Quantum Fields on Twistor Space

Roland Bittleston[a]

[a] *Perimeter Institute for Theoretical Physics,*
51 Caroline Street, Waterloo, Ontario, Canada

E-mail: `rbittleston@perimeterinstitute.ca`

Abstract: These lectures introduce quantum field theory on twistor space, with gauge theory as the motivating example. I'll begin by reviewing the Penrose-Ward transformation and how it can be used to recast the self-dual sector of gauge theory as a holomorphic field theory on twistor space. This holomorphic theory is sick at one-loop: it suffers from a gauge anomaly. Cancelling the anomaly yields a quantum integrable theory on space-time. I will then elucidate the Costello-Paquette correspondence, which leverages this integrability to compute gauge theory amplitudes and form factors using chiral algebra techniques.

Contents

1 Lecture I **1**
1.1 The Self-Dual Sector of Gauge Theory 2
1.2 Twistor Space 3
1.3 Twistor Action for Self-Dual Yang-Mills 5

2 Lecture II **6**
2.1 Twistorial Anomalies 6
2.2 Restoring Integrability 7
2.3 Space-Time Interpretation 8

3 Lecture III **9**
3.1 Deforming away from Self-Duality 9
3.2 Chiral Algebras 10
3.3 Quantum Deformation 12

1 Lecture I

➤ **Big picture:** A new notion of quantum integrability in four dimensions. Celestial chiral algebras play the role of quantum groups. Can be used to compute loop amplitudes inaccessible by other methods.

➤ **Goals of these lectures:**

- Introduce twistor space and the Ward correspondence
- Understand one-loop anomaly to integrability
- Derive quantum corrected celestial chiral algebra and compute a loop amplitude

➤ **Main references:**

- arXiv: 2111.08879 (Quantizing Local Holomorphic Field Theories on Twistor Space)
- arXiv: 2201.02595 (Celestial Holography meets Twisted Holography: 4d Amplitudes from Chiral Correlators)
- arXiv: 2204.05301 (On the Associativity of One-Loop Corrections to the Celestial OPE)

1.1 The Self-Dual Sector of Gauge Theory

➤ The dynamical field of non-abelian gauge theory on $\mathbb{R}^4$ is a connection 1-form $A \in \Omega^1(\mathbb{R}^4, \mathfrak{g})$ with field strength

$$F(A) = \mathrm{d}A + \frac{1}{2}[A, A] \in \Omega^2(\mathbb{R}^4, \mathfrak{g})\,. \tag{1.1}$$

➤ In four-dimensions the Hodge star operator maps 2-forms to 2-forms

$$(*F)_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma} \tag{1.2}$$

and $*^2 = 1$. Can decompose $F(A)$ into its $+1$ and -1 eigenspaces

$$F(A) = F_+(A) + F_-(A)\,. \tag{1.3}$$

➤ The self-dual Yang-Mills (sdYM) equations are

$$F_-(A) = \frac{1}{2}\big(F(A) - *F(A)\big) = 0\,. \tag{1.4}$$

These are *integrable*. They imply the full Yang-Mills equations since

$$\mathrm{D} * F(A) = \mathrm{D}F(A) = 0 \tag{1.5}$$

by the Bianchi identity.

➤ Yang-Mills theory on $\mathbb{R}^4$ with the flat metric δ has action

$$S_{\mathrm{YM}}[A] = -\frac{1}{2g^2}\int_{\mathbb{R}^4} \mathrm{d}^4x\,\mathrm{tr}(F_{\mu\nu}(A)F^{\mu\nu}(A)) = \frac{2}{g^2}\int_{\mathbb{R}^4}\mathrm{tr}\big(F_-(A)\wedge F_-(A)\big) + \theta\text{-term}\,. \tag{1.6}$$

➤ Up to a θ term can be rewritten in a chiral first order form

$$\int_{\mathbb{R}^4}\mathrm{tr}(B\wedge F(A)) - \frac{g^2}{8}\int_{\mathbb{R}^4}\mathrm{tr}(B\wedge B) \simeq \frac{2}{g^2}\int_{\mathbb{R}^4}\mathrm{tr}\big(F_-(A)\wedge F_-(A)\big) \tag{1.7}$$

where $B \in \Omega^2_-(\mathbb{R}^4, \mathfrak{g})$ obeys $*B = -B$.

➤ In the limit $g^2 \to 0$ recover sdYM theory. Equations of motion are

$$* F(A) = F(A)\,, \quad \mathrm{d}B + [A, B] = 0\,. \tag{1.8}$$

B represents a linearised negative-helicity gluon propagating freely on the self-dual background determined by A.

➤ The tree amplitudes of this theory vanish for generic kinematics. The one-loop amplitudes are finite. There are no connected higher loop amplitudes on combinatorial grounds.

- Can get to full Yang-Mills in perturbation theory around the self-dual sector by inserting the operator $\mathrm{tr}(B^2)(x)$ multiple times and integrating over its position

$$e^{-S_{\mathrm{YM}}[A]} = \sum_{k=0}^{\infty} \frac{g^{2k}}{k!2^{3k}} \int_{\mathbb{R}^{4k}} \mathrm{d}^4x_1 \dots \mathrm{d}^4x_k \, \mathrm{tr}(B^2)(x_1) \dots \mathrm{tr}(B^2)(x_k) e^{-S_{\mathrm{sdYM}}[A,B]} \,. \quad (1.9)$$

  The simplest non-vanishing trees appear at $k = 1$; these are the famous MHV amplitudes.

- We want understand sdYM at the quantum level with (at least) one insertion of $\mathrm{tr}(B^2)$. Currently not clear why this is any better than ordinary perturbation theory.

1.2 Twistor Space

- Choose a complex structure on $\mathbb{R}^4$ compatible with the metric δ and orientation d^4x. Let $(u^{\dot{0}}, u^{\dot{1}}) \in \mathbb{C}^2$ be holomorphic co-ordinates, and $(\bar{u}^0, \bar{u}^1)$ their complex conjugates.

