Using tractors and to solve Killing type equations and yield (super)integrable systems.

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Work with Simon Goodwin, Thomas Leistner and Jonathan Kress,

Links: e.g.

G. + Leistner, . . *prolongation* . . *Killing* . , Ann. Mat. Pura Appl. (2019)

G. + Macbeth, *Detecting Einstein geodesics* . . Diff. Geom. Appl (2014)

G. *Laplacian operators.* . ., Math. Ann. 2006

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Symmetries and hidden symmetries

Recall that on a (psuedo-) Riemannian manifold (M, g) a vector field $k \in \mathfrak{X}$ is an *infintesimal symmetry* or **Killing vector** field if

$$\mathcal{L}_k g = 0.$$

If we fix g, it is an overdetermined linear PDE on $k_a := g_{ab}k^b$:

$$\nabla_a k_b + \nabla_b k_a = 0.$$

A higher rank $k \in \Gamma(\odot^\ell T^*M)$ is a **Killing tensor** (or "hidden symmetry") if

$$symmetric - part(\nabla k) = 0 \quad \text{i.e.} \quad \nabla_{(a_1} k_{a_2 \cdots a_{\ell+1})} = 0$$

Both provide **first integrals** for **geodesics** i.e. curves $\gamma : \mathbb{R} \to M$

$$\nabla_u u = 0, \quad \text{where} \quad u := \dot{\gamma}\,,\ \nabla = \text{Levi-Civita}(g),$$

as

$$k(u, \cdots, u) = k_{a_2 \cdots a_{\ell+1}} u^{a_2} \cdots u^{a_{\ell+1}}$$

is **necessarily constant** along any geodesic: $u \cdot (k(u, \cdots, u)) = 0$.

♡ Why we love Killing tensors ♡

Killing tensors i.e. $k \in \Gamma(\odot^{\ell} T^*M)$ such that

$$\nabla_{(a_1} k_{a_2 \cdots a_{\ell+1})} = 0$$

have many other related applications: If a linear differential operator $L : C^\infty \to C^\infty$ is a **symmetry of** the Laplacian Δ^g, i.e. $[L, \Delta^d] = 0$ (so preserves its spectrum) then its leading symbol is a Killing tensor $L = k^{a_1 \cdots a_\ell} \nabla_{a_1} \cdots \nabla_{a_\ell} + lots$.

Families of Killing tensors are used for the construction of **separation of variables** coordinates (for the Laplacian and other natural operators) and in **superintegrable systems** – where there are enough (compatible) Killing tensors so that the Hamilton-Jacobi equations can be solved algebraically.

Classical mechanics formulation

For a Riemannian manifold (including Euclidean space $\mathbb{E}^n$) the geodesics – the trajectories – are determined by a Hamiltonian of the form

$$\mathcal{H} = g_q(p, p), \qquad q \in M,\ p \in T^*M$$

The resulting Hamilton equations are just the geodesic equations. First integrals are then functions k on phase space s.t.

$$\{\mathcal{H}, k\} = 0, \quad \text{where } \{\cdot, \cdot\} \text{ is Poisson bracket on } T^*M$$

If k takes the form $k = k_q(p, \cdots, p)$, monomial in p at each point q then k_q is a Killing tensor.

E.g. If we have $2n - 1$ such k that are functionally independent – superintegrable – then the unparametrised paths are determined algebraically (and, in suitable settings, bounded orbits are closed).

Potentials

In practice the Hamiltonians studied are usually kinetic plus *potential*

$$\boxed{\mathcal{H}(p,q) = g_q(p,p) + V(q).} \tag{1}$$

E.g. The Kepler system where g is the Euclidean metric and $V = k/r$, where r radial distance. [(1) quantises to a Schrödinger operator $\Delta^g + V$ on functions, Kepler $\rightsquigarrow$ Hydrogen atom]

Problem: First integrals for (1) **? It's no longer in the form** $g_q(p,p)$ **so trajectories are not just geodesics**. One can attack this directly. Or:

Enter conformal geometry: In its simplest form, the **Stäckel transform** of the Hamiltonian (1) is

$$\tilde{\mathcal{H}} = \tilde{g}_q(p,p), \qquad \tilde{g} = V \cdot g \quad \text{conformally related to } g$$

Constants of motion for $\mathcal{H}$ can be mapped to constants of motion for $\tilde{\mathcal{H}}$, in particular, this mapping preserves independence and therefore superintegrability.

