On continuous solutions to scalar balance laws

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Introduction

We consider the balance law

$$u_t + f(u)_x = g(t,x) \in L^\infty(\mathbb{R}^2), \quad u \in C(R^2, R),\ f : \mathbb{R} \to \mathbb{R}. \quad (1)$$

If u is smooth and g continuous, then the PDE is equivalent to

$$u_t + \lambda(u) u_x = g, \quad \lambda := \frac{df}{du}$$

$$\frac{d\gamma}{dt} = \lambda(u), \quad \frac{d}{dt} u(t, \gamma(t)) = g(t, \gamma(t)). \quad (2)$$

The converse is also true: a smooth solution $u = u(t,x)$ of the above ODE yields a solution to the PDE.

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The converse is also true: a smooth solution $u = u(t,x)$ of the above ODE yields a solution to the PDE.

We are interested what of the above equivalence is valid under the assumptions u continuous and g bounded Borel function.

Remark 1

By the finite speed of propagation, the results can be restated locally.

Problems we study

We will consider the relations among the following statements: for general smooth flux f

1. *u distributional solution*

$$u_t + f(u)_x = g(t,x) \in L^\infty(\mathbb{R}^2),$$

2. *u broad solution*

$$\text{if } \gamma \ \big(\dot\gamma = \lambda(u(t,\gamma))\big) \quad \Rightarrow \quad \frac{d}{dt} u \circ \gamma = \tilde g_\gamma(t) \in L^\infty(\mathbb{R}),$$

3. there exists a *universal Borel source* $\hat g : \mathbb{R}^2 \to \mathbb{R}$

$$\int_{\mathbb{R}^2} |g - \hat g| \mathcal{L}^2 = 0 \quad \text{and} \quad \int_{\mathbb{R}} \big|\tilde g_\gamma(t) - \hat g(t, \gamma(t))\big| dt = 0.$$

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The case g continuous and f convex

If γ is a characteristic, the balance of $\mathrm{div}_{t,x}(u, f(u))$ in the region

$$\Gamma^\epsilon := \left\{t \in [t_1, t_2], \gamma(t) \leq x \leq \gamma(t) + \epsilon\right\}$$

yields

$$\begin{aligned} \int_{\Gamma^\epsilon} g(t,x)dtdx = & \int_0^\epsilon \big(u(t_2, \gamma(t_2) + x) - u(t_1, \gamma(t_1) + x)\big) dx \\ & + \int_{t_1}^{t_2} \Big[f(u(t, \gamma(t) + \epsilon)) - f(u(t, \gamma(t)) \\ & \quad - \lambda(u(t, \gamma(t))(u(t, \gamma(t) + \epsilon) - u(t, \gamma(t)))\Big] dt \\ \geq & \int_0^\epsilon \big(u(t_2, \gamma(t_2) + x) - u(t_1, \gamma(t_1) + x)\big) dx, \end{aligned}$$

because $f(u') \geq f(u) + \lambda(u)(u' - u)$ by convexity.

The balance on the region

$$\Gamma^{-\epsilon} := \{t \in [t_1, t_2], \gamma(t) - \epsilon \le x \le \gamma(t)\}$$

yields the opposite inequality

$$\int_{\Gamma^{-\epsilon}} g(t,x)dtdx \le \int_{-\epsilon}^{0} \big(u(t_2,\gamma(t_2)+x) - u(t_1,\gamma(t_1)+x)\big)dx.$$

Dividing by ϵ and letting $\epsilon \to 0$ one recovers

$$u(t_2,\gamma(t_2)) - u(t_1,\gamma(t_1)) = \int_{t_1}^{t_2} g(t,\gamma(t))dt,$$

which implies

$$\frac{d}{dt} u \circ \gamma = g(t,\gamma(t)).$$

Proposition 1 (Dafermos)

If f convex, g continuous then $\hat{g} = g$.

