developments in algebraic geometry whilst the theory is still full of deep unsolved conjectures, some of which are amongst the oldest unsolved problems in Mathematics. The course will concentrate on arithmetic aspects of elliptic curves defined over the rationals, with the study of the group of rational points, and explicit determination of the rank, being the primary focus. Using elliptic curves over the rationals as an example, we will be able to introduce many of the basic tools for studying arithmetic properties of algebraic varieties.

Learning Outcomes

On completing the course, students should be able to understand and use properties of elliptic curves, such as the group law, the torsion group of rational points, and 2-isogenies between elliptic curves. They should be able to understand and apply the theory of fields with valuations, emphasising the p-adic numbers, and be able to prove and apply Hensel's Lemma in problem solving. They should be able to understand the proof of the Mordell–Weil Theorem for the case when an elliptic curve has a rational point of order 2, and compute ranks in such cases, for examples where all homogeneous spaces for descent-via-2-isogeny satisfy the Hasse principle. They should also be able to apply the elliptic curve method for the factorisation of integers.

Synopsis

Non-singular cubics and the group law; Weierstrass equations.
Elliptic curves over finite fields; Hasse estimate (stated without proof).
p-adic fields (basic definitions and properties).
1-dimensional formal groups (basic definitions and properties).
Curves over p-adic fields and reduction mod p.
Computation of torsion groups over $\mathbb{Q}$; the Nagell–Lutz theorem.
2-isogenies on elliptic curves defined over $\mathbb{Q}$, with a $\mathbb{Q}$-rational point of order 2.
Weak Mordell–Weil Theorem for elliptic curves defined over $\mathbb{Q}$, with a $\mathbb{Q}$-rational point of order 2.
Height functions on abelian groups and basic properties.
Heights of points on elliptic curves defined over $\mathbb{Q}$; statement (without proof) that this gives a height function on the Mordell–Weil group.
Mordell–Weil Theorem for elliptic curves defined over $\mathbb{Q}$, with a $\mathbb{Q}$-rational point of order 2.

Explicit computation of rank using descent via 2-isogeny.
Public keys in cryptography; Pollard's $(p-1)$ method and the elliptic curve method of factorisation.

Reading

1. J.W.S. Cassels, *Lectures on Elliptic Curves*, LMS Student Texts 24, Cambridge University Press, 1991.
2. N. Koblitz, *A Course in Number Theory and Cryptography*, Graduate Texts in Mathematics 114, Springer, 1987.
3. J.H. Silverman and J. Tate, *Rational Points on Elliptic Curves*, Undergraduate Texts in Mathematics, Springer, 1992.
4. J.H. Silverman, *The Arithmetic of Elliptic Curves*, Graduate Texts in Mathematics 106, Springer, 1986.

Further Reading

1. A. Knapp, *Elliptic Curves. Mathematical Notes 40*, Princeton University Press, 1992.
2. G, Cornell, J.H. Silverman and G. Stevans (editors), *Modular Forms and Fermat's Last Theorem*, Springer, 1997.
3. J.H. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*, Graduate Texts in Mathematics 151, Springer, 199

C10.1 Stochastic Differential Equations and Brownian Motion in Complex Analysis

Level: M-level **Method of Assessment:** Written Examination

Weight: Whole unit, or can be taken as either a Half-unit in C10.1a or a Half-unit in C10.1b

C10.1a: Stochastic Differential Equations — Dr Qian —16MT

Recommended Prerequisites:

A course on integration and an introduction to discrete martingales such as can be found in B10a Martingales Through Measure Theory.

Aims & Objectives

Stochastic differential equations have been used extensively in many areas of application, including finance and social science as well as Chemistry. This course develops the basic theory of Itô's calculus and stochastic differential equations, and gives a few applications.

Learning Outcomes

To develop an appreciation of stochastic calculus as a tool that can be used for defining and understanding diffusive systems.

Synopsis

Itô's calculus: stochastic integrals with respect to martingales, Itô's lemma, Levy's theorem on characteristic of Brownian motion, exponential martingales, exponential inequality, Girsanov's theorem, The Martingale Representation Theorem. Stochastic differential equations: strong solutions, questions of existence and uniqueness, diffusion processes, Cameron–Martin formula, weak solution and martingale problem. Some selected applications chosen from option pricing, stochastic filtering etc.

Reading — Main texts

1. Dr Qian's online notes.
2. B. Oksendal: *Stochastic Differential Equations: An introduction with applications*, Universitext, Springer (6th edition). Chapters II, III, IV, V, part of VI, Chapter VIII (F).
3. F. C. Klebaner: *Introduction to Stochastic Calculus with Applications*, Imperial College Press. Sections 3.1 – 3.5, 3.9, 3.12. Chapters 4, 5, 11.

Alternative reading

1. H. P. McKean: *Stochastic Integrals*, Academic Press, New York and London (1969).

Further reading

1. N. Ikeda & S. Watanabe: *Stochastic Differential Equations and Diffusion Processes*, North–Holland Publishing Company.
2. I. Karatzas and S. E. Shreve: *Brownian Motion and Stochastic Calculus (GTM 113)*, Springer–Verlag.
3. L. C. G. Rogers & D. Williams: *Diffusions, Markov Processes and Martingales Vol1 (Foundations) and Vol 2 (Ito Calculus)*, Cambridge University Press.

C10.1b: Brownian Motion in Complex Analysis — Prof. Lyons — 16HT

Recommended Prerequisites

Part A Analysis. At least one of (B10a) Martingales through Measure and (C10.1) Stochastic Differential Equations. To have attended both would be desirable.

Aims & Objectives

Randomness plays a key feature in the behaviour of many high dimensional systems and so is intimately connected with applications. However, it also plays a key role in our understanding of many aspects of pure mathematics. This course will look at the deep interaction between 2 dimensional Brownian motion and complex analysis. At the core of these interactions is the conformal invariance of Brownian motion observed by Levy and the relationship with Harmonic Functions (based on Martingales) first observed by Kakutani and Doob.

Since that time there have been many developments and connections. The Hardy spaces of Fefferman and Stein, Value Distribution Theory, and most recently the stochastic Lowener equation (a topic of current and very exciting research).

