On continuous solutions to scalar balance laws

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  Statement of the problem

Distributional to broad
  Dafermos computation in the convex case
  The non convex case

Broad to distributional
  Monotone flow
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  Continuity estimate of broad solutions

Identification of the source terms
  Uniqueness of the derivative along characteristics
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Introduction

We consider the balance law

\[ u_t + f(u)_x = g(t, x) \in L^\infty(\mathbb{R}^2), \quad u \in C(\mathbb{R}^2, \mathbb{R}), \quad f : \mathbb{R} \to \mathbb{R}. \quad (1) \]

If \( u \) is smooth and \( g \) continuous, then the PDE is equivalent to

\[ u_t + \lambda(u)u_x = g, \quad \lambda := \frac{df}{du} \]

\[ \frac{d\gamma}{dt} = \lambda(u), \quad \frac{d}{dt} u(t, \gamma(t)) = g(t, \gamma(t)). \quad (2) \]

The converse is also true: a smooth solution \( u = u(t, x) \) of the above ODE yields a solution to the PDE.
Introduction

We consider the balance law

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The converse is also true: a smooth solution \( u = u(t, x) \) of the above ODE yields a solution to the PDE.

We are interested what of the above equivalence is valid under the assumptions \( u \) continuous and \( g \) bounded Borel function.

Remark 1
By the finite speed of propagation, the results can be restated locally.
Problems we study

We will consider the relations among the following statements: for general smooth flux $f$

1. $u$ distributional solution

$$u_t + f(u)_x = g(t, x) \in L^\infty(\mathbb{R}^2),$$

2. $u$ broad solution

if $\gamma \left( \dot{\gamma} = \lambda(u(t, \gamma)) \right)$ \Rightarrow \frac{d}{dt} u \circ \gamma = \tilde{g}_\gamma(t) \in L^\infty(\mathbb{R}),$

3. there exists a universal Borel source $\hat{g} : \mathbb{R}^2 \rightarrow \mathbb{R}$

$$\int_{\mathbb{R}^2} |g - \hat{g}|L^2 = 0 \quad \text{and} \quad \int_{\mathbb{R}} |\tilde{g}_\gamma(t) - \hat{g}(t, \gamma(t))| dt = 0.$$
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The case $g$ continuous and $f$ convex

If $\gamma$ is a characteristic, the balance of $\text{div}_{t,x}(u, f(u))$ in the region

$$\Gamma^{\epsilon} := \{ t \in [t_1, t_2], \gamma(t) \leq x \leq \gamma(t) + \epsilon \}$$

yields

$$\int_{\Gamma^{\epsilon}} g(t, x) dtdx = \int_0^{\epsilon} \left( u(t_2, \gamma(t_2) + x) - u(t_1, \gamma(t_1) + x) \right) dx$$

$$+ \int_{t_1}^{t_2} \left[ f(u(t, \gamma(t) + \epsilon)) - f(u(t, \gamma(t))) - \lambda(u(t, \gamma(t)))(u(t, \gamma(t) + \epsilon) - u(t, \gamma(t))) \right] dt$$

$$\geq \int_0^{\epsilon} \left( u(t_2, \gamma(t_2) + x) - u(t_1, \gamma(t_1) + x) \right) dx,$$

because $f(u') \geq f(u) + \lambda(u)(u' - u)$ by convexity.
The balance on the region

\[ \Gamma^{-\epsilon} := \{ t \in [t_1, t_2], \gamma(t) - \epsilon \leq x \leq \gamma(t) \} \]

yields the opposite inequality

\[
\int_{\Gamma^{-\epsilon}} g(t, x) dtdx \leq \int_{-\epsilon}^{0} (u(t_2, \gamma(t_2) + x) - u(t_1, \gamma(t_1) + x)) dx.
\]

Dividing by \( \epsilon \) and letting \( \epsilon \to 0 \) one recovers

\[
u(t_2, \gamma(t_2)) - u(t_1, \gamma(t_1)) = \int_{t_1}^{t_2} g(t, \gamma(t)) dt,
\]

which implies

\[
\frac{d}{dt} u \circ \gamma = g(t, \gamma(t)).
\]

**Proposition 1 (Dafermos)**

If \( f \) convex, \( g \) continuous then \( \hat{g} = g \).
A counterexample

Let \( f \) be strictly increasing, and such that the set

\[
N := \{ u : f'(u) = f''(u) = 0 \}
\]

satisfies \( \mathcal{L}^1(N) > 0 \).