A counterexample

Let f be strictly increasing, and such that the set

$$N := \{u : f'(u) = f''(u) = 0\} \quad \text{satisfies} \quad \mathcal{L}^1(N) > 0.$$

Define

$$\tilde{f}(u) = f\left(u + \mathcal{L}^1(N \cap [0,u])\right), \quad \tilde{f}'(u) = f'(f^{-1}(\tilde{f}(u))).$$

The the function $u(x) := f^{-1}(x)$ is a solution to $u_t + f(u)_x = 1$, and the curve $\gamma(t) := \tilde{f}(t)$ is a characteristic:

$$\dot{\gamma} = \tilde{f}'(t) = f'(f^{-1}(\tilde{f}(t))) = f'(u(\gamma(t))).$$

However

$$\frac{d}{dt} f^{-1}(\tilde{f}(t)) = \mathcal{L}^1 + f_\sharp \mathcal{L}^1 \llcorner_N, \quad f_\sharp \mathcal{L}^1 \llcorner_N \perp \mathcal{L}^1.$$

Given f, partition $\mathbb{R}$ into

1. a countable family of disjoint open sets $\{I_i = (u_i^-, u_i^+)\}_{i\in\mathbb{N}}$ where $f\llcorner_{I_i}$ is either convex or concave,
2. a residual set of inflection points $\mathfrak{I}$.

Theorem 1

If $\mathcal{L}^1(\mathfrak{I}) = 0$, then u is Lipschitz along each characteristic.

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Thus

$$u \text{ distributional solution} \quad \xRightarrow{\mathcal{L}^1(\mathfrak{I})=0} \quad u \text{ broad solution}$$

otherwise counterexamples.

Proof.

Proposition 1 implies that

$$u \circ \gamma(t_1), u \circ \gamma(t_2) \in \overline{I}_i \ \left(\left|u \circ \gamma(t_2) - u \circ \gamma(t_1)\right| \leq |t_2 - t_1|\right).$$

Since $\mathcal{L}^1(\mathfrak{I}) = 0$, for $v^t := u \circ \gamma(t)$, $t_1 < t_2$, $I_{i_2} \ni v^{t_2} \geq v^{t_1} \in I_{i_1}$

$$\begin{aligned}
v^{t_2} - v^{t_1} &= \mathcal{L}^1\left(\left[v^{t_1}, v^{t_2}\right]\right) = \bigcup_i \mathcal{L}^1\left(\left[v^{t_1}, v^{t_2}\right] \cap I_i\right) \\
&= v^{t_2} - u^-_{i_2} + \sum_{I_i \subset [v^{t_1}, v^{t_2}]} \left(u^+_i - u^-_i\right) + u^+_{i_1} - v^{t_1} \\
&= v^{t_2} - v^{t^-_{i_2}} + \sum_{I_i \subset [v^{t_1}, v^{t_2}]} \left(v^{t^+_i} - v^{t^-_i}\right) + v^{t^+_{i_1}} - v^{t_1} \\
&\leq t_2 - t^-_{i_2} + \sum_{I_i \subset [v^{t_1}, v^{t_2}]} \left(t^+_i - t^-_i\right) + t^+_{i_1} - t_1 \leq t_2 - t_1.
\end{aligned}$$

□

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Monotone flow

Consider the continuous ODE in $\mathbb{R}$

$$\dot{x} = \lambda(t, x). \tag{3}$$

Proposition 2

There exists a continuous flow $\chi(t, y)$ such that

1. $t \mapsto \chi(t, y)$ *is a solution to* (3)*,*
2. $y \mapsto \chi(t, y)$ *is increasing.*

Proof.

For every point point $(\bar{t}, \bar{x})$ consider the curve

$$\gamma_{\bar{t},\bar{x}}(t) := \begin{cases} \max\{\gamma(t) : \gamma(\bar{t}) = \bar{x}\} & t \leq \bar{t}, \\ \min\{\gamma(t) : \gamma(\bar{t}) = \bar{x}\} & t \geq \bar{t}, \end{cases}$$

and choose suitable parameterization. □

Monotone approximations

Fix now two characteristics $\chi(t, y_1) \leq \chi(t, y_2)$, solutions to $\dot{x} = \lambda(u(t,x))$, and define for $u(t, \chi(t, y_1)) \leq u(t, \chi(t, y_2))$

$$u'(t,x) = u(t, \chi(t, y_1)) \vee \big(u(t,x) \wedge u(t, \chi(t, y_2))\big)$$

where $\chi(t, y_1) \leq x \leq \chi(t, \bar{y}_2)$. Let now χ' be the monotone flow for u' in this interval.