We will use the conformal properties of Brownian motion to examine and prove some deep theorems about value distributions for complex functions.

Learning Outcomes

To develop an appreciation of the role that Brownian motion and martingales can play in pure mathematics.

Synopsis

Brownian motion. Continuous martingales and Levy's characterisation in terms of Brownian motion. Conformal invariance of Brownian motion. Brownian motion tangles about two points and a proof of Picard's theorems. Harmonic functions on the disk and the solution of the Dirichlet problem. Burkholder's Inequalities in Hardy spaces. Harmonic and superharmonic functions — via probability. [Nevanlinna's Theorems].

Reading

There isn't a perfect book for this course and we will refer to research papers to a limited extent.

1. The Notes of Prof. Lyons
2. D. Burkholder, *Distribution Function Inequalities for Martingales*, Ann. Probability, 1 (1973) 19–42.

Further Reading

1. McKean, *Stochastic Integrals*, (1969). Hard, short, with much relevant material and some mistakes! Excellent for the able!
2. K. E. Petersen, *Brownian Motion, Hardy Spaces and Bounded Mean Oscillation*, LMS Lecture Note Series, 28, Cambridge University Press (1977).

3. T. K. Carne, *Brownian Motion and Nevanlinna Theory.* Proc. London. Math. Soc. 52 (1986), 349–68

4. Richard F. Bass, *Probabilistic Techniques in Analysis.* Springer–Verlag New York Inc, ISBN: 0387943870 (1995).

5. Lars Ahlfors, *Complex Analysis*, McGraw–Hill, ISBN: 0070006571, (1979).

6. Jean-Claude Gruet, *Nevanlinna Theory, Fuchsian Functions and Brownian Motion Windings*, Source: Rev. Mat. Iberoamericana 18 (2002), no. 2, 301–324.

C11.1: Graph Theory and Probabilistic Combinatorics

Level: M-level **Method of Assessment:** Written Examination

Weight: Whole-unit, or C11.1a may be taken as a Half-unit (C11.1b may not).

C11.1a: Graph Theory — Prof. Scott — 16MT

Recommended Prerequisites

None beyond elementary probability theory.

Aims & Objectives

Graphs are among the simplest mathematical structures, but nevertheless have a very rich and well-developed structural theory. Graph Theory is an important area of mathematics, and also has many applications in other fields such as computer science.

The main aim of the course are to introduce the analysis of discrete structures, and particularly the use of extremal methods.

Learning Outcomes

To develop an appreciation of extremal methods in the analysis and understanding of graphical structures.

Synopsis

Introduction. Trees. Euler circuits. Planar graphs.
Matchings and Hall's Theorem. Connectivity and Menger's Theorem.
Extremal problems. Long paths and cycles. Turán's Theorem. Erdős–Stone Theorem.
Graph colouring. The Theorem of Brooks. The chromatic polynomial.
Ramsey's Theorem.
Random walks on graphs.
Szemerédi's Regularity Lemma.

Reading

1. B. Bollobás, *Modern Graph Theory*, GTM 184, Springer–Verlag (1998).

Further Reading

1. J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, Elsevier (1976); available online at http://www.ecp6.jussieu.fr/pageperso/bondy/books/gtwa/gtwa.html.
2. R. Diestel, *Graph Theory*, third edition, GTM 173, Springer–Verlag (2005).
3. D. West, *Introduction to Graph Theory*, second edition, Prentice–Hall (2001).

C11.1b: Probabilistic Combinatorics — Dr Martin — 16HT

Recommended Prerequisites

C11.1a Graph Theory and Part A Probability.

Aims and objectives

Probabilistic combinatorics is a very active field of mathematics, with connections to other areas such as computer science and statistical physics. Probabilistic methods are essential for the study of random discrete structures and for the analysis of algorithms, but they can also provide a powerful and beautiful approach for answering deterministic questions. The aim of this course is to introduce some fundamental probabilistic tools and present a few applications.

Learning Outcomes

To develop an appreciation of probabilistic methods in discrete mathematics.

Synopsis

Spaces of random graphs. Threshold functions.
First and second moment methods. Chernoff bounds. Applications to Ramsey numbers and random graphs.
Lovasz Local Lemma. Two-colourings of hypergraphs (property B).
Poisson approximation, and application to the distribution of small subgraphs. Janson's inequality.
Concentration of measure. Martingales and the Azuma–Hoeffding inequality.
Chromatic number of random graphs.
Talagrand's inequality.

Reading

1. N. Alon and J.H. Spencer. *The Probabilistic Method*, second edition, Wiley, 2000.

Further reading:

1. B. Bollobás, *Random Graphs*, second edition, CUP, 2001.
2. M. Habib, C. McDiarmid, J. Ramirez-Alfonsin, B. Reed, ed., *Probabilistic Methods for Algorithmic Discrete Mathematics*, Springer 1998.
3. S.Janson, T. Luczak and A.Rucinski, *Random Graphs*, John Wiley and Sons, 2000.
4. M. Mitzenmacher and E. Upfal, *Probability and Computing : Randomized Algorithms and Probabilistic Analysis*, Cambridge University Press, New York (NY), 2005.
5. M. Molloy and B. Reed, *Graph Colouring and the Probabilistic Method*, Springer 2002.
6. R. Motwani and P. Raghavan, *Randomized Algorithms*, CUP 1995.

Numerical Linear Algebra and Analysis

C12.1 Numerical Linear Algebra and Continuous Optimization

Level: M-level **Method of Assessment:** Written Examination.

Weight: Whole unit, or can be taken a either a half-unit in C12.1a or C12.1b.

Recommended Prerequisites:

C12.1a Numerical Linear Algebra and Approximation — Dr Wathen — 16MT

Synopsis

Common problems in linear algebra. Matrix structure, singular value decomposition. QR factorization, the QR algorithm for eigenvalues. Direct solution methods for linear systems, Gaussian elimination and its variants. Iterative solution methods for linear systems.

Approximation theory, Chebyshev semi-iterative methods, conjugate gradients, convergence analysis using approximation theory. Preconditioning.

