Twistorial Anomalies

Penrose at 95 and Twistors at 60

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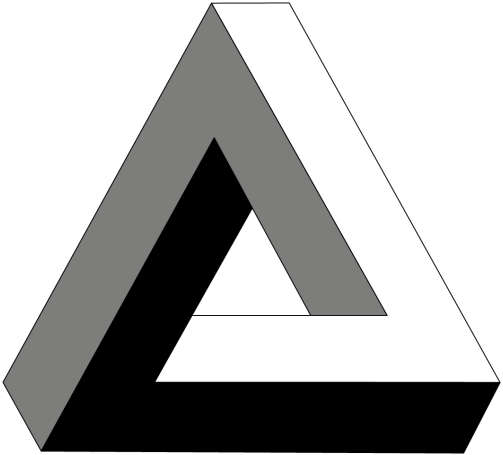


Introduction

Introduction

Twistor theory relates:

$$\text{solutions to integrable pdes} \quad \leftrightarrow \quad \text{holomorphic data}$$

This has been exploited to solve many interesting non-linear equations.

During the 'twistor uprising' the corrpespondence was formulated off-shell:

$$\text{integrable field theories} \quad \leftrightarrow \quad \text{holomorphic field theories}$$

This was exploited to great effect in the amplitudes programme.

Comparatively recently it was realised that, because holomorphic field theories are chiral, anomalies are ubiquitous in twistor theory.

Today I will explain how to compute twistorial anomalies using index theory and some of their consequences.

In particular we see simple twistor derivations of:

- Familiar space-time chiral anomalies.
- The one-loop QCD β-function.
- The 4d Weyl anomalies.
- Familiar space-time chiral anomalies.

There are also anomalies on twistor space which do not have a straightforward space-time interpretation. These can be interpreted as loop level obstructions to the integrability of:

- Self-dual gauge theory.
- Self-dual gravity.

Based on arXiv:2208.12701 & arXiv:2510.26764.

Anomalies

What is an Anomaly?

We say that a symmetry of a classical field theory has an anomaly if it does not survive quantization. This occurs when there is no regularisation of the path integral consistent with the symmetry.

Traditionally the history of the anomaly begins with the work of Adler and Bell–Jackiw.

They sought to reconcile the rapid decay of neutral pions into photons, quantified by the decay rate $\Gamma(\pi_0 \to 2\gamma)$, with the principle of chiral symmetry breaking. **[Adler, 69; Bell, Jackiw; 69]**

Adler–Bell–Jackiw Anomaly

The Adler–Bell–Jackiw (ABJ) anomaly can be derived in 4d massless QED

$$S_{\mathsf{QED}}[A,\bar{\psi},\psi] = \int_{\mathbb{R}^4} \mathrm{d}^4x \Big(-\frac{1}{4} F^{\mu\nu}F_{\mu\nu} + \mathrm{i}\bar{\psi}_{\mathrm{L}}^{\dot{\alpha}} D_{\dot{\alpha}\alpha}\psi_{\mathrm{L}}^{\alpha} + \mathrm{i}\bar{\psi}_{\mathrm{R}}^{\beta} D_{\dot{\beta}\beta}\psi_{\mathrm{R}}^{\dot{\beta}} \Big)\,.$$

The action is invariant under an axial symmetry under which the fermions transform by

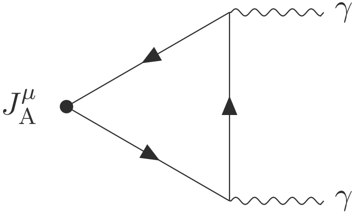
$$\psi_{\mathrm{L}} \mapsto e^{-\mathrm{i}\theta}\psi_{\mathrm{L}}\,, \qquad \psi_{\mathrm{R}} \mapsto e^{\mathrm{i}\theta}\psi_{\mathrm{R}}$$

leading to a conserved current

$$J_{\mathrm{A}}^{\dot{\alpha}\alpha} = \bar{\psi}_{\mathrm{R}}^{\alpha}\psi_{\mathrm{R}}^{\dot{\alpha}} - \bar{\psi}_{\mathrm{L}}^{\dot{\alpha}}\psi_{\mathrm{L}}^{\alpha}\,.$$

In perturbative quantum field theory correlation functions are evaluated as a sum over Feynman diagrams, to each of which is associated an integral.

