Metric geometry and analysis on boundaries of Gromov hyperbolic spaces, and applications

Lecture 1

Overview

Theory of boundaries is at the intersection of many areas:

- Metric geometry.
- Geometric group theory.
- Geometric mapping theory.
- Analysis on metric spaces.
- Dynamical systems.
- PDE and PDR.

Goals:

- Motivation for current research.
- Cover some of the foundations.
- Survey some advanced results.

Geometric group theory: a quick primer

- Problem: Understand groups. Infinite groups, usually finitely generated.
- Example: Surface group.
- Van Kampen: Generators and relations.
- Combinatorial group theory. Manipulate presentations. (Normal forms, word problem, conjugacy problem, isomorphism problem...).
- Geometric group theory: use tools from geometry (and topology, analysis, dynamics...) to understand groups.
- Geometrization 1: solvability of the word problem is directly related to the isoperimetric inequality on the surface.
- Geometrization 2: hyperbolic metric.

Groups as metric spaces

Let G be a finitely generated group.

Let $\Sigma \subset G$ be a symmetric finite generating set, $\Sigma^{-1} = \Sigma$.

Let $\Gamma = \Gamma(G, \Sigma)$ be the Cayley graph of G with respect to Σ: the vertex set is G, and $g_1, g_2 \in G$ are joined by an edge iff

$$g_1 = g_2 \sigma$$

for some $\sigma \in \Sigma$.

Give Γ the path metric where edges have length 1. Left translation induces an isometric action $G \curvearrowright \Gamma$.

Thus we have associated a metric space with the group G.

Examples:

- $\mathbb{Z}^k$. Rank k free abelian group.
- F_k. Rank k free nonabelian group.
- Surface group.

The Cayley graph depends on the choice of Σ.

Any two Cayley graphs are quasi-isometric.

Definition. A map $f : X \to Y$ between metric spaces is a **quasi-isometry** if there are constants $L \in [1, \infty)$, $A \in [0, \infty)$, such that

- For all $x_1, x_2 \in X$,

$$L^{-1}d(x_1, x_2) - A \leq d(f(x_1), f(x_2)) \leq Ld(x_1, x_2) + A$$

- For every $y \in Y$,

$$d(y, \mathrm{Im}(f)) < A .$$

Two metric spaces are **quasi-isometric** if there is a quasi-isometry between them.

Some facts about quasi-isometries

- $L = 1$, $A = 0 \iff$ isometry.
- $A = 0$ iff L-bilipschitz.
- f, g qi $\implies$ $f \circ g$ qi.
- Quasi-inverses.
- $\mathrm{QI}(X)$.
- $X \overset{qi}{\sim} Y$ iff they contain bilipschitz equivalent nets.

Examples. $\mathbb{R}^n$, $\mathbb{H}^n$.

We have a canonical quasi-isometry class of metric spaces associated with a finitely generated group G.

The Fundamental Lemma of GGT. (Milnor-Svarc)
If X is a proper geodesic space, and $G \curvearrowright X$ is a discrete, cococompact, isometric action, then G is quasi-isometric to X.

Cor. If M is a compact connected Riemannian manifold, then $\pi_1(M)$ is quasi-isometric to the universal cover $\tilde{M}$.

If $G \curvearrowright \mathbb{H}^n$ is a discrete, cocompact, isometric action, then G is quasi-isometric to $\mathbb{H}^n$.

Problem: understand metric geometry of groups.

- Q: How much is encoded in the QI class?
- Classify groups up to QI.
- Geometric mapping theory: construct/obstruct/classify QIs.
- QI invariants and QI invariant structure.
  - Growth.
  - Isoperimetric inequalities.
  - Negative curvature (aka hyperbolicity).

Exercises:

- $\mathbb{E}^k \overset{qi}{\sim} \mathbb{E}^\ell$ iff $k = \ell$.
- Show that any quasi-isometry $f : \mathbb{H}^k \to \mathbb{H}^\ell$ is at finite sup distance from a continuous quasi-isometry $\hat{f} : \mathbb{H}^k \to \mathbb{H}^\ell$.
- $\mathbb{H}^k \overset{qi}{\sim} \mathbb{H}^\ell$ iff $k = \ell$.

Lecture 2

Negative curvature 1: hyperbolic space

- Quadric model (upper sheet, projective picture).
- Ball model, upper halfspace model, normal coordinates.
- Area growth of balls, Gauss-Bonnet, slimness of triangles.
  $\downarrow$ Lecture 1

  $\downarrow$ Lecture 2

- Projection to geodesic.
- Sphere at infinity. "Ideal boundary".
- $n \geq 3$. $\mathrm{Isom}(\mathbb{H}^n) \simeq \mathrm{Mob}(S^{n-1}) \simeq \mathrm{Conf}(S^{n-1})$. (Exercise: verify this.)
- Hyperbolic manifolds and discrete subgroups.