Conformal Killing tensors

In terms of Killing tensors **finding constants of motion for $\mathcal{H} = g_q(p,p) + V(q)$ corresponds to finding Killing tensors for g that are also, after a possible trace adjustment, also Killing tensors for the conformally related $\tilde{g} = V \cdot g$.**

Weaker than the Killing equation is the **conformally invariant conformal Killing equation CKE:**

$$\nabla_{(a_1} k_{a_2 \cdots a_{\ell+1})_0} = 0 \quad \text{i.e.} \quad \boldsymbol{trace} - \boldsymbol{free} - symmetric - part(\nabla k) = 0,$$

call the (trace-free symmetric) solution $k_{a_1 \cdots a_\ell} \in \odot_0^\ell T^*M$ a conf. Killing tensor.

Hence the conformal Killing to Killing problems:

Q1: **When is a conformal Killing tensor on (M, g) actually Killing on the conformally related $(M, \widehat{g} = e^{2\omega} g)$?**

For rank 1, i.e. conformal (Killing) vector (fields), this is a classical question and linked to interesting global dynamics questions. A conformal vector field is *essential* if $\nexists\, \omega \in C^\infty$.

For rank $\ell \geq 2$ there is evidently the interesting and natural variant:
Q2: **When is a conformal Killing tensor on (M, g) the trace-free part of a Killing tensor on the conformally related $(M, \widehat{g} = e^{2\omega} g)$?**

Our main focus is Q2 with $\ell = 2$.

Many related questions that boil down to Q2: E.g. our second focus:
Q3: If $k_1, \cdots, k_s$ is a family of Killing tensors on (M, g) – **when can they each be trace adjusted** so as to be **simultaneously Killing** for the conformally related $(M, \widehat{g} = e^{2\omega} g)$?

The conf Killing equation $\nabla_{(a}k_{bc)_0} = 0$ is an overdetermined finite type linear PDE. So we should study it by prolongation. Let's do an easier "fundamental eqn" instead:

The metric $\bar{g} := \sigma^{-2}g$ is **Einstein** i.e. $\mathrm{Ric}^{\bar{g}} = \lambda\bar{g}$ iff $\exists\ \sigma > 0$ s.t.

$$\text{trace-free}(\nabla_a\nabla_b\sigma + P_{ab}\sigma) = 0. \qquad \text{(AE) – almost Einstein eqn}$$

where $\mathrm{Ric} = (n-2)P + gJ$, $J := trace^g(P)$. We drop $\sigma > 0$ and study. Best to replace σ by conformal density of weight 1 $\sigma \in \Gamma(\mathcal{E}[1])$, where $\mathcal{E}[2n] \cong (\Lambda^n(TM))^2$. Then (AE) is conformally invariant. AE is overdetermined so study by prolongation. It is equivalent to the closed first order system:

$$\begin{aligned} \nabla_a\sigma - \mu_a &= 0 \\ \nabla_a\mu_b + P_{ab}\sigma + \rho\boldsymbol{g}_{ab} &= 0 \\ \nabla_a\rho - P_{ab}\mu^b &= 0 \end{aligned}$$

$\boldsymbol{g}$ is the conformal metric, $\otimes^n\boldsymbol{g} : (\Lambda^n(TM))^2 \xrightarrow{\simeq} \mathcal{E}[2n]$.