Quantum effects are generated by diagrams with loops. Often the corresponding integral diverges and so must be regulated.

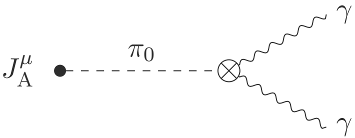


Careful regularization of the triangle diagram illustrated above below shows that in correlation functions the axial current is not in fact conserved, but rather

$$\partial_\mu J_{\mathrm{A}}^{\mu} = \frac{e^2}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \,.$$

In the context of the quark model, the fields $\psi, \bar{\psi}$ stand in for up and down (anti-)quarks. At sufficiently low energies the bilinear $\langle \bar{\psi}_{\mathrm{R}}\psi_{\mathrm{L}} + \bar{\psi}_{\mathrm{L}}\psi_{\mathrm{R}} \rangle$ is believed to acquire an expectation, spontaneously breaking chiral symmetry.

The axial current j_{A} can be used to measure the strength of chiral symmetry breaking, relating the pion decay rate to its conservation.



If the axial current is conserved, then in the limit of zero quark mass the $\pi_0 \to 2\gamma$ decay is forbidden. But the axial current is not conserved, leading to a decay rate of the correct order

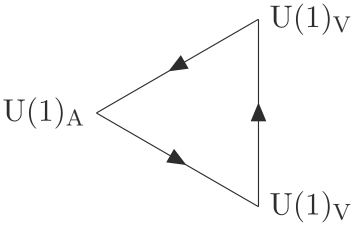
$$\Gamma(\pi_0 \to 2\gamma) = \frac{\alpha^2 m_\pi^3}{4\pi\Lambda_\pi^2}\,.$$

Chiral Anomalies

Anomalies of this type occur in theories of chiral fermions on even dimensional space-times. In four-dimensions they are generated by wheel diagrams with three external legs.

Depending on what currents couple at the vertices, different conservation laws are violated.

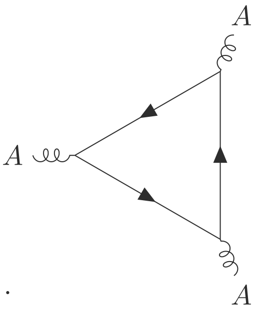
The ABJ anomaly corresponds to $\mathrm{U}(1)_{\mathrm{V}}$ at two vertices and $\mathrm{U}(1)_{\mathrm{A}}$ at one.



In a theory of N chiral fermions there is a non-Abelian chiral anomaly in the $\mathrm{SU}(N)_{\mathrm{L}}$ symmetry. This is an example of a 't Hooft anomaly. The non-Abelian current $J_{\mathrm{L,a}} = \bar{\psi}_{\mathrm{R}} \mathrm{t_a} \psi_{\mathrm{L}}$ remains conserved.

However, if we couple to a background gauge connection $\mathrm{D} = \mathrm{d} + A$ the partition function

$$\det \mathrm{D}_{+} = \int \mathcal{D}\bar{\psi}_{\mathrm{R}}\, \mathcal{D}\psi_{\mathrm{L}}\, \exp\left(-\int_{\mathbb{R}^4} \bar{\psi}_{\mathrm{R}}\, \mathrm{D}_{+} \psi_{\mathrm{L}}\right)$$



is not gauge invariant

$$\delta_\epsilon \log \det \mathrm{D}_{+} = -\int_{\mathbb{R}^4} \mathrm{tr}(\epsilon \partial_\mu J^\mu_{\mathrm{L}}) = \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \mathrm{tr_F}(\epsilon F(A)^2)\,.$$

Index Theory and Anomalies

Index Theory and Anomalies

Chiral anomalies have a beautiful topological origin. **[Atiyah, Singer, 84; Quillen, 85]**

Consider the case of N chiral fermions coupled to a background $\mathrm{SU}(N)_{\mathrm{L}}$ connection $\mathrm{D} = \mathrm{d} + A$.