Reading List

1. L N Trefethen and D Bau III, *Numerical Linear Algebra*, SIAM, 1997
2. J W Demmel, *Applied Numerial Linear Algebra*, SIAM, 1997
3. A Greenbaum, *Iterative Solution Methods for Linear Systems*, SIAM, 1997

4. G H Golub and C F van Loan, *Matrix Computations*, John Hopkins University Press, 3rd edition, 1996

5. M J D Powell, *Approximation Theory and Methods*, CUP, 1981

C12.1b Continuous Optimization — Prof. Gould — 16HT

Aims & Objectives

Optimisation deals with the problem of minimising or maximising a mathematical model of an objective function such as cost, fuel consumption etc. under a set of side constraints on the domain of definition of this function. Optimisation theory is the study of the mathematical properties of optimisation problems and the analysis of algorithms for their solution. The aim of this course is to provide an introduction to nonlinear continuous optimisation specifically tailored to the background of mathematics students.

Synopsis

[1] Preliminaries: convex sets and functions, Cholesky and QR factorisations, Sherman–Morrison–Woodbury formula, implicit function theorem, global versus local optimisation, convergence rates, optimality conditions for unconstrained optimisation.

[2–4] Line-search methods for unconstrained optimisation: steepest descent, congujate gradients, Fletcher–Reeves method, Newton–Raphson method, symmetric rank one method, Broyden–Fletcher–Goldfarb–Shanno method, practical line searches.

[5–7] Trust region methods for unconstrained optimisation: Cauchy point, dogleg method, two dimensional subspace minimisation, Steihaug's method, characterisation of exact solutions.

[8–11] Optimality conditions for constrained optimisation: convex separation, Farkas' lemma, linear programming duality, constraint qualification, Lagrangian function, Karush–Kuhn–Tucker conditions, second order optimality conditions, Lagrangian duality.

[12–16] Nonlinearly constrained optimisation: merit functions and homotopy idea, penalty function method, augmented Lagrangian method, barrier method, sequential quadratic programming.

Reading List

Lecture notes will be made available for downloading from the course webpage.
To complement the notes, reading assignments will be given from the book of J.Nocedal and S.J.Wright, *Numerical Optimisation*, Springer 1999.

C12.2 Approximation Theory and Finite Element Methods

Level: M-Level. **Method of Assesssment:** Written Examination.

Weight: Whole unit, or can be taken as either a half-unit in C12.2a or a half-unit in C12.2b.

C12.2a Approximation of Functions — Dr Sobey — 16MT

Recommended Prerequisites:

Aims & Objectives

The central idea in approximation of functions can be illustrated by the question: Given a set of functions A and an element $u \in A$, if we select a subset $B \subset A$, can we choose an element $U \in B$ so that U approximates u in some way? The course focuses on this question in the context of functions when the way we measure 'goodness' of approximation is either with an integral least square norm or with an infinity norm of the difference $u - U$. The choice of measure leads to further questions: is there a best approximation; if a best approximation exists, is it unique, how accurate is a best approximation and can we develop algorithms to generate good approximations? This course aims to give a grounding in the advanced theory of such ideas, the analytic methods used and important theorems for real functions. As well as being a beautiful subject in its own right, approximation theory is the foundation for many of the algorithms of computational mathematics and numerical analysis.

Synopsis

Introduction to approximation. Approximation in L^2. Approximation in L^∞: Oscillation Theorem, Exchange Algorithm. Approximation with splines. Rational approximation. Approximation of perodic functions.

Syllabus

Introduction to approximation. Approximation in L^2. Approximation in L^∞: Oscillation Theorem, Exchange Algorithm. Approximation with splines. Rational approximation. Approximation of perodic functions.

Reading

1. Powell, M.J.D. *Approximation Theory and Methods* (CUP).
2. Davis, P.J. *Interpolation & Approximation* (Dover).

C12.2b Finite Element Methods for Partial Differential Equations — Prof. Süli — 16HT

Recommended Prerequisites

While no formal prerequisites are assumed, students who take this course will find it helpful to attend the Michaelmas Term lecture course Partial Differential Equations for Pure and Applied Mathematicians.

Synopsis

Finite element methods represent a powerful and general class of techniques for the approximate solution of partial differential equations; the aim of this course is to provide an introduction to their mathematical theory, with special emphasis on theoretical questions such as accuracy, reliability and adaptivity; practical issues concerning the development of efficient finite element algorithms will also be discussed.

Syllabus

Elements of function spaces. Elliptic boundary value problems: existence, uniqueness and regularity of weak solutions.

Finite element methods: Galerkin orthogonality and Cea's lemma. Piecewise polynomial approximation in Sobolev spaces. Optimal error bounds in the energy norm. Variational crimes.

The Aubin–Nitsche duality argument. Superapproximation properties in mesh-dependent norms. A posteriori error analysis by duality: reliability, efficiency and adaptivity.

Finite element approximation of initial boundary value problems: stability and error analysis.

Reading List

1. S. Brenner & R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer–Verlag, Second Edition 2002 [Chapters 0,1,2,3; Chapter 4: Secs. 4.1–4.4, Chapter 5: Secs. 5.1–5.7].

2. K. Eriksson, D. Estep, P. Hansbo, & C. Johnson, *Computational Differential Equations*, CUP, 1996. [Chapters 5, 6, 8, 14 – 17].

3. C. Johnson, *Numerical Solution of Partial Differential Equations by the Finite Element Method*, CUP, 1990. [Chapters 1–4; Chapter 8: Secs. 8.1–8.4.2; Chapter 9: Secs. 9.1–9.5].

4. E. Süli, *Finite Element Methods for Partial Differential Equations*, Oxford University Computing Laboratory, 2001.

Dissertations

The guidance notes by the Projects Committee and application form are available at:

http://www.maths.ox.ac.uk/current-students/undergraduates/projects/.

You may offer either a whole-unit or a half-unit Dissertation in a Mathematical topic. You must apply to the Mathematics Project Committe in advance for approval. Proposals should be addressed to The Secretary to the Projects Committee, Room F1, The Mathematical Institute, and must be received before 12 noon on Friday of Week 3 of Michaelmas Full Term.

3 Extra Units — application required

There will be no extra units in the academic year 2007–08.

4 Other Units

MS: Statistics Half-units

MS1a: Graphical Models and Inference — lecturer tbc — 16MT

Teaching responsibility of the Department of Statistics. See the website of Statistics Department for definitive details.