Negative curvature 2: Riemannian manifolds

Let M be complete, simply connected, with sectional curvature ≤ -1

- Examples.
  - Complex hyperbolic space.
  - Homogeneous negatively curved manifolds.
  - Gromov-Thurston examples.
- Triangle comparison.
- Boundary at infinity.
- $\partial\mathbb{H}^n$, $\partial\mathbb{C}\mathbb{H}^n$.
- Exercise: $\angle_p^\infty(\gamma_1, \gamma_2) \sim e^{-R(\gamma_1,\gamma_2)}$.

Negative curvature 3: $\mathrm{CAT}(-1)$ **spaces**

Examples:

- Trees.
- 2-complexes.
- Convex subsets $C \subset \mathbb{H}^n$.
- Hyperbolic manifolds with totally geodesic boundary.

Negative curvature 4: Gromov hyperbolic spaces

- Definition.
- Hyperbolic groups.
- **Thm.** Quasi-isometry invariance of hyperbolicity.

Thm. (Approximation by trees)

For every $\delta \in [0, \infty)$, $n \in \mathbb{N}$ there is a constant $A = A(\delta, n)$ with the following property.

Suppose X is δ-hyperbolic, and $Y = \{y_1, \dots, y_n\} \subset X$. Then there is a simplicial tree T and a $(1, A)$-quasi-isometric embedding

$$f : T \to X$$

such that

- T has at most $2n$ vertices.
- $Y \subset f(T)$.
- $f(T)$ is A-quasiconvex.

Def. A map $f : X \to Y$ between metric spaces is an (L, A)-**quasi-isometric embedding** if

$$L^{-1}d(x_1, x_2) - A \leq d(f(x_1), f(x_2)) \leq Ld(x_1, x_2) + A$$

for all $x_1, x_2 \in X$.

An (L, A)-**quasigeodesic** is a quasi-isometric embedding $f : I \to Y$ where I is a connected subset of R.

Morse Lemma. For every δ, (L, A) there is a $D = D(\delta, L, A)$ with the following property.

If $\gamma : [a, b] \to X$ is an (L, A)-quasigeodesic and $\bar{\gamma} : [a, b] \to X$ is a geodesic with the same endpoints, then

$$Hd(\mathrm{Im}(\gamma), \mathrm{Im}(\bar{\gamma})) < D \, .$$

Similar statements hold for quasigeodesics $I \to X$ when $I = [0, \infty)$ or $I = R$.

Cor. (QI invariance of hyperbolicity)

If X and Y are quasi-isometric geodesic metric spaces, then X is hyperbolic iff Y is hyperbolic.

The boundary

Def. Two unit speed geodesic rays $\gamma_1, \gamma_2 : [0, \infty) \to X$ are **asymptotic** if

$$\sup_{t \in [0,\infty} d(\gamma_1(t), \gamma_2(t)) < \infty \, .$$

or equivalently, if their images have finite Hausdorff distance. This defines an equivalence relation on rays.

The **(ideal) boundary** of X is the set of equivalence classes, denoted ∂X.

Lem. The set of rays leaving the basepoint p contains a representative from every equivalence class.

If γ_1, γ_2 are rays leaving p, their overlap is

$$(\gamma_1 \mid \gamma_2) = \lim_{t\to\infty} \frac{1}{2}(d(\gamma_1(t), p) + d(\gamma_2(t), p) - d(\gamma_1(t), \gamma_2(t))) \in [0, \infty) \cup \{\infty\}\,.$$

When X is proper, there will exist a geodesic $\eta : \mathbb{R} \to X$ which is asymptotic to γ_1 as $t \to -\infty$, and to γ_2 as $t \to \infty$.

Then $(\gamma_1 \mid \gamma_2)$ agrees with $d(p, \eta)$ up to error 4δ.

Def. A **visual metric** is a distance function

$$d : \partial X \times \partial X \to [0, \infty)$$

such that for every pair of rays γ_1, γ_2 leaving p,

$$C^{-1}\, e^{-a(\gamma_1|\gamma_2)} \leq d(\gamma_1, \gamma_2) \leq C\, e^{-a(\gamma_1|\gamma_2)}$$

- If X is CAT(-1) then $\angle_p^\infty$ is a visual metric with parameter 1.
- For a sufficiently small, visual metrics always exist.