The partition function is

$$\det \mathrm{D}_+ = \int \mathcal{D}\bar{\psi}_{\mathrm{R}}\, \mathcal{D}\psi_{\mathrm{L}}\, \exp\left(- \int_M \bar{\psi}_{\mathrm{R}}\, \mathrm{D}_+ \psi_{\mathrm{L}} \right).$$

Since D_+ maps left-handed to right-handed, the partition function is not a number.

As D varies in the space of connections modulo gauge transformations, $\mathcal{A}/\mathcal{G}$, the partition function transforms as a section of a line bundle $\mathcal{L}$.

To interpret $\det \mathrm{D}_+$ as a function on $\mathcal{A}/\mathcal{G}$ we need to trivialise $\mathcal{L}$. But this is not always possible: The topological obstruction is the first Chern class $\mathrm{c}_1(\mathcal{L})$. Fortunately this can be computed using the Atiyah–Singer index theorem.

The idea is to view the theory as being defined over a family of space-times which keeps track of the background gauge connection

$$X = M \times \mathcal{A}/\mathcal{G}\,.$$

The fermions ψ are valued in a vector bundle $\mathcal{E} = (E \times \mathcal{A})/\mathcal{G}$ over this space, which is equipped with a natural connection

$$\mathbb{A} = A - \Theta(\delta A)$$

where $\Theta(\mathrm{D}\epsilon) = \epsilon$. Then the first Chern class is

$$\mathrm{c}_1(\mathcal{L}) = \left[\int_M \widehat{A}(T_{M/X})\,\mathrm{Ch}(\mathcal{E}) \right]_2 .$$

The integrand is the anomaly polynomial. Here it is a six-form.

Relation to the Triangle Diagram

Let's unpack this formula. For $M = \mathbb{R}^4$ the tangent bundle is trivial, so all contributions are from the Chern character.

Writing $\mathbb{F}(\mathbb{A})$ for the curvature of $\mathbb{A}$ we have

$$\mathrm{c}_1(\mathcal{L}) \propto \int_{\mathbb{R}^4} \mathrm{tr}_{\mathbb{C}^N}\left(\mathbb{F}(\mathbb{A})^3\right) .$$

The anomalous triangle diagram can be recovered by first pulling back $\mathcal{L}$ to $\mathcal{A}$, yielding the trivial line bundle. Taking the primitive potential for $\mathrm{c}_1(\mathcal{L})$ and contracting with the tangent vector to a gauge orbit $\delta A = \mathrm{D}_A \epsilon$ yields

$$\delta_\epsilon \log \det \mathrm{D}_+ = \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \mathrm{tr}_{\mathbb{C}^N}\left(\epsilon F(A)^2\right) .$$

Conformal Anomalies

Conformal Anomalies

It's also possible for conformal symmetries to have anomalies.

Classically massless QCD

$$S_{\mathsf{QCD}}[A] = -\frac{1}{g^2}\int_M \mathrm{tr}(F\wedge *F) + \text{massless fermions and scalars}\,.$$

is conformally invariant; the stress tensor has vanishing trace

$$T_{\mu\nu} = -\mathrm{tr}(F_{\mu\rho}F_\nu{}^\rho) + \frac{1}{4}\eta_{\mu\nu}\mathrm{tr}(F^2) + \text{fermion, scalar contributions}\,.$$

But at loop level

$$\eta^{\mu\nu}\langle T_{\mu\nu}\rangle = \frac{\beta(g)}{2g}\mathrm{tr}(F^2)$$

where $\beta(g)$ is the β-function. This is an anomaly to scale invariance.