Define

\[
\tilde{f}(u) = f(u + \mathcal{L}^1(N \cap [0, u])), \quad \tilde{f}'(u) = f'(f^{-1}(\tilde{f}(u))).
\]

The function \( u(x) := f^{-1}(x) \) is a solution to \( u_t + f(u)_x = 1 \), and the curve \( \gamma(t) := \tilde{f}(t) \) is a characteristic:

\[
\dot{\gamma} = \tilde{f}'(t) = f'(f^{-1}(\tilde{f}(t))) = f'(u(\gamma(t))).
\]

However

\[
\frac{d}{dt}f^{-1}(\tilde{f}(t)) = \mathcal{L}^1 + f_\# \mathcal{L}^1 \perp N, \quad f_\# \mathcal{L}^1 \perp N \perp \mathcal{L}^1.
\]
Given $f$, partition $\mathbb{R}$ into

1. a countable family of disjoint open sets $\{I_i = (u_i^-, u_i^+)\}_{i \in \mathbb{N}}$ where $f_{\perp I_i}$ is either convex or concave,

2. a residual set of inflection points $\mathcal{I}$.

**Theorem 1**

*If $\mathcal{L}^1(\mathcal{I}) = 0$, then $u$ is Lipschitz along each characteristic.*
Given \( f \), partition \( \mathbb{R} \) into

1. a countable family of disjoint open sets \( \{ I_i = (u_i^-, u_i^+) \}_{i \in \mathbb{N}} \)
   where \( f_{|I_i} \) is either convex or concave,
2. a residual set of inflection points \( \mathcal{I} \).

**Theorem 1**

If \( L^1(\mathcal{I}) = 0 \), then \( u \) is Lipschitz along each characteristic.

Thus

\[ u \text{ distributional solution} \quad \overset{L^1(\mathcal{I})=0}{\Rightarrow} \quad u \text{ broad solution} \]

otherwise counterexamples.
Proof.
Proposition 1 implies that

\[ u \circ \gamma(t_1), u \circ \gamma(t_2) \in \overline{I}_i \left( |u \circ \gamma(t_2) - u \circ \gamma(t_1)| \leq |t_2 - t_1| \right). \]

Since \( \mathcal{L}^1(I) = 0 \), for \( v^t := u \circ \gamma(t), t_1 < t_2, l_{i_2} \ni v^{t_2} \geq v^{t_1} \in I_{i_1} \)

\[ v^{t_2} - v^{t_1} = \mathcal{L}^1([v^{t_1}, v^{t_2}]) = \bigcup_{i} \mathcal{L}^1([v^{t_1}, v^{t_2}] \cap I_i) \]

\[ = v^{t_2} - u_{i_2}^- + \sum_{l_i \subset [v^{t_1}, v^{t_2}]} (u_i^+ - u_i^-) + u_{i_1}^+ - v^{t_1} \]

\[ = v^{t_2} - v^{t_{i_2}}^- + \sum_{l_i \subset [v^{t_1}, v^{t_2}]} (v_i^+ - v_i^-) + v_{i_1}^+ - v^{t_1} \]

\[ \leq t_2 - t_{i_2}^- + \sum_{l_i \subset [v^{t_1}, v^{t_2}]} (t_i^+ - t_i^-) + t_{i_1}^+ - t_1 \leq t_2 - t_1. \]
Monotone flow

Consider the continuous ODE in $\mathbb{R}$

$$\dot{x} = \lambda(t, x). \quad (3)$$

Proposition 2

There exists a continuous flow $\chi(t, y)$ such that

1. $t \mapsto \chi(t, y)$ is a solution to (3),
2. $y \mapsto \chi(t, y)$ is increasing.

Proof.