- In these co-ordinates the anti-self-dual (asd) 2-forms are $\mathrm{d}u^{\dot{0}} \wedge \mathrm{d}u^{\dot{1}}$, $\mathrm{d}\bar{u}^0 \wedge \mathrm{d}\bar{u}^1$ and the Kähler form $\omega = \frac{\mathrm{i}}{2}(\mathrm{d}u^{\dot{0}} \wedge \mathrm{d}\bar{u}^0 + \mathrm{d}u^{\dot{1}} \wedge \mathrm{d}\bar{u}^1)$. The sdYM equations become

$$\begin{aligned}
&F^{2,0}(A) = 0 \implies F_{u^{\dot{0}}u^{\dot{1}}}(A) = \partial_{u^{\dot{0}}} A_{u^{\dot{1}}} - \partial_{u^{\dot{1}}} A_{u^{\dot{0}}} + [A_{u^{\dot{0}}}, A_{u^{\dot{1}}}] = 0 \,, \\
&F^{0,2}(A) = 0 \implies F_{\bar{u}^0\bar{u}^1}(A) = 0 \,, \\
&\omega \wedge F^{1,1}(A) = 0 \implies F_{u^{\dot{0}}\bar{u}^0}(A) + F_{u^{\dot{1}}\bar{u}^1}(A) = 0 \,.
\end{aligned} \quad (1.10)$$

- These equations hold iff the operators

$$\bar{D}_0 = \partial_{\bar{u}^0} - z\partial_{u^{\dot{1}}} + A_{\bar{u}^0} - zA_{u^{\dot{1}}} \,, \quad \bar{D}_1 = \partial_{\bar{u}^1} + z\partial_{u^{\dot{0}}} + A_{\bar{u}^1} + zA_{u^{\dot{0}}} \quad (1.11)$$

  commute ($[\bar{D}_0, \bar{D}_1] = 0$) for all z. This is a Lax pair for the sdYM equations - a hallmark of integrability. Notice that $[\bar{D}_0, \partial_{\bar{z}}] = [\bar{D}_1, \partial_{\bar{z}}] = 0$ also.

- Now suppose we incorporate z into the geometry, that is, work on $\mathbb{R}^4 \times \mathbb{CP}^1$. We will show that $\bar{D}_0, \bar{D}_1, \partial_{\bar{z}}$ determine a holomorphic vector bundle on this space in an appropriate complex structure.

- First step is to give $\mathbb{R}^4 \times \mathbb{CP}^1$ a complex structure. We interpret $\bar{\partial}_0 = \partial_{\bar{u}^0} - z\partial_{u^{\dot{1}}}, \bar{\partial}_1 = \partial_{\bar{u}^1} + z\partial_{u^{\dot{0}}}$ and $\partial_{\bar{z}}$ as Cauchy-Riemann operators on $\mathbb{R}^4 \times \mathbb{CP}^1$ (in the patch $z \neq \infty$). Holomorphic co-ordinates are

$$v^{\dot{0}} = u^{\dot{0}} - z\bar{u}^1 \,, \quad v^{\dot{1}} = u^{\dot{1}} + z\bar{u}^0 \,, \quad z \,. \quad (1.12)$$

- In the other patch $\tilde{z} = 1/z$ we use Cauchy-Riemann operators

$$\tilde{\partial}_{\dot{0}} = \partial_{u^{\dot{0}}} + \tilde{z}\partial_{\bar{u}^1} \,, \quad \tilde{\partial}_{\dot{1}} = \partial_{u^{\dot{1}}} - \tilde{z}\partial_{\bar{u}^0} \,, \quad \partial_{\bar{\tilde{z}}} \quad (1.13)$$

  for which the holomorphic co-ordinates are

$$\tilde{v}^{\dot{0}} = -\bar{u}^1 + \tilde{z}u^{\dot{0}} = \frac{v^{\dot{0}}}{z} \,, \quad \tilde{v}^{\dot{1}} = \bar{u}^0 + \tilde{z}u^{\dot{1}} = \frac{v^{\dot{1}}}{z} \,. \quad (1.14)$$

Meromorphic functions of z with poles of at worst order m at $z = \infty$ are holomorphic sections of the line bundle $\mathcal{O}(m)$. So with this complex structure $\mathbb{R}^4 \times \mathbb{CP}^1$ is

$$\mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{CP}^1 \tag{1.15}$$

as a complex manifold. This is twistor space, denoted $\mathbb{PT}$.

➤ There are a number of ways to define a holomorphic vector bundle on a complex manifold X:

- A smooth, complex vector bundle on X whose transition functions are holomorphic maps into $\mathrm{GL}_n(\mathbb{C})$.
- A smooth, complex vector bundle with a $(0,1)$-form connection $a \in \Omega^{0,1}(X, \mathfrak{gl}_n(\mathbb{C}))$ such that
$$f^{0,2}(a) = \bar{\partial} a + a \wedge a = 0\,. \tag{1.16}$$

➤ These are equivalent: equation (1.16) implies that in a sufficiently nice patch it's possible to find a gauge in which $a = 0$. When we go from a patch $\mathcal{U}$ to a patch $\mathcal{V}$ with the transition map $h : U \cap V \to \mathrm{GL}_n(\mathbb{C})$ we have

$$h(\bar{\partial}_{\mathcal{U}} + a_{\mathcal{U}})h^{-1} = (\bar{\partial}_{\mathcal{V}} + a_{\mathcal{V}})\,. \tag{1.17}$$

Choosing patches and gauges so that $a_{\mathcal{U}} = a_{\mathcal{V}} = 0$ we learn that h is holomorphic.