Properties of visual metrics

Suppose $G \curvearrowright X$ is a discrete, cocompact, isometric action on a proper hyperbolic metric space. Then with respect to any visual metric on ∂X:

- ∂X is a compact metric space.
- ∂X is **approximately self-similar**. There is an $L \in [1, \infty)$ such that for all $r \in (0, \mathrm{diam}(\partial X))$, and every $x \in \partial X$, the rescaled ball $\frac{1}{r} B(x, r)$ is L-bilipschitz to an open set $U \subset \partial X$ of unit size.
- ∂X is **Ahlfors Q-regular** for some Q. There is a $C \in (0, \infty)$ such that for every ball $B(x, r) \subset \partial X$,

$$C^{-1} r^Q \leq \mathcal{H}^Q(B(p, r)) \leq C r^Q$$

  if $r \in [0, \mathrm{diam}(\partial X)]$.

QIs and visual metrics

Thm. Any qi $f : X \to X'$ between Gromov hyperbolic spaces induces a quasisymmetric homeomorphism $\partial f : \partial X \to \partial X'$.

Def. A homeomorphism $Z \overset{\phi}{\to} Z'$ between metric spaces is **quasisymmetric** if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that for all triples of distinct points $p, x, y \in Z$, we have

$$\frac{d(\phi(p), \phi(x))}{d(\phi(p), \phi(y))} \leq \eta\left(\frac{d(p, x)}{d(p, y)}\right)$$

Examples.

- Any bilipschitz homeomorphism is quasisymmetric.
- The map $[0, \infty) \to [0, \infty)$ given by $x \mapsto x^{\alpha}$, where $\alpha > 0$.

Thm. An (L, A)-quasi-isometry $f : X \to Y$ between hyperbolic spaces induces an η-quasi-Mobius homeomorphism $\partial f : \partial X \to \partial Y$, where $\eta = \eta(L, A)$.

Def. If Z is a metric space, and $x, y, z, w \in Z$ are distinct points, the **metric cross-ratio** is

$$[x, y, z, w] = \frac{d(x, z)}{d(x, w)} \frac{d(y, w)}{d(y, z)} .$$

Def. A homeomorphism $f : Z \to Z'$ between metric spaces is **quasi-Mobius** if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that for every 4-tuple of distinct points $\{x, y, z, w\} \subset Z$,

$$[f(x), f(y), f(z), f(w)] \leq \eta([x, y, z, w]) .$$

Exercises on QS/QM/QC

- QS $\implies$ QM $\implies$ QC, quantitatively.
- Compositions/inverses of QS/QM are QS/QM.
- QS sends balls to quasiballs.
- QM $\implies$ QS (nonquantitatively) in compact spaces.

Sketch of proof

Suppose $f : X \to Y$ is a quasi-isometry.

Step 1. There is an induced bijection $\partial f : \partial X \to \partial Y$.

Step 2. ∂f respects distances between geodesics.

Step 3. ∂f quasi-preserves additive cross-ratios.

$$\langle 1, 2, 3, 4 \rangle = (1, 3) - (1, 4) + (2, 4) - (2, 3)$$

Step 4. ∂f quasi-preserves metric cross-ratios.

Lecture 3

Metric geometry of groups:

- Q: How much is encoded in the QI class?
- Classify groups up to QI.
- Geometric mapping theory: construct/obstruct/classify QIs.
- QI invariants and QI invariant structure.

Thm. If X, Y are hyperbolic and $f : X \to Y$ is a QI, then there is an induced QM homeomorphism $f : \partial X \to \partial Y$.

- Q: How much is encoded in the QM class?
- Classify boundaries (of groups) up to QM homeomorphism.
- Geometric mapping theory: construct/obstruct/classify QM homeomorphisms.
- QM invariants and QM invariant structure.

Q: So what?

Thm. (Mostow rigidity, three versions of the hyperbolic case)

Suppose $n \geq 3$.

- Suppose M_1, M_2 are closed hyperbolic n-manifolds. If $\pi_1(M_1) \simeq \pi_1(M_2)$, then M_1 is isometric to M_2.
- Any two discrete, cocompact, isometric actions $G \curvearrowright \mathbb{H}^n$ are isometrically conjugate.
- Any two Mobius actions $G \curvearrowright S^{n-1}$ which are discrete and cocompact on triples of points are conjugate by a Mobius transformation.

Corollaries.

- Every conjugacy invariant of the action $G \curvearrowright \mathbb{H}^n$ may be viewed as an invariant of the group G. **Example:** geometric invariants of $\mathbb{H}^n/G$, such as volume, lengths of closed geodesics, etc.

- Suppose $G \curvearrowright \mathbb{H}^n$ is faithful, so $G \hookrightarrow \mathrm{Isom}(\mathbb{H}^n)$. Every automorphism $G \to G$ is induced by conjugation by an isometry $\phi : \mathbb{H}^n \to \mathbb{H}^n$ which normalizes G. Therefore

$$\mathrm{Aut}(G) \simeq N(G, \mathrm{Isom}(\mathbb{H}^n))$$

$$\mathrm{Out}(G) = \mathrm{Aut}(G)/\,\mathrm{Inn}(G) \simeq N(G, \mathrm{Isom}(\mathbb{H}^n))/G\,.$$

- $|\,\mathrm{Out}(G)| < \infty$.