For every point point $(\bar{t}, \bar{x})$ consider the curve

$$\gamma_{\bar{t}, \bar{x}}(t) := \begin{cases} 
\max\{\gamma(t) : \gamma(\bar{t}) = \bar{x}\} & t \leq \bar{t}, \\
\min\{\gamma(t) : \gamma(\bar{t}) = \bar{x}\} & t \geq \bar{t},
\end{cases}$$

and choose suitable parameterization. \qed
Monotone approximations

Fix now two characteristics \( \chi(t, y_1) \leq \chi(t, y_2) \), solutions to \( \dot{x} = \lambda(u(t, x)) \), and define for \( u(t, \chi(t, y_1)) \leq u(t, \chi(t, y_2)) \)

\[
u'(t, x) = u(t, \chi(t, y_1)) \lor (u(t, x) \land u(t, \chi(t, y_2)))
\]

where \( \chi(t, y_1) \leq x \leq \chi(t, \bar{y}_2) \). Let now \( \chi' \) be the monotone flow for \( u' \) in this interval.
Monotone approximations

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\]

where \( \chi(t, y_1) \leq x \leq \chi(t, \bar{y}_2) \). Let now \( \chi' \) be the monotone flow for \( u' \) in this interval. Fixing a characteristic curve \( \chi'(t, y') \) in between, define

\[
u''(t, x) = \begin{cases} 
u'(t, x) \land \nu'(t, \chi'(t, y')) & \chi(t, y_1) \leq x \leq \chi'(t, y'), \\ \nu'(t, x) \lor \nu'(t, \chi'(t, y')) & \chi'(t, y') < x \leq \chi(t, y_2), \end{cases}
\]

and let \( \chi'' \) be the new monotone flow with \( \chi''(t, y') = \chi'(t, y') \).
Monotone approximations

Fix now two characteristics $\chi(t, y_1) \leq \chi(t, y_2)$, solutions to $\dot{x} = \lambda(u(t, x))$, and define for $u(t, \chi(t, y_1)) \leq u(t, \chi(t, y_2))$

$$u'(t, x) = u(t, \chi(t, y_1)) \lor (u(t, x) \land u(t, \chi(t, y_2)))$$

where $\chi(t, y_1) \leq x \leq \chi(t, \bar{y}_2)$. Let now $\chi'$ be the monotone flow for $u'$ in this interval.

Fixing a characteristic curve $\chi'(t, y')$ in between, define

$$u''(t, x) = \begin{cases} u'(t, x) \land u'(t, \chi'(t, y')) & \chi(t, y_1) \leq x \leq \chi'(t, y'), \\ u'(t, x) \lor u'(t, \chi'(t, y')) & \chi'(t, y') < x \leq \chi(t, y_2), \end{cases}$$

and let $\chi''$ be the new monotone flow with $\chi''(t, y') = \chi'(t, y')$.

By repeating countably many times, we obtain a function $u_{\text{mon}}$ such that $x \mapsto u_{\text{mon}}(t, x)$ increasing, and

$$u \circ \gamma \ 1\text{-Lipschitz} \quad \Rightarrow \quad u_{\text{mon}} \circ \chi_{\text{mon}} \ 1\text{-Lipschitz}.$$
If $\chi_{\text{mon}}$, $u_{\text{mon}}$ are monotone, with $\dot{\chi}_{\text{mon}} = \lambda(u_{\text{mon}})$, then by writing

$$
\int dy u_{\text{mon}}(t) dt = \int \nu_y(dt) m(dy),
$$

one obtains $d_y \chi_{t_{\text{mon}}} = \lambda'(u_{\text{mon}})d_y u_{\text{mon}}(t) \in \mathcal{M}(\mathbb{R})$ and

$$
\int dy \chi_{\text{mon}}(t) dt = \int \left( \int_0^t \lambda'(u_{\text{mon}}(s)) d_y u_{\text{mon}}(s) ds \right) dt \\
= \int \left( \int_0^t \lambda'(u_{\text{mon}}(s)) \nu_y(ds) \right) m(dy) dt.
$$

Thus the disintegration of $\int dy \chi_{\text{mon}}(t) dt$ along characteristics is a.c. w.r.t. time.