➤ We will use the second definition. To define a locally it's enough to supply differential operators

$$\begin{aligned} \bar{D}_0 &= \bar{\partial}_0 + a_0 = \bar{\partial}_0 + \bar{\partial}_0 \lrcorner\, a\,, \\ \bar{D}_1 &= \bar{\partial}_1 + a_1 = \bar{\partial}_1 + \bar{\partial}_1 \lrcorner\, a\,, \\ \bar{D}_{\bar{z}} &= \partial_{\bar{z}} + a_{\bar{z}} = \partial_{\bar{z}} + \partial_{\bar{z}} \lrcorner\, a\,. \end{aligned} \tag{1.18}$$

Equation (1.16) holds if these differential operators commute with one another.

➤ We've seen that the sdYM equations supply suitable candidates with $a_{\bar{z}} = 0$. Can interpret this as a gauge condition, but necessary to assume this gauge exists. This yields the Ward correspondence:

$$\text{sdYM connections on } \mathbb{R}^4 \leftrightarrow \text{holomorphic vector bundles}^* \text{ on } \mathbb{PT}\,. \tag{1.19}$$

*trivialisable on twistor lines.

➤ The sdYM equations depend on the conformal structure of space-time. Somehow this must be encoded in the complex structure of twistor space.

➤ A point $x = (u^{\dot{0}}, u^{\dot{1}}) \in \mathbb{R}^4$ determines a complex line $(v^{\dot{0}}, v^{\dot{1}}) = (u^{\dot{0}} - z\bar{u}^1, u^{\dot{1}} + z\bar{u}^0)$ in twistor space. But in fact, there are lines corresponding to points in complexified space-time, i.e., can take $(\bar{u}^0, \bar{u}^1) \to (\tilde{u}^0, \tilde{u}^1)$ to be independent of $u^{\dot{0}}, u^{\dot{1}}$. These lines intersect when the corresponding points in $\mathbb{C}^4$ are null separated.

- Indeed the line $(v^{\dot{0}}, v^{\dot{1}}) = (u^{\dot{0}} - z\tilde{u}^1, u^{\dot{1}} + z\tilde{u}^0)$ intersects the line corresponding to the origin $(v^{\dot{0}}, v^{\dot{1}}) = (0,0)$ if there's a $z \in \mathbb{CP}^1$ for which

$$u^{\dot{0}} = z\tilde{u}^1 \,, \quad u^{\dot{1}} = -z\tilde{u}^0 \,. \tag{1.20}$$

This only happens if

$$u^{\dot{0}}\tilde{u}^0 + u^{\dot{1}}\tilde{u}^1 = 0 \,, \tag{1.21}$$

i.e., $(u^{\dot{0}}, u^{\dot{1}}, \tilde{u}^0, \tilde{u}^1) \in \mathbb{C}^4$ is null separated from the origin.

1.3 Twistor Action for Self-Dual Yang-Mills

- We can write down an action imposing (1.16) as an equation of motion by introducing a Lagrangian multiplier $b \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O}(-4) \otimes \mathfrak{g})$

$$S_{\mathrm{hBF}}[a,b] = \int_{\mathbb{PT}} \mathrm{d}z \wedge \mathrm{d}v^{\dot{0}} \wedge \mathrm{d}v^{\dot{1}} \wedge \mathrm{tr}\big(b \wedge \bar{\partial}a + b \wedge a \wedge a\big) \,. \tag{1.22}$$

The notation $\mathcal{O}(-4)$ indicates that b has a zero of order 4 at $z = \infty$. This compensates the fourth order pole at $z = \infty$ in

$$\mathrm{d}z \wedge \mathrm{d}v^{\dot{0}} \wedge \mathrm{d}v^{\dot{1}} = -\frac{\mathrm{d}\tilde{z} \wedge \mathrm{d}\tilde{v}^{\dot{0}} \wedge \mathrm{d}\tilde{v}^{\dot{1}}}{\tilde{z}^4} \,. \tag{1.23}$$

- This is classically equivalent to sdYM theory on space-time:

  - First gauge fix $a_{\bar{z}} = 0$.
  - Then integrate out the components of b in the $\mathbb{R}^4$ directions to learn that $[\bar{D}_0, \partial_{\bar{z}}] = [\bar{D}_1, \partial_{\bar{z}}] = 0$. These are solved by

  $$a_0 = A_{\bar{u}^0}(x) - zA_{u^{\dot{1}}}(x) \,, \quad a_1 = A_{\bar{u}^1}(x) + zA_{u^{\dot{0}}}(x) \,. \tag{1.24}$$

  The action becomes

  $$\int_{\mathbb{R}^4} \mathrm{d}^4x \int_{\mathbb{CP}^1_x} \mathrm{d}z \wedge \mathrm{d}\bar{z}\, b_{\bar{z}} \big(F_{\bar{u}^0\bar{u}^1}(A) + zF_{\bar{u}^0 u^{\dot{0}}}(A) + zF_{\bar{u}^1 u^{\dot{1}}}(A) + z^2 F_{u^{\dot{0}}u^{\dot{1}}}(A)\big) \,. \tag{1.25}$$

  - Upon making the identifications

  $$B_{u^{\dot{0}}u^{\dot{1}}} = \int_x \mathrm{d}z\, b \,, \quad B_{\bar{u}^0 u^{\dot{0}}} = B_{\bar{u}^1 u^{\dot{1}}} = \int_x \mathrm{d}z\, zb \,, \quad B_{\bar{u}^0\bar{u}^1} = \int_x \mathrm{d}z\, z^2 b \tag{1.26}$$

  we recover $S_{\mathrm{sdYM}}[A,B]$. (Here $\int_{\mathbb{CP}^1_x} = \int_x$.) This final step tells us how b is related to B in the gauge $a_{\bar{z}} = 0$. It is the linear *Penrose transform* for a field of spin $s = 1$ and helicity $h = -1$.