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where $\chi(t, y_1) \leq x \leq \chi(t, \bar{y}_2)$. Let now χ' be the monotone flow for u' in this interval.

Fixing a characteristic curve $\chi'(t, y')$ in between, define

$$u''(t, x) = \begin{cases} u'(t, x) \wedge u'(t, \chi'(t, y')) & \chi(t, y_1) \leq x \leq \chi'(t, y'), \\ u'(t, x) \vee u'(t, \chi'(t, y')) & \chi'(t, y') < x \leq \chi(t, y_2), \end{cases}$$

and let χ'' be the new monotone flow with $\chi''(t, y') = \chi'(t, y')$.

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and let χ'' be the new monotone flow with $\chi''(t, y') = \chi'(t, y')$. By repeating countably many times, we obtain a function u^{mon} such that $x \mapsto u^{\mathrm{mon}}(t,x)$ increasing, and

$$u \circ \gamma \text{ 1-Lipschitz} \quad \Rightarrow \quad u^{\mathrm{mon}} \circ \chi^{\mathrm{mon}} \text{ 1-Lipschitz.}$$

If $\chi^{\rm mon}$, $u^{\rm mon}$ are monotone, with $\dot\chi^{\rm mon} = \lambda(u^{\rm mon})$, then by writing

$$\int d_y u^{\rm mon}(t) dt = \int \upsilon_y(dt) m(dy),$$

one obtains $d_y \chi^{\rm mon}_t = \lambda'(u^{\rm mon}) d_y u^{\rm mon}(t) \in \mathcal{M}(\mathbb{R})$ and

$$\begin{aligned}\int d_y \chi^{\rm mon}(t) dt = & \int \bigg(\int_0^t \lambda'(u^{\rm mon}(s)) d_y u^{\rm mon}(s) ds \bigg) dt \\ = & \int \bigg(\int_0^t \lambda'(u^{\rm mon}(s)) \upsilon_y(ds) \bigg) m(dy) dt.\end{aligned}$$

Thus the disintegration of $\int d_y \chi^{\rm mon}(t) dt$ along characteristics is a.c. w.r.t. time.
Being the parameterization y arbitrary, we can take $m \leq \mathcal{L}^1$, and

$$\chi^{{\rm mon},a}(t,y) = \chi^{\rm mon}(t,y) + ay \quad \big(\text{i.e. enlarging } [\chi(t,y_1), \chi(t,y_2)]\big)$$

we have $a \leq \chi^{{\rm mon},a}_y \leq (1+a)$.

The balance for $\phi(t, \chi^{-1}(t,x))$ is estimated by

$$\int \left((\phi_t - \lambda\phi_x)u^{\text{mon}} + \phi_x f(u^{\text{mon}})\right) dxdt$$
$$= \int \phi_t u^{\text{mon}} \chi_y dydt + \int \phi_y \left(f(u^{\text{mon}}) - \lambda(u^{\text{mon}})u^{\text{mon}}\right) dydt$$
$$= -\int \phi \frac{d}{dt}(u^{\text{mon}} \circ \chi^{\text{mon}})\chi_y dydt$$

because if $u_y \in \mathcal{M}(\mathbb{R})$ then

$$d_y(f(u) - \lambda(u)u) = -u\lambda'(u)d_y u = -ud_y\chi_t.$$

Proposition 3

If u is a 1-Lipschitz broad solution such that $x \mapsto u(t,x)$ is monotone, then is it also a distributional solution with source term $g \in [-1,1]$.

By repeating this procedure on locally finitely many sheets

$$\mathbb{R}^2 = \cup_{j\in\mathbb{N}}\big[\chi(t, y_j), \chi(t, y_{j+1})\big]$$

we obtain a family of continuous locally BV solutions $u^{\{y_j\}}$ converging to u in C^0. Hence

Theorem 2

The function u is a distributional solution with source term g bounded by 1 in L^∞.