Being the parameterization $y$ arbitrary, we can take $m \leq \mathcal{L}^1$, and

$$
\chi_{\text{mon},a}(t, y) = \chi_{\text{mon}}(t, y) + ay \quad \text{(i.e. enlarging } [\chi(t, y_1), \chi(t, y_2)])
$$

we have $a \leq \chi_{y_{\text{mon}},a} \leq (1 + a)$. 
The balance for $\phi(t, \chi^{-1}(t, x))$ is estimated by

\[
\int \left( (\phi_t - \lambda \phi_x)u_{\text{mon}} + \phi_x f(u_{\text{mon}}) \right) dx dt \\
= \int \phi_t u_{\text{mon}} \chi_y dy dt + \int \phi_y (f(u_{\text{mon}}) - \lambda (u_{\text{mon}})u_{\text{mon}}) dy dt \\
= -\int \phi \frac{d}{dt} (u_{\text{mon}} \circ \chi_{\text{mon}}) \chi_y dy dt
\]

because if $u_y \in \mathcal{M}(\mathbb{R})$ then

\[
d_y (f(u) - \lambda(u)u) = -u \lambda'(u) d_y u = -ud_y \chi_t.
\]

Proposition 3

If $u$ is a 1-Lipschitz broad solution such that $x \mapsto u(t, x)$ is monotone, then is it also a distributional solution with source term $g \in [-1, 1]$. 
By repeating this procedure on locally finitely many sheets

\[ \mathbb{R}^2 = \bigcup_{j \in \mathbb{N}} [\chi(t, y_j), \chi(t, y_{j+1})] \]

we obtain a family of continuous locally BV solutions \( u_{\{y_j\}} \) converging to \( u \) in \( C^0 \). Hence

**Theorem 2**

*The function \( u \) is a distributional solution with source term \( g \) bounded by 1 in \( L^\infty \).*
By repeating this procedure on locally finitely many sheets

$$\mathbb{R}^2 = \bigcup_{j \in \mathbb{N}} [\chi(t, y_j), \chi(t, y_{j+1})]$$

we obtain a family of continuous locally BV solutions $u^{\{y_j\}}$ converging to $u$ in $C^0$. Hence

**Theorem 2**

*The function $u$ is a distributional solution with source term $g$ bounded by 1 in $L^\infty$.*

Thus

$$u \text{ distributional solution} \iff u \text{ broad solution}.$$  

**Remark 2**

Since $u^{\{y_j\}} \in \text{BV} \cap C^0$, then in the sense of measures

$$u_t^{\{y_j\}} + \lambda(u^{\{y_j\}})u_x^{\{y_j\}} = g^{\{y_j\}} L^2.$$
Entropy equation

For continuous BV solution we have for \( q' = \eta' \lambda \)

\[
\eta(u)_t + q(u)_x = \eta'(u)(u_t + \lambda(u)u_x) = \eta'(u)g(t,x), \quad (4)
\]

and since entropy solutions are stable w.r.t. strong convergence, we conclude that

**Corollary 1**

*The solution \( u \) is entropic if \( L^1(I) = 0 \).*
Entropy equation

For continuous BV solution we have for \( q' = \eta' \lambda \)

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\eta(u)_t + q(u)_x = \eta'(u)(u_t + \lambda(u)u_x) = \eta'(u)g(t, x),
\]  

(4)

and since entropy solutions are stable w.r.t. strong convergence, we conclude that

Corollary 1

The solution \( u \) is entropic if \( L^1(I) = 0 \).

In the general case, the entropy equation (4) holds if \( \eta \) is linear in a neighborhood of \( I \). Since \( \text{int } I = \emptyset \), we can approximate every \( \eta \) with a family \( \eta^n \) linear in a neighborhood of \( I \), and thus

Proposition 4

If \( u \) is a continuous solution to a balance laws with \( L^\infty \) source term, then it is entropic.
Continuity estimate in the strictly convex case

Let \( u \) be a broad solution and \( f \) strictly convex, and consider

\[
u(t, x_1) = \bar{u} + v, \quad u(t, x_2) = \bar{u} - v, \quad x_1 < x_2, \quad v > 0.
\]

To avoid the shock formation, the best situation is

\[
u \circ \gamma_1(t + s) = \bar{u} + v - \|g\|_\infty s, \quad u \circ \gamma_2(t + s) = \bar{u} - v + \|g\|_\infty s
\]

\[
\gamma_1 = x_1 + f(\bar{u} + v) - f(u \circ \gamma_1(t + s)), \quad \gamma_2 = x_2 + f(u \circ \gamma_2(t + s)) - f(\bar{u} - v)
\]