- There is an $K = K(G)$ such that if G is a finite index subgroup of a torsion-free group $\hat{G}$, then

$$[\hat{G} : G] < K$$

- The abstract commensurator of G coincides with the commensurator in $\mathrm{Isom}(\mathbb{H}^n)$:

$$\mathrm{Comm}(G) \simeq \mathrm{Comm}(G, \mathrm{Isom}(H^n)) \, .$$

Thm. (Sullivan, Gromov, Tukia, Cannon-Swenson) Suppose G is a finitely generated group, and G is quasi-isometric to $\mathbb{H}^n$, $n \geq 3$.

Then there is a discrete, cocompact, isometric action $G \curvearrowright \mathbb{H}^n$.

Thus $\text{Isom}(\mathbb{H}^n)$ is a single group which (virtually) contains any finitely generated group in the quasi-isometry class of $\mathbb{H}^n$.

Remark. In fact, the proof provides (by analysis) a construction of $\text{Isom}(\mathbb{H}^n) \simeq \text{Mob}(S^{n-1})$ just starting from G.

Properties of quasiMöbius homeomorphisms between open subsets of S^n, $n \geq 2$:

- (Differentiability) If $f : U \to V$ is a QM homeomorphism, then for a.e. $x \in U$, $Df(x)$ is defined and nonsingular.
- QM homeomorphisms are absolutely continuous w.r.t. Lebesgue measure:

$$|X| = 0 \quad \Longleftrightarrow \quad |f(X)| = 0 \, .$$

- ("Liouville" theorem) If $f : S^n \to S^n$ is QM and $Df(x)$ is conformal for a.e. $x \in S^n$, then $f \in \mathrm{Mob}(S^n)$.

- (Uniformization) If g is a measurable Riemannian metric on S^2 such that

$$C^{-1} g_{S^2} \leq g \leq C g_{S^2}$$

a.e., then g is conformally equivalent to g_{S^2} by a QM homeomorphism $f : S^2 \to S^2$:

$$Df(x) : (T_x S^2, g_{S^2}(x)) \to (T_{f(x)} S^2, g(x))$$

is conformal for a.e. $x \in S^2$.

- (Compactness) If $\{f_k : S^n \to S^n\}$ is a sequence of η-QM homeomorphisms, then a subsequence either (a) converges uniformly to an η-QM homeomorphism, or (b) converges uniformly to a constant map in compact subsets of $S^n \setminus \{x\}$, for some $x \in S^n$.

Sketch of proof of Mostow rigidity

Suppose

$$G \overset{1}{\curvearrowright} H^n, \quad G \overset{2}{\curvearrowright} H^n$$

are two discrete, cocompact, isometric actions.

Thus we have two associated Mobius actions

$$G \overset{1}{\curvearrowright} S^{n-1}, \quad G \overset{2}{\curvearrowright} S^{n-1}$$

Step 1. There is a quasi-isometry

$$f : \mathbb{H}^n \to \mathbb{H}^n$$

which is G-equivariant for the respective G-actions.

Step 2. There is an induced boundary homeomorphism

$$\partial f : S^{n-1} \to S^{n-1}$$

which is QC, and G-equivariant for the respective Mobius actions

$$G \overset{1}{\curvearrowright} S^{n-1}, \quad G \overset{2}{\curvearrowright} S^{n-1}$$

Step 3. Suppose $D(\partial f)(x)$ is not conformal almost everywhere. Then on the set where $D(\partial f)(x)$ is not conformal, one has a measurable subbundle

$$V \subset TS^{n-1}$$

corresponding to the maximal stretch subspace of $D(\partial f)$. This will be G-invariant w.r.t. the action

$$G \overset{1}{\curvearrowright} S^{n-1} .$$

Step 4. If $x_0 \in S^{n-1}$ is a point of approximate continuity of V, we may use the expanding dynamics of the action

$$G \overset{1}{\curvearrowright} S^{n-1}$$

to see that in some upper half-space model for H^n, the subbundle V has constant coefficients. Then V extends canonically to a smooth subbundle except at one singular point $y \in S^{n-1}$. Then y must be fixed by G, which is a contradiction. Thus $D(\partial f)(x)$ is conformal for a.e. $x \in S^{n-1}$.