Method of assessment: Written Examination:4 examination questions, $1\frac{3}{4}$ hour paper.

Weight: Half-unit

Recommended Prerequisites

OBS1 Applied statistics and OBS2 Statistical Inference would be helpful but not essential.

Aims & Objectives

Graphical models have become increasingly important in many areas where statistics play a role. They enable the description and analysis of complex stochastic systems via their natural modularity, expressed in terms of (mathematical) graphs which encode conditional independence structure. The modules correspond typically to well-understood, classical models. This course builds upon and develops the specific theory and computational tools needed in the analysis of graphical models for categorical and multivariate Gaussian data as well as Bayesian graphical models for complex stochastic systems.

Synopsis

Topics in MT07 include:

1. Conditional independence and Markov properties.
2. Log-linear graphical models for categorical data.
3. Gaussian graphical models.
4. Graphical models for complex stochastic systems

Reading

1. D. Edwards, *Introduction to Graphical Models* (2nd ed.), Springer–Verlag, New York (2002).
2. S. L. Lauritzen, *Graphical Models*, Oxford University Press, Oxford (1996).
3. P. J. Green, N. L. Hjort and S. Richardson, eds. *Highly Structured Stochastic Systems*, Oxford University Press, Oxford (2003).

MS1b: Statistical Data Mining — lecturer tbc — 12HT and 4 one-hour computer labs

Teaching responsibility of the Department of Statistics. See the website of Statistics Department for definitive details.

Method of assessment: Written Examination.

Weight: Half-unit

Recommended Prerequisites

Part A Probability and Statistics. OBS1 Applied Statistics would be an advantage.

Aims & Objectives

'Data mining' is now widely used to find interesting patterns in large databases, for example in insurance, in marketing and in many scientific fields. With large amounts of data we can search for quite subtle patterns.

This course concentrates on the statistical tools used to identify patterns, and then to identify those which are interesting not just the result of chance associations.

Synopsis

Fundamentals of pattern recognition, machine learning and data mining.

Exploratory methods: principal components analysis, biplots, independent component analysis, multidimensional scaling.

Cluster Analysis: K-means, hierarchical methods, vector quantisation, self-organising maps.

Linear discriminant analysis, logistic discrimination, linear separation and perceptrons.

Classification trees. Splitting criteria, existence of pruning sequences. V-fold cross-validation.

Feed-forward neural networks. Universal approximation properties, back-propagation, training algorithms, assessment of fit.

Reading

1. C. Bishop, *Neural Networks for Pattern Recognition*, Oxford UP (1995).
2. D. Hand, H. Mannila, P. Smyth, *Principles of Data Mining*, MIT Press (2001).
3. I. H. Witten and E. Franke, *Data Mining. Practical Machine Learning Tools and Techniques with Java Implementations*, Morgan Kaufmann (2000).

Further Reading

1. B. D. Ripley, *Pattern Recognition and Neural Networks*, Cambridge UP (1996).

MS2a: Bioinformatics and Computational Biology — lecturer tbc — 16MT

Teaching responsibility of the Department of Statistics. See the website of Statistics Department for definitive details.

Method of assessment: Written Examination.

Weight: Half-unit.

Recommended Prerequisites

None. In particular, no previous knowledge of Genetics will be necessary.

Aims & Objectives

Modern molecular biology generates large amounts of data, such as sequences, structures and expression data, that needs different forms of statistical analysis and modelling to be properly interpreted. The fields of Bioinformatics and Computational Biology have this as their subject matter and there is no sharp boundary between them. Bioinformatics has an applied flavour while Computational Biology is viewed as the study of the models, statistical methodology and algorithms needed to do bioinformatics analysis. This course aims to present core topics of these fields with and emphasis on modelling and computation.

Synopsis

Fundamental Data Structures in Biology: Sequences, Genes and RNA secondary structure.

Stochastic Models of Sequence and Genome Evolution including models of single nucleotide/amino acid/codon evolution.

Phylogenies: enumerating phylogenies, the probability of sequences related by a specified phylogeny, the minimal number of events needed to explain a data set (Parsimony).

Likelihood and algorithms (Markov Chain Monte Carlo) for inference based on the likelihood. Software packages for sample-based inference.

Alignment Algorithms. Comparing 2 strings, an arbitrary number of strings, find segments of high similarity in 2 strings.

Common Patterns in a set of Sequences.

Reading

1. C. Semple and M. Steel, *Phylogenetics*, Oxford University Press (2003).

2. Durbin et al., *Biological Sequence Analysis*, Cambrigde University Press (1998).

3. T. Jiang et al., (editors) *Current Topics in Computational Biology*, MIT Press, (2003).

4. M. S. Waterman et al., *Computational Genome Analysis: An Introduction*, Springer (2004).

MS2b: Stochastic Models in Mathematical Genetics — lecturer tbc — 16HT

Teaching responsibility of the Department of Statistics. See the website of Statistics Department for definitive details.

Method of assessment: Written Examination.

Weight: Half-unit

Aims & Objectives

The aim of the lectures is to introduce modern Stochastic models in Mathematical Population Genetics that describe the distribution of gene frequencies and ancestry in a population or sample of genes. Stochastic and Graph theoretic properties of coalescent and gene trees are studied in the first eight lectures. Diffusion process models of gene frequencies and their applications are studied in the second eight lectures.

Synopsis

Evolutionary models in Mathematical Genetics:

The Wright–Fisher model. The Genealogical Markov chain describing the number ancestors back in time of a collection of genes.

The Coalescent process describing the stochastic behaviour of the ancestral tree of a collection of genes. Mutations on ancestral lineages in a coalescent tree. Inferring the time to the most recent common ancestor in a sample of genes from the number of mutations occurring to the genes. Models with a variable population size.

The frequency spectrum and age of a mutation. Ewens' sampling formula for the probability distribution of the allele configuration of genes in a sample in the infinitely-many-alleles model. Hoppe's urn model for the infinitely-many-alleles model.

The infinitely-many-sites model of mutations on DNA sequences. Gene trees as perfect phylogenies describing the mutation history of a sample of DNA sequences. Graph theoretic constructions and characterizations of gene trees from DNA sequence variation. Gusfield's construction algorithm of a tree from DNA sequences. Examples of gene trees from data. The probability distribution of a gene tree.