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Thus

$$u \text{ distributional solution} \quad \Longleftarrow \quad u \text{ broad solution.}$$

Remark 2

Since $u^{\{y_j\}} \in \mathrm{BV} \cap C^0$, then in the sense of measures

$$u_t^{\{y_j\}} + \lambda(u^{\{y_j\}})u_x^{\{y_j\}} = g^{\{y_j\}}\mathcal{L}^2.$$

Entropy equation

For continuous BV solution we have for $q' = \eta' \lambda$

$$\eta(u)_t + q(u)_x = \eta'(u)\big(u_t + \lambda(u)u_x\big) = \eta'(u)g(t,x), \tag{4}$$

and since entropy solutions are stable w.r.t. strong convergence, we conclude that

Corollary 1

The solution u is entropic if $\mathcal{L}^1(\mathfrak{I}) = 0$.

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In the general case, the entropy equation (4) holds if η is linear in a neighborhood of $\mathfrak{I}$. Since $\mathrm{int}\, \mathfrak{I} = \emptyset$, we can approximate every η with a family η^n linear in a neighborhood of $\mathfrak{I}$, and thus

Proposition 4

If u is a continuous solution to a balance laws with L^∞ source term, then it is entropic.

Continuity estimate in the strictly convex case

Let u be a broad solution and f strictly convex, and consider

$$u(t,x_1) = \bar{u} + v, \;\; u(t,x_2) = \bar{u} - v, \quad x_1 < x_2, v > 0.$$

To avoid the shock formation, the best situation is

$$u \circ \gamma_1(t+s) = \bar{u} + v - \|g\|_\infty s, \;\; u \circ \gamma_2(t+s) = \bar{u} - v + \|g\|_\infty s$$

$$\gamma_1 = x_1 + f(\bar{u}+v) - f(u \circ \gamma_1(t+s)), \; \gamma_2 = x_2 + f(u \circ \gamma_2(t+s)) - f(\bar{u}-v)$$

At the meeting point $u \circ \gamma_i = \bar{u}$, i.e.

$$x_2 - x_1 \geq f(u_1) + f(u_2) - 2f\left(\frac{u_1+u_2}{2}\right). \tag{5}$$

Lemma 1

If f is strictly convex, then u satisfies (5). In particular, if $f = u^2/2$, then u is $1/2$-Hölder continuous.

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Uniqueness of $\{\tilde{g}_\gamma(t) : \gamma(t) = x\}$

The source term $\tilde{g}$ is a priori a function of the characteristic,

$$\tilde{G}(t,x) := \left\{\tilde{g}_\gamma(t) : \gamma(t) = x\right\} \quad \text{is a multifunction.}$$

Theorem 3

Up to a residual set N negligible along each characteristic, it holds

$$\sharp\{\tilde{g}(t) : \gamma(t) = x\} \leq 1.$$

For the proof, we subdivide the each interval I_i of convexity/concavity into

- closed intervals with non empty interior where f is linear,
- open intervals where f is strictly convex.

Proof.

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Inflection points. Since $\mathcal{L}^1(\mathfrak{I}) = 0$, for all $u \circ \gamma$ Lipschitz

$$\frac{d}{dt} u \circ \gamma \llcorner_{u \circ \gamma \in \mathfrak{I}} = 0 \quad \mathfrak{L}^1 - \text{a.e.}.$$

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Linear intervals. Begin λ constant, the characteristic curves do not overlaps so that $\tilde{g}$ is uniquely defined.

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Linear intervals. Begin λ constant, the characteristic curves do not overlaps so that $\tilde{g}$ is uniquely defined.

Strictly convex intervals. If $\tilde{g}$ is a Borel selection of $\tilde{G}$, since f is strictly convex, it is enough to prove that for fixed $\epsilon, \delta > 0$, $\bar{\gamma}$ the following set is negligible:

$$\Big\{ t : \underbrace{\frac{d}{dt} \lambda\big(u \circ \bar{\gamma}(t+s)\big) \leq \lambda(u \circ \gamma(t) + (\tilde{g} \circ \gamma(t) - \epsilon)s), |s| < \delta}_{\text{the derivative of } u \circ \gamma \text{ is } \leq \tilde{g} - \epsilon \text{ in a neighborhood of size } \delta} \Big\}.$$

The points in this set must have a distance of at least 2δ, otherwise at the crossing the curves $\tilde{\gamma}$ are transversal. □

Broad solution not differentiable $\mathcal{L}^2$-a.e. (t, x)

Since $g \in L^\infty$, then $g(t, \gamma(t))$ is meaningless, so that one cannot compute directy $\tilde{g}$ from g.