The one-loop contribution to the β-function is one of the most famous formulas in high energy physics. For $\mathrm{SU}(N_c)$ gauge theory coupled to N_f massless fundamental Diracs and N_s massless complex scalars it reads

$$\beta(g) = -\left(\frac{11}{3}N_c - \frac{2}{3}N_f - \frac{1}{6}N_s\right)\frac{g^3}{16\pi^2} + \mathrm{O}(g^5) = -b\frac{g^3}{16\pi^2} + \mathrm{O}(g^5)\,.$$

The one-loop β-function was first computed for pure Yang–Mills in **[Khriplovich, 69]**, the full expression appeared independently in **[Gross, Wilczek, 73; Politzer, 73]**. In general the β-function receives higher-loop contributions.

Remark

The crucial overall minus sign is responsible for the asymptotic freedom of QCD at high energies, making it viable as a model of strong interactions.

On a curved background

$$g^{\mu\nu}\langle T_{\mu\nu}\rangle = \frac{1}{(4\pi)^2}(cC^2 - aE) + \frac{\beta(g)}{2g}\mathrm{tr}(F^2)\,.$$

$C^2 = C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}$ for $C_{\mu\nu\rho\sigma}$ the Weyl tensor and E is the Euler density **[Capper, Duff, 74]**. The one-loop contributions of n_s complex scalars, n_f Weyl fermions and n_g gluons are

$$a = \frac{1}{180}\left(31n_g + \frac{11}{4}n_f + n_s\right), \qquad c = \frac{1}{60}\left(6n_g + \frac{3}{2}n_f + n_s\right).$$

Just like the β-function, the a and c coefficients receive higher order loop corrections.

Remark

Scale anomalies do not have the same topological origin as the chiral anomalies since the dilation group $\mathbb{R}^{\times}_{>0}$ is topologically trivial.

Field Theories on Twistor Space

Twistor Theory

Twistor theory is a powerful tool for studying self-dual theories. It establishes a correspondence:

solutions of self-dual pdes on particular space-times	$\leftrightarrow$	holomorphic data on a certain complex three-manifolds

These three-manifolds are twistor spaces.

Example

The twistor space of $(\mathbb{R}^4, \delta)$, denoted $\mathbb{PT}$, is the total space of the rank two holomorphic vector bundle

$$\mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{CP}^1 \,.$$

Points of complexified space-time $x \in \mathbb{C}^4$ are degree one rational curves $\mathbb{CP}^1_x \hookrightarrow \mathbb{PT}$, and those on the Euclidean slice are fixed by an anti-holomorphic involution.

Penrose Transform

The Penrose transform identifies

$$\begin{array}{c}\text{solutions to massless helicity } s\\ \text{free field equations on space-time}\end{array} \leftrightarrow \begin{array}{l}\text{cohomology classes}\\ H^1(\mathbb{PT};\mathcal{O}(2s-2))\end{array}$$

[Penrose, 69] To describe quantum fields on twistor space we'll need an off-shell formulation. **[Woodhouse, 85]**

This is achieved by resolving the cohomology $H^1(\mathbb{PT};\mathcal{O}(2s-2))$ as a Dolbeault complex

$$\Omega^{0,\bullet}(\mathbb{PT};\mathcal{O}(2s-2)) = \Omega^0(\mathbb{PT};\mathcal{O}(2s-2)) \xrightarrow{\bar\partial} \Omega^{0,1}(\mathbb{PT},\mathcal{O}(2s-2)) \xrightarrow{\bar\partial} \dots$$

So that off-shell fields of helicity s and $-s$ are represented by

$$\chi \in \Omega^{0,\bullet}(\mathbb{PT};\mathcal{O}(2s-2))\,, \qquad \widetilde{\chi} \in \Omega^{0,\bullet}(\mathbb{PT};\mathcal{O}(-2s-2))\,.$$

These are naturally paired by the action

$$S[\chi,\widetilde{\chi}] = \int_{\mathbb{PT}} \widetilde{\chi}\wedge\bar\partial\chi\,.$$