Diffusion process models in Mathematical Genetics:

Introduction to diffusion processes. The stochastic process describing the distribution of the gene frequency of an allele forward in time for a two-allele model. The Moran model. The diffusion process limit from the Moran model. The generator of a diffusion process with two allele types.

Heuristic introduction to Stochastic differential equations. Examples in using the diffusion process generator and Stochastic differential equations. The mean time to absorption or fixation of an allele.

The genealogy of the diffusion process describing the gene frequency of an allele. The underlying infinite-particle coalescent.

Two allele models with mutation and selection. Stationary distributions of diffusion process. Sampling from the stationary distribution.

A brief introduction to diffusion process models with more than two types. The Dirichlet distribution describing the stationary distribution of allele frequencies. The Poisson–Dirichlet process.

Reading

1. R. Durrett, *Probability Models for DNA Sequence Evolution*, Springer (2002).
2. W. J. Ewens, *Mathematical Population Genetics*, 2nd ed, Springer (2004).
3. J. R. Norris, *Markov Chains*, Cambridge University Press (1999).
4. M. Slatkin and M. Veuille, *Modern Developments in Theoretical Population Genetics*, Oxford Biology (2002).
5. S. Tavaré and O. Zeitouni, *Lectures on Probability Theory and Statistics*, *Ecole d'Eté de Probabilités de Saint-Flour XXXI — 2001*, Lecture Notes in Mathematics 1837. Springer (2004).

MS3b: Lévy Processes and Finance — lecturer tbc — 16HT

Teaching responsibility of the Department of Statistics. See the website of Statistics Department for definitive details.

Method of assessment: Written Examination.

Weight: Half-unit.

Recommended Prerequisites

Part A Probability is a prerequisite. BS3a/OBS3a Applied Probability or B10 Martingales and Financial Mathematics would be useful, but are by no means essential; some material from these courses will be reviewed without proof.

Aims & Objectives

Lévy processes form a central class of stochastic processes, contain both Brownian motion and the Poisson process, and are prototypes of Markov processes and semimartingales. Like Brownian motion, they are used in a multitude of applications ranging from biology and physics to insurance and finance. Like the Poisson process, they allow to model abrupt moves by jumps, which is an important feature for many applications. In the last ten years Lévy processes have seen a hugely increased attention as is reflected on the academic side by a number of excellent graduate texts and on the industrial side realising that they provide versatile stochastic models of financial markets. This continues to stimulate further research in both theoretical and applied directions. This course will give a solid introduction to some of the theory of Lévy processes as needed for financial and other applications.

Synopsis

Review of (compound) Poisson processes, Brownian motion (informal), Markov property. Connection with random walks, [Donsker's theorem], Poisson limit theorem. Spatial Poisson processes, construction of Lévy processes.
Special cases of increasing Lévy processes (subordinators) and processes with only positive jumps. Subordination. Examples and applications. Financial models driven by Lévy processes. Stochastic volatility. Level passage problems. Applications: option pricing, insurance ruin, dams.
Simulation: via increments, via simulation of jumps, via subordination. Applications: option pricing, branching processes.

Reading

1. J.F.C. Kingman: *Poisson Processes*, Oxford University Press (1993), Ch.1–5, 8.
2. A.E. Kyprianou: *Introductory Lectures on Fluctuations of Lévy Processes with Applications*, Springer (2006), Ch. 1–3, 8–9.
3. W. Schoutens: *Lévy Processes in Finance: Pricing Financial Derivatives*, Wiley (2003).

Further reading

1. J. Bertoin: *Lévy Processes*, Cambridge University Press (1996), Sect. 0.1–0.6, I.1, III.1–2, VII.1.
2. K. Sato: *Lévy Processes and Infinite Divisibility*,Cambridge University Press (1999), Ch. 1–2, 4, 6, 9.

Computer Science: Half Units

Please see the Computing Laboratory website for full, up-to-date course information.

Please note that these three courses will be examined by mini-project (as for MSc students). Mini-projects will be handed out to candidates on the last Friday of the term in which the subject is being taught, and you will have to hand it in to the Exam Schools by noon on Monday of Week 1 of the following term. The mini-project will be designed to be completed in about three days. It will include some questions that are more open-ended than those on a standard sit-down exam. The work you submit should be your own work, and include suitable references.

Please note that the Computer Science courses in Part C are 50% bigger than those in earlier years, i.e. for each Computer Science course in the 3rd year undergraduates are expected to undertake about 10 hours of study per week, but 4th year courses will each require about

15 hours a week of study. Lecturers are providing this extra work in a variety of ways, e.g. some will give 16 lectures with extra reading, classes and/or practicals, whereas others will be giving 24 lectures, and others still will be doing something in between. Students will need to look at each synopsis for details on this.

CCS1a: Categories, Proofs and Processes — lecturer t.b.c — 20 lectures (plus extra reading) MT

Level: M-level. **Method of Assessment:** Mini-project.

Weight: Half-unit.

Recommended Prerequisites: Some familiarity with basic discrete mathematics: sets, functions, relations, mathematical induction. Basic familiarity with logic: propositional and predicate calculus. Some first acquaintance with abstract algebra: vector spaces and linear maps, and/or groups and group homomorphisms. Some familiarity with programming, particularly functional programming, would be useful but is not essential.

Aims & Objectives

Category Theory is a powerful mathematical formalism which has become an important tool in modern mathematics, logic and computer science. One main idea of Category Theory is to study mathematical 'universes', collections of mathematical structures and their structure-preserving transformations, as mathematical structures in their own right, i.e. categories — which have their own structure-preserving transformations (functors). This is a very powerful perspective, which allows many important structural concepts of mathematics to be studied at the appropriate level of generality, and brings many common underlying structures to light, yielding new connections between apparently different situations.

Another important aspect is that set-theoretic reasoning with elements is replaced by reasoning in terms of arrows. This is more general, more robust, and reveals more about the intrinsic structure underlying particular set-theoretic representations.