Step 5. By the Liouville theorem, ∂f is a Mobius transformation, i.e. $\partial f = \partial \bar{f}$ for a unique isometry $\bar{f} : \mathbb{H}^n \to \mathbb{H}^n$. Since $\partial \bar{f}$ is equivariant w.r.t. the two Mobius actions

$$G \overset{1}{\curvearrowright} S^{n-1}, \quad G \overset{2}{\curvearrowright} S^{n-1}$$

it follows that $\bar{f}$ is equivariant w.r.t. the two isometric actions

$$G \overset{1}{\curvearrowright} \mathbb{H}^n, \quad G \overset{2}{\curvearrowright} \mathbb{H}^n$$

Sullivan, Gromov, Tukia rigidity

- G finitely generated, with Cayley graph Γ.
- G QI to $\mathbb{H}^n$.
- $G \xrightarrow{f} \mathbb{H}^n$ QI.
- $G \curvearrowright \Gamma$ left action on the Cayley graph Γ.
- $G \curvearrowright \partial G$ boundary action. This is η-QM.
- Conjugate by $\partial f : \partial G \to \partial \mathbb{H}^n \simeq S^{n-1}$
  $\rightsquigarrow \quad G \curvearrowright S^{n-1} \quad \eta'$-QM action.
- $[g_{S^{n-1}}]$ standard conformal structure.
- $\{g \cdot [g_{S^{n-1}}] \mid g \in G\}$ $\quad G$-orbit.
- $[h]$ center of mass.
- $G \curvearrowright (S^{n-1}, h)$ is a conformal action.
- Apply uniformization $n = 3$, or use blow-up argument ($n \geq 3$).

Motivating questions

One would like to have analogs of Mostow rigidity and the Sullivan/Gromov/Tukia rigidity theorem for other groups.

Q: Can one find canonical geometries for (some classes) of groups?

For non-hyperbolic groups, a variety of structures can be used to prove rigidity theorems – coarse topology, topology of asymptotic cones, coarse differentiation.

In the hyperbolic case, the only known approach is typically via analysis on the boundary.

Lecture 4

Motivating questions

One would like to have analogs of Mostow rigidity and the Sullivan/Gromov/Tukia rigidity theorem for other groups.

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For non-hyperbolic groups, a variety of structures can be used to prove rigidity theorems – coarse topology, topology of asymptotic cones, coarse differentiation.

In the hyperbolic case, the only known approach is typically via analysis on the boundary.

Q's:

- Which metric spaces have a good theory of QM/QS/QC homeomorphisms?
- What are some QM invariants?

Properties of QM/QS/QC homeomorphisms between open subsets of S^n, $n \geq 2$:

- Every quasi-isometry $\mathbb{H}^{n+1} \to \mathbb{H}^{n+1}$ induces a QM homeomorphism $S^n \to S^n$, and conversely.
- Compositions and inverses of QM homeomorphisms are QM.
- QM homeomorphisms belong to $W^{1,n}_{\mathrm{loc}}$.
- (Differentiability) If $f : U \to V$ is a QM homeomorphism, then for a.e. $x \in U$, $Df(x)$ is defined and nonsingular.

- QM homeomorphisms are absolutely continuous w.r.t. Lebesgue measure:

$$|X| = 0 \quad \Longleftrightarrow \quad |f(X)| = 0\,.$$

- (Compactness) If $\{f_k : S^n \to S^n\}$ is a sequence of uniformly QM homeomorphisms, then some subsequence either (a) converges uniformly to a quasiconformal homeomorphism, or (b) converges uniformly to a constant map in compact subsets of $S^n \setminus \{x\}$, for some $x \in S^n$.

- (Liouville theorem) If $f : S^n \to S^n$ is QM and $Df(x)$ is conformal for a.e. $x \in S^n$, then f is a Mobius transformation.

- (Uniformization) If g is a measurable Riemannian metric on S^2 such that

$$C^{-1} g_0 \leq g \leq Cg$$

  a.e., then g is conformally equivalent to g_0 by a quasiconformal homeomorphism $f : S^2 \to S^2$:

$$Df(x) : (T_x S^2, g_0(x)) \to (T_{f(x)} S^2, g(x))$$

  is conformal for a.e. $x \in S^2$.

- If $f : S^n \to S^n$ is a homeomorphism, then TFAE:
  - f is QM.
  - f is QC.
  - f quasi-preserves n-energy.
  - f quasi-preserves n-capacity.
  - $f \in W^{1,n}$ and $\exists K > 0$ such that

$$J_f(x) := \det(Df(x)) > K \|Df(x)\|$$

    for a.e. x.

Some motivation and observations

Let Z be a Riemannian manifold.

If $u \in C^1(Z)$ the n-**energy** is:

$$E(u) := \int_Z |\nabla u|^n \, dV$$

- A C^1 diffeomorphism $S^n \supset X \to Y \subset S^n$ is conformal iff it preserves n-energy.
- A C^1 diffeomorphism $S^n \supset X \to Y \subset S^n$ is quasiconformal iff it quasipreserves n-energy.