Ward Transform

Both the Ward transform and the non-linear graviton can be written in the Dolbeault language. The Ward transform identifies

$$\begin{array}{c}\text{self-dual gauge connections } \mathrm{D} \text{ on} \\ \text{a vector bundle } E \text{ over space-time}\end{array} \leftrightarrow \begin{array}{c}\text{holomorphic vector} \\ \text{bundles } V \to \mathbb{PT}\end{array}$$

[Ward, 77] A holomorphic structure on V is determined by a nilpotent Dolbeault operator

$$\bar{\partial}_V : \Gamma(\mathbb{PT}; V) \to \Omega^{1,0}(\mathbb{PT}; V)\,.$$

Almost holomorphic deformations of this structure can be written as

$$\bar{\mathrm{D}} = \bar{\partial}_V + a$$

for $a \in \Omega^{0,1}(\mathbb{PT}; \mathrm{End}\, V)$ a partial connection. They are holomorphic when

$$\bar{\mathrm{D}}^2 = \bar{\partial}_V a + a^2 = 0\,.$$

Non-Linear Graviton

The non-linear graviton identifies

$$\text{self-dual conformal geometries } (M,[g]) \quad \leftrightarrow \quad \text{complex three-manifolds } Z$$

[Penrose, 76] Similarly almost complex structure deformations of Z can be written as

$$\bar{\mathrm{D}} = \bar{\partial}_Z + v$$

for $v \in \Omega^{0,1}(Z, T_Z)$ a Beltrami differential. Consistency of the Cauchy-Riemann equations necessitates

$$\bar{\partial}_Z v + \frac{1}{2}[v,v] = 0\,.$$

By demanding further structure on the complex manifolds Z can recover self-dual vacuum Einstein geometries, etc.

The linear Penrose transform can be adapted to describe coloured massless fields on self-dual gauge and gravitational backgrounds. The cohomology class is upgraded to

$$H^1(\mathbb{PT}; \mathbb{C}^N \otimes \mathcal{O}(2s-2)) \to H^1(Z; V \otimes K_Z^{(1-s)/2})$$

For off-shell twistor fields

$$\chi \in \Omega^{0,\bullet}(Z; V \otimes K_Z^{(1-s)/2})\,, \qquad \widetilde{\chi} \in \Omega^{0,\bullet}(Z; V^* \otimes K_Z^{(1+s)/2})\,,$$

the partial connection a couples to the current

$$j = \widetilde{\chi} \otimes \chi$$

and the Beltrami v couples to the stress tensor

$$t = \frac{1}{2}\langle \widetilde{\chi}, \partial\chi \rangle_V - \frac{1}{2}\langle \partial\widetilde{\chi}, \chi \rangle_V + \frac{s}{2}\partial\langle \widetilde{\chi}, \chi \rangle_V\,.$$

The Index Theorem on Twistor Space

One-Loop QCD Partition Function

The twistor bundle describing positive helicity linearised fluctuations of QCD around a self-dual background is

$$R = (\operatorname{End} V \otimes \mathcal{O}) \oplus \Pi\Big((V \oplus V^*) \otimes K_Z^{1/4}\Big)^{\oplus N_f} \oplus \Big(V \otimes K_Z^{1/2}\Big)^{\oplus N_s}$$

with off-shell fields $\phi \in \Omega^{0,\bullet}(Z;R)$, $\widetilde{\phi} = \Omega^{0,\bullet}(Z;R^* \otimes K_Z)$.

The one-loop partition function of QCD on a self-dual background coincides with the one-loop determinant on twistor space

$$\mathcal{Z}_{\mathsf{QCD}}^{\text{one-loop}} = \det \bar{\mathrm{D}} = \int \mathcal{D}\widetilde{\phi}\,\mathcal{D}\phi\, \exp\left(-\int_Z \langle \widetilde{\phi}, \bar{\mathrm{D}}\phi\rangle_R\right).$$

for $\bar{\mathrm{D}}$ the Dolbeault operator on a non-trivial Ward bundle V over a curved twistor space Z. We would like the understand the anomalous properties of this determinant.