Category theory has many important connections to logic. We shall in particular show how it illuminates the study of formal proofs as mathematical objects in their own right. This will involve looking at the Curry–Howard isomorphism between proofs and programs, and at Linear Logic, a resource-sensitive logic. Both of these topics have many important applications in Computer Science.

Category theory has also deeply influenced the design of modern (especially functional) programming languages, and the study of program transformations. One exciting recent development we will look at will be the development of the idea of coalgebra, which allows the formulation of a notion of coinduction, dual to that of mathematical induction, which provides powerful principles for defining and reasoning about infinite objects.

This course will develop the basic ideas of Category Theory, and explore its applications to the study of proofs in logic, and to the algebraic structure of programs and programming languages.

Learning Outcomes

To master the basic concepts and methods of categories.

To understand how category-theory can be used to structure mathematical ideas, with the concepts of functoriality, naturality and universality; and how reasoning with objects and arrows can replace reasoning with sets and elements. To learn the basic ideas of using commutative diagrams and unique existence properties.

To understand the connections between categories and logic, focussing on structural proof theory and the Curry–Howard isomorphism.

To understand how some basic forms of computational processes can be modelled with categories.

Synopsis

Categories: 10 lectures.
1 each for the following topics:
Categories
Monics, epics, isomorphisms
Products and coproducts
Limits and colimits
Functors and natural transformations
Universal arrows
Adjunctions
Monads
Cartesian closed categories
Symmetric monoidal closed categories
Connection to Logic: 4 lectures
Algebras and coalgebras: 4 lectures.

Syllabus

- Introduction to category theory. Categories, functors, natural transformations. Isomorphisms. monics and epics. Products and coproducts. Universal constructions. Cartesian closed categories. Symmetric monoidal closed categories. The ideas will be illustrated with many examples, from both mathematics and Computer Science.

- Introduction to structural proof theory. Natural deduction, simply typed lambda calculus, the Curry–Howard correspondence. Introduction to Linear Logic. The connection between logic and categories.

- Further topics in category theory. Algebras, coalgebras and monads. Connections to programming (structural recursion and corecursion), and to programming languages (monads as types for computational effects).

Reading List

Lecture notes will be provided.

The following books provide useful background reading.

1. *Basic Category Theory for Computer Science*, Pierce.
2. *Conceptual Mathematics*, Lawvere and Schanuel.
3. *Categories for the Working Mathematician*, Saunders Mac Lane.
4. *Categories and Types*, R. Crole.
5. *Proofs and Types*, Girard, Lafont and Taylor.
6. *Algebra of Programming*, Bird and de Moor.
   Of these, the book by Pierce provides a very accessible and user-friendly first introduction to the subject.

CCS3b: Quantum Computer Science — Dr Coecke — 20 lectures (plus extra reading) HT

Level: M-level. **Method of Assessment:** Mini-project.

Weight: Half-unit.

Recommended Prerequisites:

We do not assume any prior knowledge of quantum mechanics. Some knowledge of basic linear algebraic notions such as vector spaces and matrices is however a pre-requisite. The course notes do comprise an overview of this material so we advise students with a limited background in linear algebra to consult the course notes before the course starts.

Aims & Objectives

Both physics and computer science have been very dominant scientific and technological disciplines in the previous century. Quantum Computer Science aims at combining both and hence promises to play an important similar role in this century. Combining the existing expertise in both fields proves to be a non-trivial but very exciting interdisciplinary journey. Besides the actual issue of building a quantum computer or realizing quantum protocols it involves a fascinating encounter of concepts and formal tools which arose in distinct disciplines.

This course provides an interdisciplinary introduction to the emerging field of quantum computer science, explaining (very) basic quantum mechanics (including finite dimensional Hilbert spaces and the tensor product thereof), quantum entanglement, its structure and its physical consequences (e.g. non-locality, no-cloning principle), and introduces qubits. We give detailed discussions of some key algorithms and protocols such as Shor's factorization algorithm, quantum teleportation and quantum key exchange, and analyze the challenges these pose for computer science, mathematics etc. We also provide a more conceptual semantic analysis of some of the above. Other important issues such as quantum information theory (including mixed states) will also be covered (although not in great detail). We also discuss alternative computational paradigms and models for the circuit model, we argue the need for high-level methods, provide some recent results concerning graphical language and categorical semantics for quantum informatics and delineate the remaining scientific challenges for the future.

Learning Outcomes

The student will know by the end of the course what quantum computing and quantum protocols are about, why they matter, and what the scientific prospects concerning are. This includes a structural understanding of some basic quantum mechanics, knowledge of important algorithms such as Shor's algorithm and important protocols such as quantum teleportation. He/she will also know where to find more details and will be able to access these. Hence this course also offers computer science and mathematics students a first stepping-stone for research in the field, with a particular focus on the newly developing field of quantum computer science semantics, to which Oxford University Computing Laboratory has provided pioneering contributions.

Synopsis

1. Historical and physical context
1.1 The birth of quantum mechanics
1.2 The status of quantum mechanics
1.3 The birth of quantum informatics
1.4 The status of quantum informatics
2 Qubits vs. bits
2.1 Acting on qubits
2.2 Describing a qubit with complex numbers
2.3 Describing two qubits
3 von Neumanns pure state formalism
3.1 Hilbert space
3.2 Matrices
3.3 Tensor structure
3.4 Dirac notation
4 Protocols from entanglement
4.1 Bell-base and Bell-matrices

4.2 Teleportation and entanglement swapping
5 The structure of entanglement
5.1 Map-state duality and compositionality
5.2 The logic of bipartite entanglement
5.3 Quantifying entanglement
5.4 Trace
6 Algorithms and gates
6.1 Special gates
6.2 The Deutch–Jozsa algorithm
6.3 Grover's algorithm
6.4 Shor's factoring algorithm
6.4.1 Period finding
6.4.2 Factoring and code-breaking
6.5 Quantum key distribution
7 Mixed states
8 Quantum logic and Gleason's theorem
9 Mixed operations
10 More on tensors
11 Graphical language for quantum informatics
11.1 Symmetric monoidal categories
11.2 dagger-compact categories

Syllabus

Finite dimensional vector space, inner-product, complex numbers, linear adjoints, unitary maps, projectors, trace, tensor product of Hilbert spaces, Dirac notation, bit, qubit, entanglement, map-state duality, no-cloning, quantum circuits, quantum gates, Shor's algorithm, Grover's algorithm, quantum teleportation, quantum key-exchange, teleportation and measurement based quantum computing, decoherence, mixed states, quantum information, quantum logic, quantum categorical semantics.