2 Lecture II

2.1 Twistorial Anomalies

➤ Last lecture wrote down a twistor action for sdYM. This lecture we will find that it does not exist at the quantum level; it suffers from a gauge anomaly.

➤ Recall that gauge anomalies arise when we have a gauge invariant action, but there's no regularization which retains this symmetry at loop level. Usually associated with chiral fermions.

➤ Since the kinetic term on twistor space is $\bar{\partial}$, all fields are chiral and contribute to anomalies. It's perhaps strange that a field can contribute to its own anomaly - this is because we should define QFT with a cutoff and require gauge invariance of the effective action.

➤ Recall the twistor action

$$\int_{\mathbb{PT}} \mathrm{d}z \mathrm{d}v^{\dot{0}} \mathrm{d}v^{\dot{1}} \mathrm{tr}(b \wedge \bar{\partial} a + b \wedge a \wedge a)\,. \tag{2.1}$$

This has a gauge symmetry with parameter $\epsilon \in \Omega^0(\mathbb{PT}, \mathfrak{g})$

$$\delta a = \bar{\partial}\epsilon + [a, \epsilon]\,, \quad \delta b = [b, \epsilon] \tag{2.2}$$

➤ The symmetry with parameter ϵ is anomalous. There are two ways of seeing this:

– direct Feynman diagram computation,

– index theory.

In these lectures I will concentrate on the former.

➤ For simplicity let's work in the patch $z \neq \infty$ which looks like $\mathbb{C}^3$. Since anomalies are local, this will also be an anomaly on $\mathbb{PT}$.

➤ Then the gauge variation of the box diagram is anomalous. To evaluate this we follow the following steps:

– Pick the metric $|z|^2 + |v^{\dot{0}}|^2 + |v^{\dot{1}}|^2$ and fix Lorenz gauge

$$\partial_z a_{\bar{z}} + \partial_{v^{\dot{0}}} a_{\bar{v}^0} + \partial_{v^{\dot{1}}} a_{\bar{v}^1} = 0\,. \tag{2.3}$$

Propagator is a Bochner-Martinelli kernel

$$\langle a \wedge b \rangle_0 \propto \frac{\bar{z}\mathrm{d}\bar{v}^0 \mathrm{d}\bar{v}^1 + \bar{v}^0 \mathrm{d}\bar{v}^1 \mathrm{d}\bar{z} - \bar{v}^1 \mathrm{d}\bar{v}^0 \mathrm{d}\bar{z}}{(|z|^2 + |v^{\dot{0}}|^2 + |v^{\dot{1}}|^2)^2}\,. \tag{2.4}$$

– Employ a heat kernel regularisation with length scales $0 < l \ll L$.

– Evaluate integral in limit $l \to 0$ (which defines the theory at scale L) followed by $L \to 0$ (the UV limit).

- The result is

$$\int_{\mathbb{C}^3} \mathrm{tr}_{\mathrm{Ad}}(\epsilon \partial a \wedge \partial a \wedge \partial a) \tag{2.5}$$

where $\partial = \mathrm{d}z\partial_z + \mathrm{d}v^{\dot{0}}\partial_{v^{\dot{0}}} + \mathrm{d}v^{\dot{1}}\partial_{v^{\dot{1}}}$.

- In the index theory approach, we interpret the one-loop partition function around some background a_0 as a section of a line bundle over the moduli space of holomorphic vector bundles. For the partition function to be a number this bundle would need to be trivial, but we can check that it has non-vanishing first Chern class. (This essentially reproduces the above formula).

2.2 Restoring Integrability

- In order to restore integrability we need to cancel the anomaly. There are a few ways to do this:
  - couple to appropriate Grassmann odd fields,
  - Green-Schwarz mechanism,
  - couple to an infinite tower of higher spin fields,
  - add a non-local term on twistor space.

- In the first case want fields to obey spin-statistics on space-time. Can be achieved using the linear Penrose transform Weyl fermions in the representation R

$$\begin{aligned} &s = 1/2\,, \quad h = +1 \text{ represented by } H^1(\mathbb{PT}, \mathcal{O}(-1) \otimes R)\,, \\ &s = 1/2\,, \quad h = +1 \text{ represented by } H^1(\mathbb{PT}, \mathcal{O}(-3) \otimes R^*)\,. \end{aligned} \tag{2.6}$$

- In the $h = +1$ case can use a Dolbeault representative

$$\chi \in \Pi\Omega^{0,1}(\mathbb{PT}; \mathcal{O}(-1) \otimes R)\,. \tag{2.7}$$

The corresponding left-handed space-time fermion is

$$\Psi_{\dot{\alpha}} = \int_x \mathrm{d}z \, \frac{\partial}{\partial v^{\dot{\alpha}}} \chi\,. \tag{2.8}$$

- In the $h = -1$ case can use a Dolbeault representative

$$\tilde{\chi} \in \Pi\Omega^{0,1}(\mathbb{PT}; \mathcal{O}(-3) \otimes R^*)\,. \tag{2.9}$$

The corresponding right-handed space-time fermion is

$$\tilde{\Psi}^{\alpha} = \int_x \mathrm{d}z \begin{pmatrix} 1 \\ z \end{pmatrix}^{\alpha} \tilde{\chi}\,. \tag{2.10}$$

- Can write down a twistor action reproducing the usual action for Weyl fermions on space-time

$$\int_{\mathbb{PT}} \mathrm{d}z\mathrm{d}v^{\dot{0}}\mathrm{d}v^{\dot{1}} \, \tilde{\chi}_i(\bar{\partial} + a)^i{}_j \chi^j = \int_{\mathbb{R}^4} \mathrm{d}^4x \, \tilde{\Psi}^{\alpha}_i \sigma^{\mu}_{\dot{\alpha}\alpha}(\partial_\mu + A_\mu)^i{}_j \Psi^{j\dot{\alpha}}\,. \tag{2.11}$$