The Index Theorem on Twistor Space

Similar to the Weyl operator, the $\bar{\mathrm{D}}$ operator increases form degree. Its determinant is not a number, but a section of a line bundle $\mathcal{L}$ over the moduli space X of twistor spaces Z and Ward bundles V.

The first Chern class of this line bundle can be computed using the Hirzeburch–Riemann–Roch theorem.

$$\mathrm{c}_1(\mathcal{L}) = \left[\frac{1}{2}\int_{\mathcal{Z}} \mathrm{Td}(\mathcal{T}_{Z/X})\,\mathrm{Ch}\big(\mathcal{R}\oplus(\mathcal{K}_Z\otimes\mathcal{R}^*)\big)\right]_2 .$$

Remark

Note that fields of all helicities can contribute to the anomaly: This includes gluons with $s=\pm 1$ and (in principle) gravitons with $s=\pm 2$.

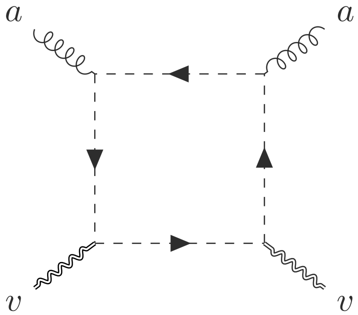
Anomalous Box Diagram

Since twistor space is a complex three-manifold, the integrand is an eight-form.

The Feynman diagrams responsible for this anomalous behaviour are wheels with four vertices.

Remark

$\mathbb{PT}$ is a non-trivial complex manifold. Its Dolbeault cohomology can contribute to the anomaly polynomial.



Space-Time Anomalies from Twistor Space

One-Loop QCD β-Function

We can probe the one-loop QCD β-function by evaluating the scale dependence of $\mathcal{Z}_{\text{QCD}}^{\text{one-loop}}$ on a instanton background with charge k.

To see why, consider the action of the dilation operator Δ

$$\Delta \log \mathcal{Z}_{\text{QCD}}^{\text{one-loop}} = -\int_{\mathbb{R}^4} \mathrm{d}^4x \, \langle T^{\mu}{}_{\mu} \rangle = \frac{b}{2(4\pi)^2} \int_{\mathbb{R}^4} \mathrm{d}^4x \, \mathrm{tr}\big(F(A)^2\big)$$

where we've used the one-loop trace identity for QCD.

But on an instanton background

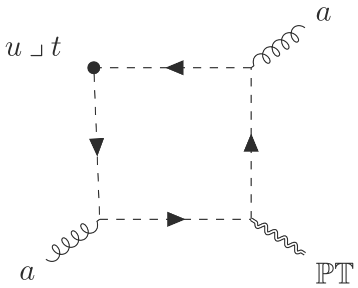
$$\frac{1}{2(4\pi)^2} \int_{\mathbb{R}^4} \mathrm{d}^4x \, \mathrm{tr}\big(F(A)^2\big) = \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \mathrm{tr}(F \wedge F) = -k$$

so that $\Delta \log \mathcal{Z}_{\text{QCD}}^{\text{one-loop}} = -bk$.

We can evaluate this scale dependence using the index theorem on twistor space.

The dilation operator lifts to a holomorphic vector field $u = -\mu^{\dot\alpha}\partial/\partial\mu^{\dot\alpha}$ on $\mathbb{PT}$. The anomalous scale dependence is generated by a box diagram with $u \lrcorner t$ inserted at a vertex.

In practice this is computed using the equivariant index theorem with parameter q keeping track of the dilation action.



Remark

Note that on twistor space dilations complexify to a $\mathbb{C}^\times$ action.