Reading List

Lecture notes which cover the whole course and which provide detailed pointers to additional reading will be made available. Standard books on the subject that might be of use are:

1. Gruska, J. (1999) *Quantum Computing.* McGraw–Hill.

2. Nielsen, M. and Chuang, I. L. (2000) *Quantum Computation and Quantum Information.* Cambridge University Press.

3. Kitaev, A. Yu., Shen, A. H. and Vyalyi, M. N. (2001) *Classical and Quantum Computing.* Graduate Studies in Mathematics 47, American Mathematical Society.

On-line available courses elsewhere which can be consulted are:

1. By Braunstein: http://www.weizmann.ac.il/chemphys/schmuel/comp/comp.html

2. By Bub: http://arxiv.org/abs/quant-ph/0512125
3. By Preskill http://www.theory.caltech.edu/people/preskill/ph229/

A different angle which is also very much reflected in the course is available at:

1. http://arxiv.org/abs/quant-ph/0510032
2. http://arxiv.org/abs/quant-ph/0506132

CCS4b — Automata, Logics and Games — Prof. Ong — 16 lectures (plus extra reading) HT

Level: M-level. **Method of Assessment:** Mini-project.

Weight: Half-unit.

Recommended Prerequisites: There are no prerequisites, though familiarity with finite automata, B1 Logic, and the basics of complexity theory would be helpful.

Aims & Objectives

To introduce the mathematical theory underpinning the Computer-Aided Verification of computing systems. The main ingredients are:

- Automata (on infnite words and trees) as a computational model of state-based systems.
- Logical systems (such as temporal and modal logics) for specifying operational behaviour.
- Two-person games as a conceptual basis for understanding interactions between a system and its environment.

Learning Outcomes

At the end of the course students will be able to:

1. Describe in detail what is meant by a Buchi automaton, and the languages recognised by simple examples of Buchi automata.
2. Use linear-time temporal logic to describe behaviourial properties such as recurrence and periodicity, and translate LTL formulas to Buchi automata.

3. Use S1S to define omega-regular languages, and give an account of the equivalence between S1S definability and Buchi recognisability.

4. Explain the intuitive meaning of simple modal mu-calculus formulas, and describe the correspondence bewteen property-checking games and modal mu-calculus model checking.

Synopsis/Syllabus

Automata on infinite words. Buchi automata: Closure properties. Determinization and Rabin automata.
Nonemptiness and Nonuniversality problems for Buchi automata.
Linear temporal logic and alternating Buchi automata.
Modal mu-calculus: Fundamental Theorem, decidability and finite model property. Parity Games and the Model-Checking Problem: memoryless determinacy, algorithmic issues.
Monadic Second-order Logic and its relationship with the modal mu-calculus.

Reading List

Selected parts from:

1. J. Bradeld and C. P. Stirling. *Modal logics and* $^{1}-$*calculi.* In J. Bergstra, A. Ponse, and S. Smolka, editors, Handbook of Process Algebra, pages 293–332. Elsevier, North–Holland, 2001.

2. B. Khoussainov and A. Nerode. *Automata Theory and its Applications.* Progress in Computer Science and Applied Logic, Volume 21. Birkhauser, 2001.

3. C. P. Stirling. *Modal and Temporal Properties of Processes.* Texts in Computer Science. Springer–Verlag, 2001.

4. W. Thomas. *Languages, Automata and Logic.* In G. Rozenberg and A. Salomaa, editors, Handbook of Formal Languages, volume 3. Springer–Verlag, 1997.

5. M. Y. Vardi. *An Automata — Theoretic Approach to Linear Temporal Logic.* In Logics for Concurrency: Structure versus Automata, ed. F. Moller and G. Birtwistle, LNCS vol. 1043, pp. 238–266, Springer–Verlag, 1996.
The only copy of the above book is in the RSL on open shelves. However, the article in question is available in pdf form online at this address:
http://folli.loria.fr/cds/1998/pdf/degiacomo-nardi/vardi.pdf

Philosophy

Rise of Modern Logic — no lectures in 2007-08 but tutorials will be given for those wishing to take it.

For further details on the Philosophy courses (including details of method of assessment, etc.) please refer to the Philosophy Lectures Prospectus which is published at http://www.philosophy.ox.ac.uk/ prior to the start of each term (usually in 0th Week).

There will be no Rise of Modern Logic lectures in 2007/08. Any students who want to take this option in this year will need to study it in tutorials, which will be given in Michaelmas Term by Dr Isaacson. Please contact academic.administrator@maths.ox.ac.uk if you require further details.

Method of Assessment: A three hour examination paper in Trinity Term and a 5000 word essay to be submitted by Week 0 in Trinity Term.

Weight: Whole unit.

Essential Prerequisites

Philosophy of Mathematics, B1 Foundations

Synopsis

The eight lectures shall concentrate on Frege's and Russell's logicism, the origins of set theory (Cantor and Zermelo), Hilbert's programme and the impact of Gödel's theorems on it. Further topics will include the rise of first-order logic and intuitionism.

Further details can be found in the Mathematics & Philosophy Handbook.

Reading

Much of the relevant material is contained in

Jean van Heijenoort (ed.), *From Frege to Godel. A Source Book in Mathematical Logic, 1879–1931*, (Harvard University Press).

A reading list will be available shortly at my website.

5 Language Classes

Language courses in French offered by the University Language Centre.

Students in the FHS Mathematics, FHS Mathematics and Statistics and Mathematics and Philosophy may apply to take language classes. In 2007–2008, French language classes will be run in MT and HT.

Students wishing to take language classes should attend the qualifying test on Monday of Week 1 Michaelmas Term from 5-7pm in the Language Centre, Woodstock Road.

Two levels of courses are offered, a lower level for those with a good pass at GCSE, and a higher level course for those with A/S or A level. Acceptance on either course will depend on satisfactory performance in the Preliminary Qualifying Test held in 0th Week of Michaelmas Term.