- ➤ Contribute to the twistorial anomaly with opposite sign and in representation R, so cocycle is modified to

$$\int_{\mathbb{PT}} \mathrm{tr}_{\mathrm{Ad}\oplus\Pi R}(\epsilon\partial a \wedge \partial a \wedge \partial a) = \int_{\mathbb{PT}} \mathrm{tr}_{\mathrm{Ad}}(\epsilon\partial a \wedge \partial a \wedge \partial a) - \mathrm{tr}_{\mathrm{R}}(\epsilon\partial a \wedge \partial a \wedge \partial a)\,. \quad (2.12)$$

- ➤ Clearly vanishes when R = Ad, corresponding to $\mathcal{N} = 1$ SUSY, but there are other possibilities. Vanishing is equivalent to the trace identity $\mathrm{tr}_{\mathrm{Ad}}(X^4) = \mathrm{tr}_{\mathrm{R}}(X^4)$.

- ➤ Consider, e.g., $G = \mathrm{SL}_2(\mathbb{C})$ and look for $R = \mathrm{F}^{\oplus N_f} \oplus (\mathrm{F}^*)^{\oplus N_f}$. This is the self-dual sector of SU_2 gauge theory with N_f fundamental Diracs. Can easily check that

$$\mathrm{tr}_{\mathrm{Ad}}(X^4) = 16\mathrm{tr}_{\mathrm{F}}(X^4) \quad (2.13)$$

so $N_f = 8$ will do. There exist many other examples.

- ➤ Might worry that we miss something by working on $\mathbb{C}^3$ rather than $\mathbb{PT}$. Indeed we do: the usual chiral gauge anomaly on space-time comes from a mixed anomaly with the background complex structure on $\mathbb{PT}$. For $G = \mathrm{SL}_n(\mathbb{C})$ with $n \geq 3$ this can be evaded using Dirac fermions.

2.3 Space-Time Interpretation

- ➤ Twistorial anomalies do not represent gauge anomalies on space-time. What do they tell us?

- ➤ To understand these need to think about amplitudes. Positive helicity gluon scattering states in sdYM can be represented by

$$A_{u^{\dot\alpha}} = \mathrm{i}t_{\mathsf{a}}\frac{\tilde\kappa_{\dot\alpha}}{w}e^{\mathrm{i}x\cdot p}\,, \quad A_{\bar u^{\alpha}} = 0 \quad (2.14)$$

where $p_\mu\sigma^{\mu\dot\alpha\alpha} = \tilde\kappa^{\dot\alpha}(1,w)^\alpha$ is a complexified null momentum. In particular

$$x\cdot p = (u^{\dot 0} - w\bar u^1)\tilde\kappa_{\dot 0} + (u^{\dot 1} + w\bar u^0)\tilde\kappa_{\dot 1}\,. \quad (2.15)$$

Negative helicity states are represented by

$$B_{u^{\dot 0}u^{\dot 1}} = e^{\mathrm{i}x\cdot p}\,, \quad B_{u^{\dot 0}\bar u^0} = B_{u^{\dot 1}\bar u^1} = we^{\mathrm{i}x\cdot p}\,, \quad B_{\bar u^0\bar u^1} = w^2e^{\mathrm{i}x\cdot p}\,. \quad (2.16)$$

- ➤ These lift to twistor representatives localised at points on the $\mathbb{CP}^1$ base, in particular

$$a = t_{\mathsf{a}}\delta^{(2)}(z-w)e^{\mathrm{i}v^{\dot\alpha}\tilde\kappa_{\dot\alpha}}\mathrm{d}\bar z\,, \quad b = t_{\mathsf{a}}\delta^{(2)}(z-w)e^{\mathrm{i}v^{\dot\alpha}\tilde\kappa_{\dot\alpha}}\mathrm{d}\bar z\,, \quad (2.17)$$

and similarly for b.

- ➤ Holomorphic BF theory on twistor space does not know about the metric, so we can choose to evaluate Feynman diagrams in whichever gauge we like. A natural choice is

$$\delta + r^2 g_{\mathbb{CP}^1} \quad (2.18)$$

where δ is the flat metric on $\mathbb{R}^4$ and $g_{\mathbb{CP}^1}$ is the Fubini-Study metric on $\mathbb{CP}^1$. Scaling up r we can make the propagation in the $\mathbb{CP}^1$ direction arbitrarily difficult. This means that states supported at different values of z can cannot talk to one another, and amplitudes vanish for generic kinematics.

➤ But although the trees vanish in sdYM, the loops do not. As a result, sdYM theory cannot arise as QFT on twistor space. This is the four-dimensional interpretation of the anomaly: the non-vanishing loop amplitudes.

➤ When the twistorial anomaly vanishes other nice properties hold:

 – Conformal symmetries complexify, e.g., protects operators from acquiring anomalous dimensions.
 – Correlation functions of local operators are analytic functions of position with poles on the complexified light cone.
 – Chiral algebra bootstrap, subject of next lecture.

3 Lecture III

3.1 Deforming away from Self-Duality

➤ Have seen that to deform one step away from the self-dual sector we need to add the integral of $\mathrm{tr}(B^2)$. This will give us access to two-minus tree, one-minus one-loop and all-plus two-loop amplitudes.

➤ Inserting the operator $\mathrm{tr}(B^2)$ at some point x breaks translation invariance, but integrating over position restores it. In an amplitude the integral over x generates the momentum conserving δ-function. It's therefore enough to evaluate amplitudes in the presence of the operator $\mathrm{tr}(B^2)(0)$.

➤ We'd like to uplift $\mathrm{tr}(B^2)(0)$ to twistor space. In the trivial gauge background (in the gauge $a_{\bar{z}} = 0$

$$\mathrm{tr}(B^2)(x) \simeq \int_{x=0} \int_{x=0} \mathrm{d}z\mathrm{d}z'\,(z-z')^2 \mathrm{tr}\big(b \wedge b'\big)\,. \tag{3.1}$$

But our scattering states are not in this gauge - we need a gauge invariant expressions.