The tractor connection and D-operator

The above system can be collected into a linear connection – in fact the conformally invariant **tractor bundle** $\mathcal{T} = \mathcal{E}[1] \oplus\!\!\!\!\!+\, T^*M[1] \oplus\!\!\!\!\!+\, \mathcal{E}[-1]$ with invariant $X : \mathcal{E}[-1] \to \mathcal{T}$, and **connection** $\nabla^{\mathcal{T}}$. Given $\overline{g} \in \boldsymbol{c}$ this is given by

$$\mathcal{T} \overset{\overline{g}}{=} \mathcal{E}[1] \oplus T^*M[1] \oplus \mathcal{E}[-1], \qquad \mathcal{E}[1] := (\Lambda^n TM)^{\frac{2}{2n}}$$

$$\nabla_a^{\mathcal{T}}(\sigma, \mu_b, \rho) = (\nabla_a \sigma - \mu_a,\ \nabla \mu_b + P_{ab}\sigma + \boldsymbol{g}_{ab}\rho,\ \nabla_a \rho - P_{ab}\mu^b),$$

and $\nabla^{\mathcal{T}}$ preserves a conformally invariant **tractor metric** h

$$\mathcal{T} \ni V = (\sigma, \mu_b, \rho) \mapsto 2\sigma\rho + \mu_b\mu^b = h(V, V).$$

There is also a 2^{nd} order conformally invariant **Thomas operator** :

$$\Gamma(\mathcal{E}[w]) \ni f \mapsto D_A f \overset{\overline{g}}{=} \begin{pmatrix} wf \\ \nabla_a f \\ -(n + 2w - 2)^{-1}(\Delta f + w\mathrm{J} f) \end{pmatrix}$$

where J is $\mathrm{trace}^{\overline{g}}(P_{ab})$, so a number times $\mathrm{Sc}(\overline{g})$.

The scale tractor

If $I_A \stackrel{g}{=} (\rho, \mu_a, \sigma)$ is a parallel tractor then $\mu_a = \nabla_a \sigma$, and $\rho = -\frac{1}{n+1}(\Delta\sigma + \mathrm{J}\sigma)$. This gives the first statement of:

Proposition

I **parallel** *implies $I_A = D_A\sigma$. So $I \neq 0 \Rightarrow \sigma$ is nonvanishing on an open dense set $M_{\sigma\neq 0}$. On $M_{\sigma\neq 0}$, $\bar{g} = \sigma^{-2}\mathbf{g}$* **is Einstein**. *Conversely if $\bar{g} = \sigma^{-2}\mathbf{g}$ is Einstein then $I := D\sigma$ is parallel.*

Drop parallel. The **scale tractor** $I := D\sigma$:

Now if $k^a \in \Gamma(TM)$ then its half-prolongation

$$K_A := \begin{pmatrix} 0 \\ k^a \\ -\frac{1}{n}\nabla_a k^a \end{pmatrix} \quad \in \Gamma(\mathcal{T})[1]$$

is an invariant tractor and $\boxed{I_A K^A = 0 \text{ iff } \nabla\cdot k = 0.}$ As, in the scale $g = \sigma^{-2}\mathbf{g}$, $I_A = (\rho \;\; 0 \;\; \sigma)$.

Full prolongation of conformal K vectors

Theorem (G. Math. Ann 2006)

$k \in \Gamma(TM)$ is a conformal Killing vector iff

or equivalently $D_B K_C = \mathbb{K}_{BC} := D_{[A}K_{B]}$ *half*

$\nabla_a \mathbb{K}_{BC} = \mathit{Curvature} \cdot \mathbb{K}_{BC}$ *full*

Proof.

First part: Calculate – or think about conformally flat case first.
Second part: **Conf flat case:** $\Rightarrow$ Using first part

$$D_A D_B K_C = D_A \mathbb{K}_{BC} = D_{(A} D_{B)} K_C \doteq 0.$$

$\Leftarrow$: top slot of $\nabla_a \mathbb{K}_{BC} = 0$ is $\nabla_{(a} k_{b)_0} = 0$. □

Corollary: a conformal Killing vector is Killing for an Einstein metric $\sigma^{-2}\boldsymbol{g}$ iff

$$\boxed{I^A \mathbb{K}_{AB} = 0} \qquad I_A = \frac{1}{n} D_A \sigma$$

Purely algebraic condition in c flat case !! Corank($\mathbb{K}$) $\rightsquigarrow$ upper bound on number of Einstein scales . ..