Performance on the course will not contribute to the class of degree awarded. However, upon successful completion of the course, students will be issued with a certificate of achievement which will be sent to their college.

Places at these classes are limited, so students are advised that they must indicate their interest at this stage. If you are interested but are unable to attend this presentation for some reason please contact the Academic Administrator in the Mathematical Institute (academic.administrator@maths.ox.ac.uk; (2)75330) as soon as possible.

Aims and rationale

The general aim of the language courses is to develop the student's ability to communicate (in both speech and writing) in French to the point where he or she can function in an academic or working environment in a French-speaking country.

The courses have been designed with general linguistic competence in the four skills of reading, writing, speaking and listening in mind. There will be opportunities for participants to develop their own particular interests through presentations and assignments.

Form and Content

Each course will consist of a thirty-two contact hour course, together with associated work. Classes will be held in the Language Centre at two hours per week in Michaelmas and Hilary Terms.

The content of the courses is based on coursebooks together with a substantial amount of supplementary material prepared by the language tutors. Participants should expect to spend an additional two hours per week on preparation and follow-up work.

Each course aims to cover similar ground in terms of grammar, spoken and written language and topics. Areas covered will include:

Grammar:

- all major tenses will be presented and/or revised, including the subjunctive
- passive voice
- pronouns
- formation of adjectives, adverbs, comparatives
- use of prepositions
- time expressions

Speaking

- guided spoken expression for academic, work and leisure contact (e.g. giving presentations, informal interviews, applying for jobs)
- expressing opinions, tastes and preferences
- expressing cause, consequence and purpose

Writing

- Guided letter writing for academic and work contact
- Summaries and short essays

Listening

- Listening practice of recorded materials (e.g. news broadcasts, telephone messages, interviews
- developing listening comprehension strategies

Topics (with related readings and vocabulary, from among the following)

- life, work and culture
- the media in each country
- social and political systems
- film, theatre and music
- research and innovation
- sports and related topics
- student-selected topics

Teaching staff

The courses are taught by Language Centre tutors or Modern Languages Faculty instructors.

Teaching and learning approaches

Each course uses a communicative methodology which demands active participation from students. In addition, there will be some formal grammar instruction. Students will be expected to prepare work for classes and homework will be set. Students will also be given training in strategies for independent language study, including computer-based language learning, which will enable them to maintain and develop their language skills after the course.

Entry

Two classes at (probably at Basic and Threshold levels) will be formed according to level of French at entry. The minimum entry standard is a good GCSE Grade or equivalent. Registration for each option will take place at the start of Michaelmas Term. A preliminary qualifying test is then taken which candidates must pass in order to be allowed to join the course.

Learning Outcomes

The learning outcomes will be based on internationally agreed criteria for specific levels (known as the ALTE levels), which are as follows:

Basic Level (corresponds to ALTE Level 2 "Can-do" statements)

- Can express opinions on professional, cultural or abstract matters in a limited way, offer advice and understand instructions.
- Can give a short presentation on a limited range of topics.
- Can understand routine information and articles, including factual articles in newspapers, routine letters from hotels and letters expressing personal opinions.
- Can write letters or make notes on familiar or predictable matters.

Threshold Level (corresponds to ALTE Level 3 "Can-do" statements)

- Can follow or give a talk on a familiar topic or keep up a conversation on a fairly wide range of topics, such as personal and professional experiences, events currently in the news.
- Can give a clear presentation on a familiar topic, and answer predictable or factual questions.
- Can read texts for relevant information, and understand detailed instructions or advice.
- Can make notes that will be of reasonable use for essay or revision purposes.

- Can make notes while someone is talking or write a letter including non- standard requests.

Assessment

There will a preliminary qualifying test in Michaelmas Term. There are three parts to this test: Reading Comprehension, Listening Comprehension, and Grammar. Candidates who have not studied or had contact with French for some time are advised to revise thoroughly, making use of the Language Centre's French resources.

Students' achievement will be assessed through a variety of means, including continuous assessment of oral performance, a written final examination, and a project or assignment chosen by individual students in consultation with their language tutor.

Reading comprehension, writing, listening comprehension and speaking are all examined. The oral component may consist of an interview, a presentation or a candidate's performance in a formal debate or discussion.

6 Registration

LECTURES AND CLASSES: Students will have to register in advance for the lectures and classes they wish to take. Students will have to register by Wednesday of Week 9 of Trinity Term 2007 using the form found at https://www.maths.ox.ac.uk/current-students/undergraduates/forms/. In the case of unexpected excessive demand for intercollegiate class places for an option, the lecturer may apply for a quota to be imposed. Students should check their e-mail regularly over summer, as those affected by the introduction of such a quota will be contacted by email, and presented with an opportunity to make a case for a place on a quota and to name an alternative option.

LECTURES: Some combinations of subjects are not advised and lectures may clash. Details are given below. We will use the information on your registration forms to aim to keep clashes to a minimum. However, because of the large number of options available in Part C some clashes are inevitable, and we must aim to accommodate the maximum number of student preferences.

Lecture Timetabling in Part C, 2007–8

The Teaching Committee has agreed that the following clashes be allowed.

C1.1 Model Theory & Godel's Incompleteness Theorems	may clash with	C6.1 Solid Mechanics
C1.2 Analytic Topology Axiomatic Set Theory		C6.2 Elasticity and Plasticity
C2.1 Lie Algebras & Representation Theory of Symmetric Groups		C6.3 Perturbation Methods & Applied Complex Variables
C3.1 Lie Groups & Differentaible Manifolds		C8.1 Mathematics and the Environment & Mathematical Physiology
C9.1 Analytic Number Theory & Elliptic Curves		All Statistics options
		C12.1 Numerical Linear Algebra and Continuous Optimization
		C12.2 Approximation Theory and Finite Element Methods

Mathematics vs. Computing		
C6.1 Solid Mechanics	may clash with	CCS1a Categories, Proofs and Processes
C6.2 Elasticity and Plasticity		
C6.3 Perturbation Methods and Applied Complex Variables		
C3.1 Lie Groups and Differentiable Manifolds		

Other and Non-mathematical subjects		
All Statistics options	may clash with	All Philosophy papers