$f : X \to Y$ conformal, $u \in C^1(Y)$. Then

$$J_f := |\det Df| = \|Df\|^n , \quad |\nabla(u \circ f)| = (|\nabla u| \circ f)\|Df\|$$

$$\begin{aligned} E(u) &= \int_Y |\nabla u|^n \, d\mathcal{L} \\ &= \int_X (|\nabla u|^n \circ f)\, J_f \, d\mathcal{L} \\ &= \int_X (|\nabla u|^n \circ f)\, \|Df\|^n \, d\mathcal{L} \\ &= \int_X |\nabla(u \circ f)|^n \, d\mathcal{L} \\ &= E(u \circ f)\,. \end{aligned}$$

Def. If $E, F \subset S^n$ are disjoint compact subsets, let

$$\mathrm{Cap}(E, F) := \inf\{E(u) \mid u\Big|_E \equiv 0\,; u\Big|_F \equiv 1\}\,.$$

This is quasipreserved by quasiconformal diffeomorphisms.

The Loewner property

For compact connected sets, the capacity is equivalent to a geometry quantity – the relative distance.

Def. If Z is a metric space, and $E, F \subset Z$, then the **relative distance** between E and F is

$$reldist(E, F) = \frac{d(E, F)}{\min(\mathrm{diam}(E), \mathrm{diam}(F))}\,.$$

Loewner property:

$$\alpha(reldist(E, F)) \leq \mathrm{Cap}(E, F) \leq \beta(reldist(E, F))$$

where

$$\alpha : [0, \infty) \to (0, \infty)$$

$$\beta : [0, \infty) \to [0, \infty) \cup \{\infty\}$$

is a function with

$$\lim_{t\to\infty} \beta(t) = 0$$

Let Z be a connected Riemannian n-manifold for $n \geq 2$.

If Z is compact, it has the Loewner property.

The **Ferrand cross-ratio** of a 4-tuple x, y, z, w is

$$\inf\{\mathrm{Cap}(E,F) \mid \{x,y\} \subset E\,,\ \{z,w\} \subset F\,,\\ E, F \subset Z \text{ compact, connected}\}$$

A quasiconformal homeomorphism quasipreserves this cross-ratio.

If the Loewner property holds, one can relate the Ferrand cross-ratio with the metric cross-ratio, and conclude that a K-QC homeomorphism is η-QM, where $\eta = \eta(K, Z)$.

Lichnerowicz Conjecture. (Ferrand) If the conformal group of a compact Riemannian manifold Z is noncompact, then Z is conformally equivalent to the standard sphere.

Modulus in metric measure spaces

Let Z be a metric space, and suppose μ is a Borel measure on Z.

Let Γ be a family of paths in Z.

Let

$$\rho : Z \to [0, \infty) \cup \{\infty\}$$

be a Borel measurable function.

Def. The ρ-**length of a path** $\gamma : I \to Z$ is

$$\mathrm{length}_\rho(\gamma) = \int_\gamma \rho \, ds \,.$$

The Q-**mass of** ρ is

$$\mathrm{Mass}_Q(\rho) = \int_Z \rho^Q \, d\mu \,.$$

ρ is Γ-**admissible** if

$$\mathrm{length}_\rho(\gamma) \geq 1$$

for every rectifiable path $\gamma \in \Gamma$.

Def. The Q-**modulus of** Γ is the infimal mass of Γ-admissible functions:

$$\mathrm{Mod}_Q(\Gamma) = \text{inf } \{ \ \mathrm{Mass}_Q(\rho) \mid \rho \text{ is } \Gamma\text{-admissible}\}$$

The Q-**modulus of a pair of subsets** $E, F \subset Z$ is the Q-modulus of the family of paths joining E to F.

Suppose (Z, μ), (Z', μ') are Riemannian n-manifolds equipped with Riemannian measures.

Let Γ be a path family in Z, $f : Z \to Z'$ be a diffeomorphism, and

$$\Gamma' = \{f \circ \gamma \mid \gamma \in \Gamma\} .$$

If f is conformal, then

$$\mathrm{Mod}_n(\Gamma') = \mathrm{Mod}_n(\Gamma) .$$

If f is K-quasiconformal, then

$$K^{-n}\,\mathrm{Mod}_n(\gamma) \leq \mathrm{Mod}_n(\Gamma') \leq K^n\,\mathrm{Mod}_n(\Gamma) .$$

Recall:

A metric space is Q-**regular** if an r-ball has Q-dimensional Hausdorff measure comparable to r^Q.

Modulus bounds

Lem. Suppose Z is Ahlfors Q-regular. Then the Q-modulus of the pair

$$(B(p,r)\ ,\ Z \setminus B(p,R))$$

is

$$\leq \frac{C}{(\log \frac{R}{r})^{Q-1}}$$

if $R \in [2r, \infty)$.