Rank 2 conformal Killing tensors

Lemma: (Half-prolongation)

$$k_{ab} \mapsto K_{AB} = \begin{pmatrix} 0 & 0 & 0 \\ \rho_a & k_{ab} & 0 \\ \rho & \rho_b & 0 \end{pmatrix} \qquad \begin{aligned} \rho_a &= \tfrac{2}{n+2} \nabla^b k_{ab} \\ \rho &= \tfrac{1}{n+1} \left(\tfrac{1}{n+2} \nabla^c \nabla^d k_{cd} + P^{cd} k_{cd} \right) \end{aligned}$$

is an isomorphism between sections of $\mathcal{E}_{(ab)_0}[4]$ and sections of $\mathcal{E}_{(AB)_0}[2]$ that satisfy the equations

$$X^A K_{AB} = 0 \quad \text{and} \quad \mathbb{D}^A K_{AB} = 0. \tag{2}$$

Theorem

$\Gamma(S^2_0 T^* M)[4] \ni k_{ab}$ *is conf Killing iff*

$$D_{(A} K_{BC)} \overset{conf.flat}{=} 0.$$

For a conf. Killing tensor k_{bc}, we say a scale $\sigma \in \Gamma(\mathcal{E}[1])$ is a **strong Killing scale (SKS)** if it is Killing for $g = \sigma^{-2}\boldsymbol{g}$, i.e. $\nabla^b k_{bc} = 0$.

By inspection of the half-prolongation above we have:

Proposition: 1. A scale $\sigma \in \Gamma(\mathcal{E}[1])$ is a SKS iff $I^A K_{AB} = X_B F$ for some $F \in \mathcal{E}[1]$.

2. $\boxed{\sigma \text{ is an Einstein SKS iff } I^A K_{AB} = 0 \text{ and } I_A \text{ is parallel}}$ for the conformal tractor connection.

We say that $\sigma \rightsquigarrow I = D\sigma$ is a **Killing scale (KS)** if k_{bc} is the trace-free part of a Killing tensor for $g = \sigma^{-2}\boldsymbol{g}$.

Proposition: Given a conformal Killing tensor k_{bc}, σ is a KS iff $\exists$ $\lambda \in \Gamma(\mathcal{E}[2])$ s.t.

$$I^A K_{AB} = \tfrac{\sigma}{2} D_B \lambda - \lambda I_B + X_B F \ . \qquad (3)$$

If so $\overline{k}_{bc} = k_{bc} + \lambda g$ is the Killing tensor.

Proof: The display is the same as $\rho_a = \frac{2}{n+2}\nabla^b k_{ab} = -\nabla_a \lambda$ being a gradient $\leftrightsquigarrow$ Bertrand-Darboux $(\star)$

Recall Thm above $\nabla_{(a}k_{bc)_0} \Leftrightarrow D_{(A}K_{BC)} = 0$ if c-flat.
Corollary: [cf e.g. BGG Machinery Čap et al] On a conformally flat manifolds, there is a one-to-one correspondence:

$$\text{conformal Killing tensors } k_{bc} \leftrightarrow \mathbb{K}_{ABCD} \text{ parallel tractors}$$

where the $\mathbb{K}_{ABCD}$ have Weyl tensor symmetries. Say $\mathbb{K} \in \mathbb{W}[0]$
It's the proof here that's important:
Proof: $\Rightarrow$: $D_B D_C K_{DE}$ is symmetric on BC and DE, but $D_{(C}K_{DE)} = 0$ (and $D_B D_C K_{DE}$ is trace-free) whence

$$\mathbb{K}_{BCDE} := D_B D_C K_{DE} \in \overline{\mathbb{W}}[0] = \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array}_{\,0} \qquad \text{alt. Weyl space.}$$

$D_A D_B D_C K_{DE}$ is symmetric on ABC and DE but $D_{(C}K_{DE)} = 0$.
Thus $D_A D_B D_C K_{DE} = 0$ whence

$$\mathbb{K}_{BCDE} \qquad \text{is parallel}$$

$\Leftarrow$: top slot of $\nabla_a D_B D_C K_{DE} = 0$ is $\nabla_{(a}k_{bc)_0} = 0$.