➤ This can be achieved by gluing the two copies of b together with a frame field $g(z, z')$ obeying

$$(\partial_{\bar{z}} + a_{\bar{z}})g(z, z') = 0\,, \quad g(z', z') = \mathrm{id.}\,. \tag{3.2}$$

We can solve this explicitly to get

$$g(z, z') = \sum_{m=0}^{\infty} \int_{x=0} \cdots \int_{x=0} \frac{\mathrm{d}z_1\mathrm{d}z_2 \ldots \mathrm{d}z_m}{(z-z_1)(z_1-z_2)\ldots(z_m-z')} a_1 \wedge a_2 \cdots \wedge a_m\,. \tag{3.3}$$

➤ Then

$$\operatorname{tr}(B^2) = \sum_{m,n\geq 1} \int_{x=0} \cdots \int_{x=0} \frac{\mathrm{d}z_1\mathrm{d}z_2 \ldots \mathrm{d}z_{m+n}}{z_{12}z_{23}\ldots z_{(m+n)1}} z_{1(m+1)}^4 \operatorname{tr}(b_1 \wedge a_2 \ldots a_m \wedge b_{m+1} \wedge a_{m+2} \ldots a_{m+n}) . \tag{3.4}$$

At this stage can plug in scattering states and recover tree MHV amplitude; however, we will instead reinterpret this formula as the correlator of some chiral CFT coupled to holomorphic BF theory.

3.2 Chiral Algebras

➤ A chiral algebra consists of holomorphic operators $\mathcal{O}_i(z)$ of conformal dimensions Δ_i together with a singular OPE

$$\mathcal{O}_i(z_1)\mathcal{O}_j(z_2) \sim \sum_{k\,:\,\Delta_i+\Delta_j>\Delta_k} \frac{C_{ij}{}^k}{z_{12}^{2(\Delta_i+\Delta_j-\Delta_k)}} \mathcal{O}_k(z_2) . \tag{3.5}$$

➤ Would like to view $\operatorname{tr}(B^2)$ as the correlator of some chiral CFT coupling to a, b. Suppose a^{c} couples to $J_{\mathsf{c}}(z)$ and b^{d} to $\tilde{J}_{\mathsf{d}}(z)$

$$I[a,b] = \int_{x=0} \mathrm{d}z \left(a^{\mathsf{c}} J_{\mathsf{c}}(z) + b^{\mathsf{d}} \tilde{J}_{\mathsf{d}}(z) \right) . \tag{3.6}$$

From this formula can read off $\Delta(J_{\mathsf{c}}(z)) = 1$, $\Delta(\tilde{J}_{\mathsf{d}}(z)) = -1$.

➤ The OPEs of $J, \tilde{J}$ can be determined by requiring gauge invariance of the coupling. The variation under $\delta a = \bar{\partial}\epsilon + [a, \epsilon]$ is

$$\delta I = \int_{x=0} \mathrm{d}z \left(\bar{\partial}\epsilon^{\mathsf{c}} J_{\mathsf{c}}(z) + f_{\mathsf{ab}}{}^{\mathsf{c}} a^{\mathsf{a}} \epsilon^{\mathsf{b}} J_{\mathsf{c}}(z) \right) \tag{3.7}$$

First term vanishes by holomorphicity of J_{c}. Second must cancel against linearised variation of the bilocal term

$$\delta(I^2) = \int_{x=0} \int_{x=0} \mathrm{d}z_1 \mathrm{d}z_2 \left(\bar{\partial}\epsilon^{\mathsf{c}}(z_1) a^{\mathsf{d}}(z_2) J_{\mathsf{c}}(z_1) J_{\mathsf{d}}(z_2) + a^{\mathsf{c}}(z_1) \bar{\partial}\epsilon^{\mathsf{d}}(z_2) J_{\mathsf{c}}(z_1) J_{\mathsf{d}}(z_2) \right) . \tag{3.8}$$

Now $J_{\mathsf{c}}(z_1)J_{\mathsf{d}}(z_2)$ may have a pole as $z_1 \to z_2$, so integrating by parts in first term can generate a local contribution. This cancels equation (3.7) if

$$J_{\mathsf{a}}(z_1) J_{\mathsf{b}}(z_2) \sim \frac{f_{\mathsf{ab}}{}^{\mathsf{c}}}{z_{12}} J_{\mathsf{c}}(z_2) . \tag{3.9}$$

This is Kac-Moody at level zero. If we'd kept b would also learn that

$$J_{\mathsf{a}}(z_1) \tilde{J}_{\mathsf{b}}(z_2) \sim \frac{f_{\mathsf{ab}}{}^{\mathsf{c}}}{z_{12}} \tilde{J}_{\mathsf{c}}(z_2) , \quad \tilde{J}_{\mathsf{a}}(z_1) \tilde{J}_{\mathsf{b}}(z_2) \sim 0 . \tag{3.10}$$

➤ To recover $\mathrm{tr}(B^2)$ need to define correlation functions compatible with these OPEs. Enough to fix the two-point function

$$\langle \tilde{J}_{\mathsf{a}}(z_1)\tilde{J}_{\mathsf{b}}(z_2)\rangle = \kappa_{\mathsf{ab}} z_{12}^2\,, \tag{3.11}$$

and require that insertions of three or more $\tilde{J}$s give zero. The remaining correlators are determined by the operator product and conformal dimensions

$$\begin{aligned}&\langle J_{\mathsf{a}_1}(z_1)\dots J_{\mathsf{a}_{n-2}}(z_{n-2})\tilde{J}_{\mathsf{a}_{n-1}}(z_{n-1})\tilde{J}_{\mathsf{a}_n}(z_n)\rangle\\ &=\sum_{\sigma\in S_{n-1}}\frac{z_{(n-1)n}^4}{z_{\sigma(1)\sigma(2)}z_{\sigma(2)\sigma(3)}\dots z_{\sigma(n-1)n}z_{n\sigma(1)}}\mathrm{tr}(t_{\mathsf{a}_{\sigma(1)}}\dots t_{\mathsf{a}_{\sigma(n-1)}}t_{\mathsf{a}_n})\,.\end{aligned} \tag{3.12}$$

The proof is by induction. Using this formula we find that $\mathrm{tr}(B^2)(x) = \langle \exp(-I[a,b])\rangle$.