For example, writing $H = \mathrm{c}_1(\mathcal{O}(1))$ for the hyperplane class we find that:

- The Todd class is

$$\mathrm{Td}_{\mathbb{C}^\times}(T_{\mathbb{PT}}) = 1 + 2H + q + \frac{11}{6}qH + \mathcal{O}(q^2)\,.$$

- The contribution of a gluon fluctuations to the Chern character is

$$\frac{1}{2}\mathrm{Ch}_{\mathbb{C}^\times}(\mathrm{End}\,\mathcal{V} \oplus (K_{\mathbb{PT}} \otimes \mathrm{End}\,\mathcal{V})) = 2N_c(1 - 2H - q + 4qH)\mathrm{ch}_2(\mathcal{V}) + \dots\,.$$

So the gluons contribute

$$\mathrm{c}_1(\mathcal{L}) = 2N_c\Big(\frac{11}{6} \times 2 - 4 + 4\Big)q \int_{\mathbb{PT}} H\mathrm{ch}_2(\mathcal{V}) = \frac{11}{3}N_c k q\,.$$

Together with similar calculations for fermions and scalars we get

$$\Delta \log \det \bar{\mathrm{D}} = -\Big(\frac{11}{3}N_c - \frac{2}{3}N_f - \frac{1}{6}N_s\Big)k\,.$$

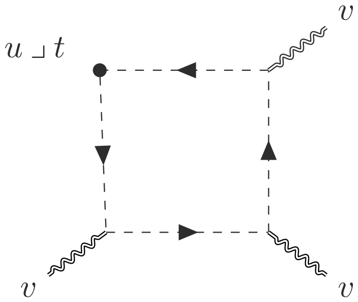
Weyl Anomalies

The same trick can be applied to the Weyl anomalies by investigating the scale dependence of $\mathcal{Z}_{\mathsf{QCD}}^{\text{one-loop}} = \det \bar{\mathrm{D}}$ on the twistor space of a Gibbons–Hawking metric.

The Weyl anomaly predicts

$$\Delta \log \mathcal{Z}_{\mathsf{QCD}}^{\text{one-loop}} = -\frac{1}{8\pi^2}\int_M \mathrm{tr}_{T_M}(C \wedge C)$$

$$= (a-c)p\,.$$



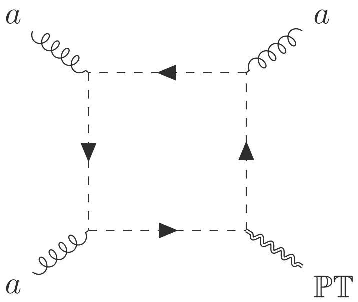
Applying the equivariant index theorem yields the correct coefficients.

Chiral Anomaly

We can also extract the familiar space-time chiral anomaly for Weyl fermions valued in non-self-dual representations.

Three legs couple to the Ward bundle and one to the background complex structure of $\mathbb{PT}$.

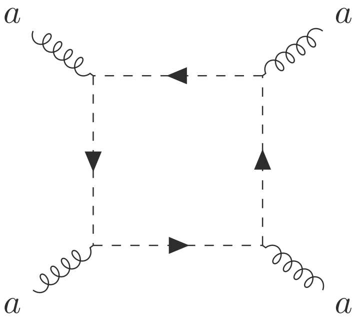
$$\begin{aligned} \mathrm{c}_1(\mathcal{L}) &= \frac{s}{2}\int_{\mathbb{PT}} \mathrm{c}_1(\mathbb{PT})\mathrm{ch}_3(\mathcal{V}) \\ &= 2s\int_{\mathbb{R}^4} \mathrm{ch}_3(\mathcal{E})\,. \end{aligned}$$



Anomalies to Integrability

Twistorial Gauge Anomaly

There are also anomalies on twistor space which do not have such straightforward space-time interpretations.



The first of these was identified by Costello and corresponds to the fourth Chern character $\mathrm{ch}_4(\mathcal{V})$. **[Costello, 21]**

It signals a local failure of invariance under gauge transformations of the Ward bundle, but not under space-time gauge transformations.