Einstein SKS if conformally flat

From the above we have at once:

Theorem

Let (M, g) be conformally flat and k_{ab} is a conformal Killing tensor with associated K_{AB} and $\mathbb{K}_{ABCD} = D_A D_C K_{BD}$. Then an Einstein SKS σ is equivalent to

$$\boxed{I_A \quad \text{is parallel and satisfies} \quad I^C \mathbb{K}_{ABCD} = 0.}$$

Proof.

$\Rightarrow$: Ein SKS means $I^B K_{BD} = 0$ and I parallel. Latter means $[D, I] = 0$.
$\Leftarrow$ $I^C \mathbb{K}_{ABCD} = 0$ means $X^A X^B I^C \mathbb{K}_{ABCD} = 0 = 2 I^C K_{CD}$. □

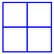
Corollary: On conformally flat spaces, for a conformal Killing tensor k_{ab} the dimension of the space of Einstein SKS is bounded by the dimension of the nullity of K_{ABCD} and is locally equal to this dimension. In particular there are conformal Killing tensors that admit no Einstein SKS.

Trace adjustment and Einstein scales??

How can we incorporate the trace adjustment etc?? First hint:

Theorem: On conformally flat manifolds there is a one-to-one correspondence between:

1. Sections r_{ab} of $\mathcal{E}_{(ab)}$ that are Killing in the Einstein scale $\sigma = X^A I_A$. and

2. Parallel tractors R_{ABCD} with Riemann tensor symmetries  and $I^A R_{ABCD} = 0$.

Proof: When I is parallel then $I^{\perp} \subset \mathcal{T}$ is the projective tractor bundle with connection induced by $\nabla^{\mathcal{T}}$ [G.+Macbeth 2014]. Then from [G.+Leistner 2019] (or BGG theory)

$$r_{ab} \overset{1-1}{\leftrightarrow} R_{ABCD}.$$

A punchline

Theorem

On a conformally flat manifold, let k_{ab} be a conformal Killing tensor with full conformal prolongation $\mathbb{W}_{ABCD}$, where $\mathbb{W}_{ABCD}$ has alg. Weyl tensor symmetries. Then k_{ab} is, after trace adjustment, Killing for the Einstein scale $\sigma = X^A I_A$ if and only if

$$0 = I^A \mathbb{W}_{AB[CD} I_{E]}.$$

Proof.

Case $I^2 \neq 0$ Sketch. $\Leftarrow$: $0 = I^A \mathbb{W}_{AB[CD} I_{E]}$ with I and $\mathbb{W}$ parallel implies that

$R_{ABCD} := I^2 \mathbb{W}_{ABCD} + (h \wedge P)_{ABCD}$, is parallel and orthogonal to I, where

$$P_{BD} := \frac{\mathsf{J}}{n}\left(I^2 h_{BD} - 2 I_B I_D\right) - I^A I^C \mathbb{W}_{ABCD}.$$

Then $R_{ABCD} \leftrightarrow r_{ab}$ Killing for the scale $\sigma = X^A I_A$ by prev slide. $\square$

The incidence relation

On conformally flat simply connected M.

Summary:

- Einstein metrics g correspond to parallel tractors I_A.
- conformal Killing tensors k_{bc} correspond to parallel tractors

$$\mathbb{W} \in \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array}_{\,0} \qquad \mathbb{W} = \mathrm{prol}(k).$$

k_{bc} is the trace-free part of a Killing tensor for g iff

$$\boxed{0 = I^A \mathbb{W}_{AB[CD} I_{E]}.} \qquad (\star)$$

Given $\mathbb{W}_{ABCD}$ we can solve for the variety of Is that solve $(\star)$.

Given I we can solve for the $\mathbb{W}$ that solve $(\star)$.

Can we use this to construct superintegrable systems? Especially new and maximal ?

Yes, yes, and yes but more subtle!