➤ This is not the only correlator compatible with the OPEs, for example, could require that the two-point function vanishes but

$$\langle \tilde{J}_{\mathsf{a}}(z_1)\tilde{J}_{\mathsf{b}}(z_2)\tilde{J}_{\mathsf{c}}(z_3)\rangle_{\mathrm{tr}(B^3)} = f_{\mathsf{abc}} z_{12}z_{23}z_{31}\,. \tag{3.13}$$

Insertions of four or more $\tilde{J}$s are required to give zero. This correlator generates the operator $\mathrm{tr}(B^3)$. A family of correlators compatible with the OPE is known as a *conformal block.*

➤ But this is not the most general possible coupling we could have considered between holomorphic BF theory and a chiral CFT. Could also couple to holomorphic derivatives $\partial^m_{v^{\dot{0}}}\partial^n_{v^{\dot{1}}}a$ and $\partial^m_{v^{\dot{0}}}\partial^n_{v^{\dot{1}}}b$ with operators $J_{\mathsf{a}}[m,n](z)$ and $\tilde{J}_{\mathsf{b}}[m,n](z)$ of conformal dimensions $1-(m+n)/2$ and $-1-(m+n)/2$.

Gauge invariance of the bulk-defect coupling yields the S-algebra

$$\begin{aligned}&J_{\mathsf{a}}[p,q](z_1)J_{\mathsf{b}}[r,s](z_2)\sim\frac{f_{\mathsf{ab}}{}^{\mathsf{c}}}{z_{12}}J_{\mathsf{c}}[p+r,q+s](z_2)\,,\\ &J_{\mathsf{a}}[p,q](z_1)\tilde{J}_{\mathsf{b}}[r,s](z_2)\sim\frac{f_{\mathsf{ab}}{}^{\mathsf{c}}}{z_{12}}\tilde{J}_{\mathsf{c}}[p+r,q+s](z_2)\,,\quad \tilde{J}_{\mathsf{a}}[p,q](z_1)\tilde{J}_{\mathsf{b}}[r,s](z_2)\sim 0\,.\end{aligned} \tag{3.14}$$

In fact *any* operator in sdYM theory can be obtained as a conformal block of this chiral CFT.

➤ To compute an amplitude in the presence of a local operator we just plug into our twistor representatives for a, b. These correspond to insertions of hard states

$$J_{\mathsf{a}}(\tilde{\kappa},w)=\sum_{m,n\geq 0}\frac{\tilde{\kappa}_{\dot{0}}^m\tilde{\kappa}_{\dot{1}}^n}{m!n!}J_{\mathsf{a}}[m,n](w)\,,\quad \tilde{J}_{\mathsf{a}}(\tilde{\kappa},w)=\sum_{m,n\geq 0}\frac{\tilde{\kappa}_{\dot{0}}^m\tilde{\kappa}_{\dot{1}}^n}{m!n!}\tilde{J}_{\mathsf{a}}[m,n](w) \tag{3.15}$$

respectively. The amplitude in the presence of $\mathcal{O}$ is the correlation function of these states, dressed by a momentum conserving δ-function generated by the space-time integral.

3.3 Quantum Deformation

- The real power of this approach is at loop level. This is because quantum corrections can be built into the chiral algebra.

- These are generated by loop contributions to the gauge variation of the bulk-defect coupling. On grounds of conformal spin and space-time dilation symmetry the simplest one-loop correction must take the form

$$J_{\mathsf{a}}[1,0](z_1)J_{\mathsf{b}}[0,1](z_2) \sim \hbar\frac{A}{z_{12}^2}f_{\mathsf{ab}}{}^{\mathsf{c}}\tilde{J}_{\mathsf{c}}\big(\tfrac{z_1+z_2}{2}\big) + \hbar\frac{B}{z_{12}}(f_{\mathsf{ae}}{}^{\mathsf{c}}f_{\mathsf{bd}}{}^{\mathsf{e}} + f_{\mathsf{be}}{}^{\mathsf{c}}f_{\mathsf{ad}}{}^{\mathsf{e}}) :J_{\mathsf{c}}\tilde{J}^{\mathsf{d}}: (z_2)\,. \tag{3.16}$$

- However, there's no choice for A, B such that this OPE is associative. To see this we compare

$$\oint_{|z_{12}|=2\epsilon}\oint_{|z_2|=\epsilon} \mathrm{d}z_1\mathrm{d}z_2\, z_1 J_{\mathsf{a}}[1,0](z_1)J_{\mathsf{b}}[0,1](z_2)J_{\mathsf{c}}(0) \tag{3.17}$$

to

$$\Big(\oint_{|z_2|=2\epsilon}\oint_{|z_1|=\epsilon} - \oint_{|z_1|=2\epsilon}\oint_{|z_2|=\epsilon}\Big)\mathrm{d}z_1\mathrm{d}z_2\, z_1 J_{\mathsf{a}}[1,0](z_1)J_{\mathsf{b}}[0,1](z_2)J_{\mathsf{c}}(0)\,. \tag{3.18}$$

By a deformation of contours argument these should be equal, but they do not agree.