Proof. Use an admissible function of the form

$$\rho(x) = \frac{c}{d(x,p)}\,.$$

Def. If Z is a metric space, and $E, F \subset Z$, then the **relative distance** between E and F is

$$reldist(E, F) = \frac{d(E, F)}{\min(\operatorname{diam}(E), \operatorname{diam}(F))}.$$

Lem. Suppose Z is Ahlfors Q-regular, and $E, F \subset Z$ are disjoint subsets. Then

$$\mathrm{Mod}_Q(E, F) \leq \beta(reldist(E, F)),$$

where

$$\beta : [0, \infty) \to [0, \infty) \cup \{\infty\}$$

is a function with

$$\lim_{t \to \infty} \beta(t) = 0$$

which depends only on the regularity constant of Z.

Def. A compact Ahlfors Q-regular space Z is Q-**Loewner** if $Q > 1$ and there is a function

$$\alpha : [0, \infty) \to (0, \infty)$$

such that for every pair of closed connected subsets $E, F \subset Z$,

$$\mathrm{Mod}_Q(E, F) \geq \alpha(\mathit{reldist}(E, F)) .$$

For brevity, a metric space is **Loewner** if it is Q-regular and Q-Loewner for some Q.

Much of the classical theory of quasiconformal homeomorphisms carries over to Loewner spaces.

Examples

- S^n.
- S^{2n-1} with the standard Carnot/subRiemannian metric.
- (Semmes) n-regular linearly locally contractible n-manifolds.
- Noncollapsed limits of manifolds with a lower Ricci curvature bound.
- Fuchsian buildings.
- Inverse limits of graphs.

Properties of Loewner spaces

- (Quasiconvexity) There is an $L \in [1, \infty)$ such that any two points $p, q \in Z$ can be joined by a path of length at most $L\, d(p, q)$.

- There is a $\lambda \in (0, 1)$ such that if $p \in Z$, $r \in (0, \mathrm{diam}(Z))$, then any two points in

$$Z \setminus B(p, r)$$

can be joined by a path in

$$Z \setminus B(p, \lambda r)\,.$$

Thm. Suppose Z, Z' are compact Q-regular, Q-Loewner spaces. Then every quasiconformal homeomorphism $Z \to Z'$ is QM. Moreover, the distortion function depends only on the quasiconformality constant.

Outline of proof.

Step 1: If $E, F \subset Z$ are compact connected subsets, then

$$reldist(f(E), f(F)) \leq \phi(reldist(E, F)) \, .$$

for some function

$$\phi : [0, \infty) \to (0, \infty) \, .$$

Step 2: The above condition implies that f is QM.

Thm. Q-modulus is QM quasi-invariant, among compact Q-regular spaces.

Corollary. The Loewner property is QM invariant, among Q-regular spaces.

Thm. Suppose Z, Z' are Q-Loewner spaces. Then every QM homeomorphism $Z \to Z'$ is absolutely continuous w.r.t. Q-dimensional Hausdorff measure.

In other words, the $\mathcal{H}^Q$ measure class is QM invariant.

Thm. Suppose Z, Z' are Q-Loewner spaces, and $f : Z \to Z'$ is QM.

Then $f \in W^{1,Q}(Z, Z')$. In particular, there is a Borel measurable function

$$\rho : Z \to [0, \infty) \cup \{\infty\}$$

with $\rho \in L^Q(Z)$ such that for every curve

$$\gamma : I \to Z$$

$$d(f(\gamma(1)), f(\gamma(0)) \leq \int_\gamma \rho \, ds \,.$$

Thm. (QM Uniformization)

Suppose Z is a Q-Loewner metric space homeomorphic to S^2. Then $Q = 2$ and Z is QM homeomorphic to S^2.

Differentiable structures on metric measure spaces

Def. Suppose Z is a metric space. A function $u : Z \to R$ is **constant to first order at** $p \in Z$ if

$$u(z) - u(p) = o(d(z,p)) \quad \text{near } p\,.$$

A set of functions $u_1, \ldots, u_k : Z \to R$ is **dependent (to first order) at** p if there is a nontrivial linear combination

$$a_1 u_1 + \ldots + a_k u_k$$

which is constant to first order at p.

Note: A function $u : R^n \to R$ is constant to first order at p iff $Du(p) = 0$ at p. $u_1, \ldots, u_k : R^n \to R$ are dependent to first order at $p \in R^n$ if some nontrivial linear combination has zero derivative at p.

Thm.

Suppose Z is a Loewner space.