At the meeting point \( u \circ \gamma_i = \bar{u} \), i.e.

\[
x_2 - x_1 \geq f(u_1) + f(u_2) - 2f\left(\frac{u_1 + u_2}{2}\right). \quad (5)
\]

Lemma 1

If \( f \) is strictly convex, then \( u \) satisfies (5). In particular, if \( f = u^2/2 \), then \( u \) is 1/2-Hölder continuous.
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Uniqueness of $\{\tilde{g}_\gamma(t) : \gamma(t) = x\}$

The source term $\tilde{g}$ is a priori a function of the characteristic,

$$\tilde{G}(t, x) := \{\tilde{g}_\gamma(t) : \gamma(t) = x\}$$

is a multifunction.

**Theorem 3**

*Up to a residual set $N$ negligible along each characteristic, it holds*

$$\#\{\tilde{g}(t) : \gamma(t) = x\} \leq 1.$$ *

For the proof, we subdivide the each interval $I_i$ of convexity/concavity into

- closed intervals with non empty interior where $f$ is linear,
- open intervals where $f$ is strictly convex.
Proof.
We have to consider 3 cases.

Inflection points.
Since $L_1(I, u \circ \gamma) = 0$, for all $u \circ \gamma$ Lipschitz $d\gamma dt u \circ \gamma \in L_1 - a.e.$.

Linear intervals.
Begin $\lambda$ constant, the characteristic curves do not overlaps so that $\tilde{g}$ is uniquely defined.

Strictly convex intervals.
If $\tilde{g}$ is a Borel selection of $\tilde{G}$, since $f$ is strictly convex, it is enough to prove that for fixed $\epsilon, \delta > 0$, $\bar{\gamma}$ the following set is negligible:

$$\left\{ t : \frac{d}{dt} \lambda(u \circ \bar{\gamma}(t + s)) \leq \lambda(u \circ \gamma(t)) + (\tilde{g} \circ \gamma(t) - \epsilon)s, |s| < \delta \right\}$$

The points in this set must have a distance of at least $2\delta$, otherwise at the crossing the curves $\tilde{\gamma}$ are transversal.
Proof.
We have to consider 3 cases.

Inflection points. Since $L^1(I) = 0$, for all $u \circ \gamma$ Lipschitz

$$\frac{d}{dt} u \circ \gamma \big|_{u \circ \gamma \in I} = 0 \quad L^1 - a.e.$$
Proof.
We have to consider 3 cases.

*Inflection points.* Since \( L^1(I) = 0 \), for all \( u \circ \gamma \) Lipschitz

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\frac{d}{dt} u \circ \gamma \big|_{u \circ \gamma \in I} = 0 \quad L^1 - \text{a.e.}
\]

*Linear intervals.* Begin \( \lambda \) constant, the characteristic curves do not overlaps so that \( \tilde{g} \) is uniquely defined.
Proof.
We have to consider 3 cases.

Inflection points. Since $L^1(I) = 0$, for all $u \circ \gamma$ Lipschitz

$$\frac{d}{dt} u \circ \gamma \quad \text{for all} \quad u \circ \gamma \in I = 0 \quad L^1 - \text{a.e.}$$

Linear intervals. Begin $\lambda$ constant, the characteristic curves do not overlaps so that $\tilde{g}$ is uniquely defined.

Strictly convex intervals. If $\tilde{g}$ is a Borel selection of $\tilde{G}$, since $f$ is strictly convex, it is enough to prove that for fixed $\epsilon, \delta > 0$, $\bar{\gamma}$ the following set is negligible:

$$\left\{ t : \frac{d}{dt} \lambda(u \circ \bar{\gamma}(t + s)) \leq \lambda(u \circ \gamma(t) + (\tilde{g} \circ \gamma(t) - \epsilon)s), |s| < \delta \right\}.$$  

the derivative of $u \circ \gamma$ is $\leq \tilde{g} - \epsilon$ in a neighborhood of size $\delta$

The points in this set must have a distance of at least $2\delta$, otherwise at the crossing the curves $\tilde{\gamma}$ are transversal.
Broad solution not differentiable $L^2$-a.e. $(t, x)$

Since $g \in L^\infty$, then $g(t, \gamma(t))$ is meaningless, so that one cannot compute directly $\tilde{g}$ from $g$. 
Since $g \in L^\infty$, then $g(t, \gamma(t))$ is meaningless, so that one cannot compute directly $\tilde{g}$ from $g$.