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On the other hand, it is possible to construct a solution u of the balance law with strictly convex flux f and source $g \in L^\infty$ such that

$$\mathcal{L}^2\Big(\Big\{(t,x) : \nexists \gamma\Big(\dot{\gamma} = \lambda(u), \gamma(t) = x, \exists \frac{du \circ \gamma}{dt}(t)\Big)\Big\}\Big) > 0.$$

Hence in general we cannot compute g directly from $\tilde{g}$, and the function g, $\tilde{g}$ live on different sets.

Existence of a universal source $\hat{g}$

However the two functions are compatible: define in fact

$$\hat{g}(t,x) := \begin{cases} \tilde{g}(t,x) & \exists \tilde{g}(t,x), \\ g(t,x) & \text{otherwise.} \end{cases}$$

Theorem 4

It holds $\|\hat{g} - g\|_\infty = 0$.

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Hence

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Proof.

Since y is an arbitrary parameterization, we can assume that

$$(t, \chi^{-1}(t, y))_\sharp \mathcal{L}^2 = \int \xi_y(t) m(dy), \quad m(dy) \leq \mathcal{L}^1.$$

Thus the sets, where we need to compare g and $\tilde{g}$ are the sets which are not negligible for both, which means

$$\begin{gathered} d_y \chi(t, \chi^{-1}(t, x)) \sim a \in (0, \infty), \\ (t, x), (t, y = \chi^{-1}(t, x)) \text{ density point of } g, \tilde{g}, \text{ respectively.} \end{gathered}$$

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For $\epsilon \ll 1$, in the set $(t, x) + [-\epsilon, \epsilon]^2$ one thus has

$$\lim_{h \to 0} \frac{1}{ah} \int_{-\epsilon}^{\epsilon} \chi(t + s, y \pm h) - \chi(t + s, y) ds = \pm 2\epsilon (1 + \mathcal{O}(\sqrt{\delta})),$$

$$\lim_{h \to 0} \frac{1}{ah} \bigg| \int_{-\epsilon}^{\epsilon} \int_{\chi(t,y)}^{\chi(t,y\pm h)} \big| g(t + s, z) - g(t, x) \big| dz ds \bigg| = \mathcal{O}(\sqrt{\delta}),$$

up to a set of y of measure $\leq \mathcal{O}(\sqrt{\delta})$, hence $\tilde{g}$ is close to g. □

The uniformly convex case

In the case f is uniformly convex outside a $\mathcal{L}^1$-negligible set, then $\tilde{g}$ determines g completely.

Theorem 5 (Rademacher)

If f uniformly convex, then the set where $\tilde{g}$ is defined is of full Lebesgue measure in (t, x).

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The above theorem can be extended to the following situation: there exists $p \geq 1$ such that for $\epsilon \ll 1$

$$\frac{1}{\epsilon^{2p}}\big(f(u+\epsilon v) - f(u) - \epsilon f'(u)v\big) \sim_{C^2} v^{2p}$$

Remark 3

The set where $p > 1$ has Lebesgue measure 0.

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Hence

$$f \text{ uniformly convex} \quad \Longrightarrow \quad \tilde{g} = \hat{g} \ \mathcal{L}^2 - \text{a.e.}.$$

Proof for Burgers equation.

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Step 1. The covering

$$Q^{\epsilon}_{t,x} := \Big\{ t \le s \le t + \epsilon/2, \chi(s, y_{x-\epsilon}) \le x \le \chi(s, y_{x+\epsilon}) \Big\}$$

satisfies Besicovitch covering property: in particular,

$$\lim_{\epsilon \to 0} \frac{1}{\mathcal{L}^2(Q^{\epsilon}_{t,x})} \int_{Q^{\epsilon}_{t,x}} |g(s,z) - g(t,x)| dsdz = 0 \quad \mathcal{L}^2 - \text{a.e. } (t,x).$$

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Step 2. In the above points, being $u(t,x)$ Lipschitz along characteristics and 1/2-Hölder in x, the rescaling

$$u^{\epsilon}(\tau, z) := \frac{1}{\epsilon}\big(u(t + \epsilon s, x + \epsilon^2 z) - u(t,x) \big)$$

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Step 3. Dafermos computation applies.

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