More concretely, suppose we try to treat a as a dynamical quantum field on twistor space with the action

$$S_{\mathsf{hBF}}[a,b] = \int_{\mathbb{PT}} \mathrm{tr}_V \Big(b \wedge (\bar\partial a + \frac{1}{2}[a,a]) \Big) \,,$$

for $b \in \Omega^{3,1}(\mathbb{PT}, \mathrm{End}(V))$ a Lagrange multiplier. **[Witten, 01; Mason 05]** The theory is anomalous under gauge transformations $\delta_\epsilon a = \bar\partial\epsilon + [a,\epsilon]$ since

$$\delta_\epsilon \log\det\bar{\mathrm{D}} \propto \int_{\mathbb{PT}} \mathrm{tr}_R(\epsilon\partial a^3) \,.$$

But the corresponding space-time theory, self-dual gauge theory, has no anomaly

$$S_{\mathsf{sdYM}}[A,B] = \frac{1}{2}\int_{\mathbb{R}^4} \mathrm{tr}_E(B \wedge (F(A) - *F(A)) \,.$$

Here $B \in \Omega^2_-(\mathbb{R}^4; \mathrm{End}(E))$ is an anti-self-dual Lagrange multiplier.

Costello interpreted as the anomaly as a quantum failure of integrability in the self-dual gauge theory.

This is supported by the fact that when the anomaly vanishes:

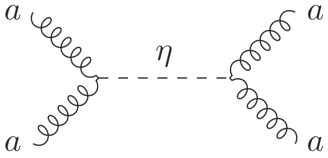
- Vanishing of amplitudes.
- Analyticity of correlation functions.
- Complexification of conformal symmetries.
- Most remarkably the infinite dimensional hidden symmetries of the self-dual Yang-Mills equations admit a non-linear deformation. **[Costello, Paquette, 21; Fernández, Paquette, 25]**

Conversely, when the anomaly is present all fail.

Local Cancellation

There are at least two local methods of cancelling the anomaly:

- Tuning the matter content of the theory so that the box diagram vanishes. For example, it vanishes in any supsersymmetric self-dual gauge theory. But also for $\mathrm{SU}(2)$ with 8 fundamental Diracs and $\mathrm{SU}(3)$ with 9 fundamental Diracs.



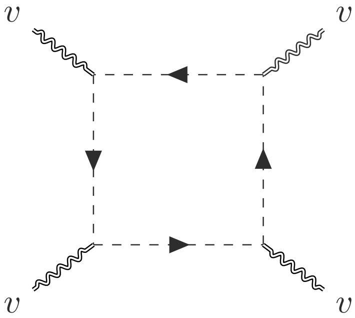
- By a Green-Schwarz mechanism involving a coupling to a ∂-closed two-form field $\eta \in \Omega^{2,\bullet}(\mathbb{PT})$ via

$$\int_{\mathbb{PT}} \eta \wedge \mathrm{tr}(a \wedge \partial a)\,.$$

  This works when then the fourth Chern character factorizes. On space-time η descends to an axion with fourth-order kinetic term. **[Mason, 87]**

Twistorial Gravitational Anomaly

The non-linear graviton suffers from a similar failure of diffeomorphism invariance on twistor space, even when restricted to self-dual Einstein backgrounds. **[RB, Sharma, Skinner, 23]**



It signals an integrability anomaly in self-dual gravity.

It vanishes in supergravity, or more generally whenever counting the degrees of freedom weighted by parity gives zero.

In this case the hidden symmetries of self-dual gravity also admit a non-linear quantum deformation. **[RB, 23]**

Conclusion

Summary

- Have seen that quantum fields on twistor space generically suffer from anomalies.
- Exploited this to provide index theoretic derivations of the one-loop QCD β-function and Weyl anomalies.
- At loop level integrability of self-dual theories can fail. Cancellation leads to a notion of quantum integrability.

Where next?

- Understand higher loop conformal properties of field theories on twistor space.
- Exploit quantum integrability to solve for partition function, correlators as functions on the moduli space.

Thank you all for listening and Happy Birthday Roger!