On the other hand, it is possible to construct a solution $u$ of the balance law with strictly convex flux $f$ and source $g \in L^\infty$ such that

$$L^2\left(\left\{(t, x) : \not\exists \gamma \left(\dot{\gamma} = \lambda(u), \gamma(t) = x, \exists \frac{du \circ \gamma}{dt}(t)\right)\right\}\right) > 0.$$ 

Hence in general we cannot compute $g$ directly from $\tilde{g}$, and the function $g$, $\tilde{g}$ live on different sets.
Existence of a universal source $\hat{g}$

However the two functions are compatible: define in fact

$$\hat{g}(t, x) := \begin{cases} \tilde{g}(t, x) & \exists \tilde{g}(t, x), \\ g(t, x) & \text{otherwise.} \end{cases}$$

**Theorem 4**

It holds $\|\hat{g} - g\|_\infty = 0$. 


Existence of a universal source \( \hat{g} \)

However the two functions are compatible: define in fact

\[
\hat{g}(t, x) := \begin{cases} 
\tilde{g}(t, x) & \exists \tilde{g}(t, x), \\
g(t, x) & \text{otherwise}.
\end{cases}
\]

**Theorem 4**

It holds \( \| \hat{g} - g \|_\infty = 0 \).

Hence

there exists a universal source \( \hat{g} \).
Proof.
Since \( y \) is an arbitrary parameterization, we can assume that

\[
(t, \chi^{-1}(t, y))\#\mathcal{L}^2 = \int \xi_y(t)m(dy), \quad m(dy) \leq \mathcal{L}^1.
\]

Thus the sets, where we need to compare \( g \) and \( \tilde{g} \) are the sets which are not negligible for both, which means

\[
d_y\chi(t, \chi^{-1}(t, x)) \sim a \in (0, \infty),
(t, x), (t, y = \chi^{-1}(t, x)) \text{ density point of } g, \tilde{g}, \text{ respectively.}
\]
Proof.
Since \( y \) is an arbitrary parameterization, we can assume that

\[
(t, \chi^{-1}(t, y)) \# \mathcal{L}^2 = \int \xi_y(t)m(dy), \quad m(dy) \leq \mathcal{L}^1.
\]

Thus the sets, where we need to compare \( g \) and \( \tilde{g} \) are the sets which are not negligible for both, which means

\[
d_y\chi(t, \chi^{-1}(t, x)) \sim a \in (0, \infty),
\]

\((t, x), (t, y = \chi^{-1}(t, x))\) density point of \( g, \tilde{g} \), respectively.

For \( \epsilon \ll 1 \), in the set \((t, x) + [-\epsilon, \epsilon]^2\) one thus has

\[
\lim_{h \to 0} \frac{1}{ah} \int_{-\epsilon}^{\epsilon} \chi(t + s, y \pm h) - \chi(t + s, y)ds = \pm 2\epsilon(1 + \mathcal{O}(\sqrt{\delta})),
\]

\[
\lim_{h \to 0} \frac{1}{ah} \left| \int_{-\epsilon}^{\epsilon} \int_{\chi(t, y)} g(t + s, z) - g(t, x)dzds \right| = \mathcal{O}(\sqrt{\delta}),
\]

up to a set of \( y \) of measure \( \leq \mathcal{O}(\sqrt{\delta}) \), hence \( \tilde{g} \) is close to \( g \).
The uniformly convex case

In the case \( f \) is uniformly convex outside a \( \mathcal{L}^1 \)-negligible set, then \( \tilde{g} \) determines \( g \) completely.

**Theorem 5 (Rademacher)**

*If \( f \) uniformly convex, then the set where \( \tilde{g} \) is defined is of full Lebesgue measure in \( (t, x) \).*

Remark 3

The set where \( p > 1 \) has Lebesgue measure 0. Hence \( f \) uniformly convex \( \implies \tilde{g} = \hat{g} \) \( L^2 \)-a.e.
The uniformly convex case

In the case $f$ is uniformly convex outside a $\mathcal{L}^1$-negligible set, then $\tilde{g}$ determines $g$ completely.