- The culprit is the twistorial anomaly we saw in lecture II. Consider a general G and R such that the anomaly vanishes. The fermions contribute two new towers of Grassmann odd states $M_i[m,n](z)$ and $\tilde{M}^j[m,n](z)$ coupling to $\partial^m_{v^{\dot 0}}\partial^n_{v^{\dot 1}}\chi^i$ and $\partial^m_{v^{\dot 0}}\partial^n_{v^{\dot 1}}\tilde{\chi}_j$ of conformal dimensions $(1-m-n)/2$ and $-(1+m+n)/2$ respectively. These have tree OPEs which are determined by gauge invariance

$$\begin{aligned} J_{\mathsf{a}}[p,q](z_1)M_i[r,s](z_2) &\sim \frac{1}{z_{12}}(t_{\mathsf{a}})^j{}_i M_j[p+r,q+s](z_2)\,,\\ J_{\mathsf{a}}[p,q](z_1)\tilde{M}^j[r,s](z_2) &\sim -\frac{1}{z_{12}}(t_{\mathsf{a}})^j{}_i \tilde{M}^i[p+r,q+s](z_2)\,,\\ M_i[p,q](z_1)\tilde{M}^j(z_2) &\sim \frac{1}{z_{12}}(t_{\mathsf{a}})^j{}_i \tilde{J}^{\mathsf{a}}[p+r,q+s](z_2)\,. \end{aligned} \tag{3.19}$$

$(t_{\mathsf{a}})^j{}_i$ is the matrix representing the basis element t_{a} in the representation R. There are also quantum corrections involving the $M_i[m,n], \tilde{M}^j[m,n]$ states. In particular the J, J OPE acquires a correction (again fixed by symmetry)

$$J_{\mathsf{a}}[1,0](z_1)J_{\mathsf{b}}[0,1](z_2) \sim \hbar\frac{C}{z_{12}}\big((t_{\mathsf{a}})^i{}_k(t_{\mathsf{b}})^k{}_j + (t_{\mathsf{a}})^k{}_j(t_{\mathsf{b}})^i{}_k\big) :M_i(z_1)\tilde{M}^j: (z_2)\,. \tag{3.20}$$

- Can show that there is a consistent chiral algebra with

$$A = -\frac{I_2(\mathrm{Ad}) - I_2(R)}{96\pi^2}\,, \quad B = C = \frac{1}{32\pi^2}\,. \tag{3.21}$$

with $\mathrm{tr}_R(X^2) = I_2(R)\mathrm{tr}(X^2)$.

- At this stage can compute the one-loop one-minus amplitude at arbitrary multiplicity. Using the loop corrected OPEs one finds that in the $\mathrm{tr}(B^2)$ conformal block

$$\langle J_{\mathsf{a}}[1,0](z_1) J_{\mathsf{b}}[0,1](z_2) \tilde{J}_{\mathsf{c}}(z_3)\rangle = -\hbar f_{\mathsf{abc}} \frac{I_2(\mathrm{Ad}) - I_2(R)}{96\pi^2} \frac{z_{13} z_{23}}{z_{12}^2} \,. \tag{3.22}$$

  To prove this we first check the singularity of the left hand side in the $z_1 \to z_2$ limit, which yields

$$\begin{aligned} &- \hbar f_{\mathsf{ab}}{}^{\mathsf{d}} \frac{I_2(\mathrm{Ad}) - I_2(R)}{96\pi^2} \left(\frac{1}{z_{12}^2} \langle \tilde{J}_{\mathsf{d}}(z_2) \tilde{J}_{\mathsf{c}}(z_3)\rangle + \frac{1}{2 z_{12}} \partial_{z_2} \langle \tilde{J}_{\mathsf{d}}(z_2) \tilde{J}_{\mathsf{c}}(z_3)\rangle \right) + \mathcal{O}(z_{12}) \\ &= -\hbar f_{\mathsf{abc}} \frac{I_2(\mathrm{Ad}) - I_2(R)}{96\pi^2} \left(\frac{z_{23}^2}{z_{12}^2} + \frac{z_{23}}{z_{12}} \right) + \mathcal{O}(z_{12}) \\ &= -\hbar f_{\mathsf{abc}} \frac{I_2(\mathrm{Ad}) - I_2(R)}{96\pi^2} \frac{z_{13} z_{23}}{z_{12}^2} + \mathcal{O}(z_{12}) \,. \end{aligned} \tag{3.23}$$

  This is the only singularity in z_1 (can check that there are no contributions from normal ordered products). Finally the correlator vanishes to first order as $z_1, z_2 \to \infty$, so the regular terms vanish.

- Can induct to get the one-loop one-minus amplitude in gauge theory with group G and anomaly cancelling fermions in the representation R. The final result is

$$\begin{aligned} &\mathcal{A}\big(1^+, \ldots, (n-1)^+, n^-\big) = \langle J_{\mathsf{a}_1}(\tilde{\kappa}_1, w_1) \ldots J_{\mathsf{a}_{n-1}}(\tilde{\kappa}_{n-1}, w_{n-1}) \tilde{J}_{\mathsf{a}_n}(\tilde{\kappa}_n, w_n)\rangle \\ &= \frac{\hbar}{96\pi^2} \sum_{\sigma \in S_{n-1}} \sum_{1 \le i \le j \le n-1} \frac{\tilde{\kappa}_i^{\dot\alpha} \tilde{\kappa}_j^{\dot\beta} \epsilon_{\dot\alpha\dot\beta} w_{in}^2 w_{jn}^2}{w_{ij} w_{\sigma(1)\sigma(2)} \cdots w_{\sigma(n-1)n} w_{n\sigma(1)}} \mathrm{tr}_{\mathrm{Ad} \oplus \Pi R}(t_{\mathsf{a}_{\sigma(1)}} \ldots t_{\mathsf{a}_{\sigma(n-1)}} t_{\mathsf{a}_n}) \end{aligned} \tag{3.24}$$

  Proof is by induction on n.