To get maximal superintegrability we need,
Non-Einstein scales, $\tilde{I} = \hat{D}\tilde{\sigma}$.
Then the equation becomes

$$3\tilde{I}^A X^B Z_c^C Z_d^D X^E \mathbb{W}_{AB[CD}\tilde{I}_{E]} = \tilde{\sigma} X^B Z_c^C Z_d^D X^E \mathbb{W}_{AB[CD}\hat{D}_{E]}\tilde{I}^A. \quad (4)$$

In general this is hard! But:

Idea: Examine scales that "lie between" Einstein scales I, J. That is metrics

$$\tilde{g} = f^{-2}(s) g$$

where g is Einstein and $s^{-2}g$ is Einstein. (cf Bertrand)
Then the scale tractor $\tilde{I} = I^f$ satisfies

$$\tilde{I}_A = (f(s) - sf'(s)) I_A + f'(s) J_A \mod X_A$$

E.g. I is a Killing scale $I^A \mathbb{W}_{AB[CD} I_{E]} = 0$ and J any other Einstein scale then (4) reduces to the form

$$X^B Z_c^C Z_d^D X^E \Big(h(s)(J^A R_{AB[CD} I_{E]}) + \ell(s)(J^A R_{AB[CD} J_{E]}) \Big) = 0$$

where R in $I^\perp$ is s.t. $\mathbb{W}$ is the trace-free part of R, and
$h(s) = (3(f(s) - sf'(s))f'(s) + sf(s)f''(s))$ and
$\ell(s) = (3f'(s)^2 - f(s)f''(s))$,

Taking into account also the need for functional independence we have this: For any Einstein scales I, J and any potential $V(s)$ (with s interpolating between the scales) we get $2n-3$ constants from $\mathbb{W}$s (effectively killing vectors) in $I^{\perp} \cap J^{\perp}$, 1 constant from $\mathcal{H}$ itself, and then we need 1 more to have maximal superintergrability! This corresponds to finding a suitable quadratic Killing tensor that is, after trace adjustment, Killing for two scales (and is a new one).

For a given $\mathbb{W}$, if I is a Killing scale, but J (Einstein) is not then $\tilde{g} = f^{-2}(s)g$ can be a Killing scale if, with $h(s) = (3(f(s) - sf'(s))f'(s) + sf(s)f''(s))$ and $\ell(s) = (3f'(s)^2 - f(s)f''(s))$, either

$$(1) \quad \ell(s) = 0 \quad \text{or} \qquad (2) \quad h(s) = s\ell(s) \quad I, J \text{ null cases}$$

E.g. with I Euclidean, the Harmonic oscillator has $g(s) = 0$ and Kepler $h(s) = s\ell(s)$, where $s = -r^2/2$.

The punchline

Key points:

1. It is easy to solve these ODEs.

2. For each solution it's straightforward to find the R which gives the extra constant of motion – in this generality: i.e. I and J Einstein scales and background conformally flat.

3. In fact:

Thm: Given two Einstein scales $\mathcal{H} = g^{-1} + V(s)$ is superintegrable iff one of the ODEs holds $\rightsquigarrow$ exactly two superintegrable $V(s)$, for each pair I, J.

This generalises a classical Theorem of Bertand which states that in Euclidean 3 spaces there are are just two radially symmetric superintegral potentials.

Enough already,

Thank you for Listening.

Abstract

Using tractors and to solve Killing type equations and yield (super)integrable systems.
The Killing equation and its generalisations for Killing tensors are important for the study of symmetry, structure deformations, and for yielding integrable solutions of Hamilton-Jacobi and Laplacian equations. These Killing equations arise in some differential complexes known as BGG complexes. Understanding this is one of the key steps to a new approach to constructing collections of solutions to these equations that we outline in this talk. The interaction of the prolonged system with scale is managed by tractors and yields the desired solutions. These then provide first integrals for equations such as the geodesic equation, other Hamiliton-Jacobi equations and for Schrodinger operators equations. The overall story demonstrates a beautiful interaction of mechanics, geometry, and algebra.
This is joint work with Simon Goodwin, Thomas Leistner, and Jonathan Kress