**Theorem 5 (Rademacher)**

*If $f$ uniformly convex, then the set where $\tilde{g}$ is defined is of full Lebesgue measure in $(t, x)$.*

The above theorem can be extended to the following situation: there exists $p \geq 1$ such that for $\epsilon \ll 1$

\[
\frac{1}{\epsilon^{2p}} (f(u + \epsilon v) - f(u) - \epsilon f'(u)v) \sim c_2 \, v^{2p}
\]

**Remark 3**

The set where $p > 1$ has Lebesgue measure 0.
The uniformly convex case

In the case $f$ is uniformly convex outside a $\mathcal{L}^1$-negligible set, then $\tilde{g}$ determines $g$ completely.

**Theorem 5 (Rademacher)**

*If $f$ uniformly convex, then the set where $\tilde{g}$ is defined is of full Lebesgue measure in $(t, x)$.*

The above theorem can be extended to the following situation: there exists $p \geq 1$ such that for $\epsilon \ll 1$

\[
\frac{1}{\epsilon^{2p}} (f(u + \epsilon v) - f(u) - \epsilon f'(u)v) \sim_{c^2} v^{2p}
\]

**Remark 3**

The set where $p > 1$ has Lebesgue measure 0.

Hence

\[
f \text{ uniformly convex } \implies \tilde{g} = \hat{g} \mathcal{L}^2 - \text{a.e.}
\]
Proof for Burgers equation.

Step 1. The covering $Q_\epsilon t, x := \{ t \leq s \leq t + \epsilon/2, \chi(s, y - \epsilon x) \leq x \leq \chi(s, y + \epsilon x) \}$ satisfies Besicovitch covering property: in particular, 
\[
\lim_{\epsilon \to 0} \frac{1}{L^2(Q_\epsilon t, x)} \int_{Q_\epsilon t, x} |g(s, z) - g(t, x)| \, ds \, dz = 0 \quad L^2 - \text{a.e. } (t, x).
\]

Step 2. In the above points, being $u(t, x)$ Lipschitz along characteristics and $1/2$-Hölder in $x$, the rescaling 
\[
u_\epsilon(\tau, z) := \frac{1}{\epsilon} (u(t + \epsilon s, x + \epsilon/2 z) - u(t, x))
\] converges strongly to a solution to $u + (u^2/2)z = g(t, x)$.

Step 3. Dafermos computation applies.
Proof for Burgers equation.

Step 1. The covering

\[ Q_{t,x}^\epsilon := \left\{ t \leq s \leq t + \epsilon/2, \chi(s, y_{x-\epsilon}) \leq x \leq \chi(s, y_{x+\epsilon}) \right\} \]

satisfies Besicovitch covering property: in particular,

\[ \lim_{\epsilon \to 0} \frac{1}{\mathcal{L}^2(Q_{t,x}^\epsilon)} \int_{Q_{t,x}^\epsilon} |g(s, z) - g(t, x)| dsdz = 0 \quad \mathcal{L}^2 \text{ a.e. } (t, x). \]
Proof for Burgers equation.

Step 1. The covering

\[ Q^\epsilon_{t,x} := \left\{ t \leq s \leq t + \epsilon/2, \chi(s, y_{x-\epsilon}) \leq x \leq \chi(s, y_{x+\epsilon}) \right\} \]

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Step 2. In the above points, being \( u(t, x) \) Lipschitz along characteristics and 1/2-Hölder in \( x \), the rescaling

\[ u^\epsilon(\tau, z) := \frac{1}{\epsilon} \left( u(t + \epsilon \tau, x + \epsilon^2 z) - u(t, x) \right) \]

converges strongly to a solution to

\[ u_s + (u^2/2)_z = g(t, x). \]
Proof for Burgers equation.

Step 1. The covering

\[ Q^\epsilon_{t,x} := \left\{ t \leq s \leq t + \epsilon/2, \chi(s, y_{x-\epsilon}) \leq x \leq \chi(s, y_{x+\epsilon}) \right\} \]

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converges strongly to a solution to

\[ u_s + \left( u^2 / 2 \right)_z = g(t, x). \]

Step 3. Dafermos computation applies.
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