Then there is a canonically associated normed measurable vector bundle T^*Z, and a bounded linear map

$$d : \mathrm{lip}(Z) \to \Gamma_{L^\infty}(T^*Z)$$

such that for every $u \in \mathrm{lip}(Z)$,

$$\{z \in Z \mid du(z) = 0\} = \{z \in Z \mid u \text{ is constant to first order at } z\}$$

up to null sets.

Lecture 5

Recall. A metric space Z is **Loewner** if for some $Q > 1$:

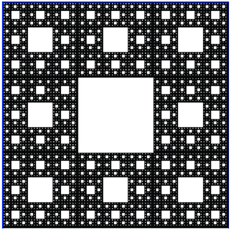
- Z is Ahlfors Q-regular.
- $\mathrm{Mod}_Q(E, F)$ is quantatively equivalent to relative distance, for $E, F \subset Z$ compact connected disjoint subsets.

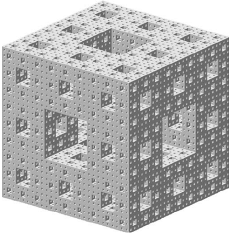
Loewner spaces have a good theory of QM/QC homeomorphisms.

Examples:

- Riemannian and sub-Riemannian manifolds.
- Boundaries of some negatively curved homogeneous spaces.
- Boundaries of Fuchsian buildings.
- Some self-similar spaces.

Nonexamples





Thm. Q-modulus is QM quasi-invariant, among compact Q-regular spaces.

Cor. The Loewner property is QM invariant, among Q-regular spaces.

Thm. Suppose Z, Z' are Q-Loewner spaces.

Then every QM homeomorphism $Z \to Z'$ is absolutely continuous w.r.t. Q-dimensional Hausdorff measure.

Thm. If Z is Q-regular and Mod_Q is nontrivial, then any metric space Z' QM homeomorphic to Z has Hausdorff dimension $\geq Q$.

Def. The **Ahlfors regular conformal dimension of** Z is:

$$\mathrm{ConfDim}(Z) := \inf\{Q' \mid Z' \overset{QM}{\longrightarrow} Z\,, \quad Z' \;\; Q'\text{-regular}\}$$

By the above:

- If Z is Q-regular and Q-modulus is nontrivial then $Q = \mathrm{ConfDim}(Z)$.
- Suppose $f : Z \to Z'$ is QM, where Z is Loewner and Z' is Ahlfors regular. Then $\mathrm{ConfDim}(Z') \geq \mathrm{ConfDim}(Z)$ with equality only if Z' is Loewner and f preserves measure classes.

Thm. Suppose Z is QM to the boundary of a hyperbolic group.

If Z is Q-regular and $Q = \mathrm{ConfDim}(Z)$, then Z is Loewner.

Needle in a haystack

Let M be a compact hyperbolic manifold with metric g.

Let d be a visual metric on $\partial\tilde{M}$.

Perturb g to another negatively curved metric g', and look at the corresponding visual metric d' on $\partial\tilde{M}$.

Then unless g' has constant curvature, the measure classes of d and d' will be mutually singular.

Necessary conditions to be QM to a Loewner space

Def. A metric space Z is **LLC** if there is an L such that for all $p \in Z$, $0 < r \leq \mathrm{diam}(X)$, the inclusions

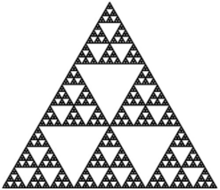
$$B(p,r) \to B(p,Lr), \quad X \setminus B(p,r) \longrightarrow X \setminus B\left(p, \frac{r}{L}\right)$$

induce the zero homomorphism on reduced 0-dimensional homology.

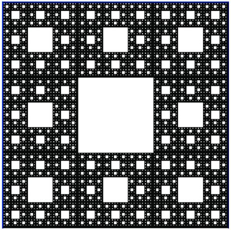
Lem. LLC is a QS invariant property.

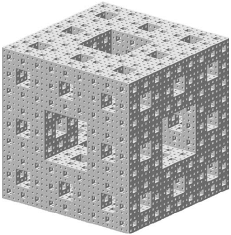
Lem. Loewner spaces are LLC.

Cor. The Sierpinski gasket is not QM to a Loewner space.



Open problem: Are the Sierpinski-carpet and Menger-sponge QM to Loewner spaces? What is their conformal dimension?





Cor. If G is a hyperbolic group, and ∂G is QM homeomorphic to a Loewner space, then ∂G is LLC.

In particular, ∂G is connected, locally connected, and has no local cut points.

Hence G cannot virtually split over a virtually cyclic group.

Thm. The above conditions are necessary, but not sufficient, for ∂G to be QM to a Loewner space.

There is a combinatorial version of modulus for a compact doubling metric space Z, defined using discrete approximations.

The combinatorial analog of the Loewner condition is called the **Combinatorial Loewner Property.**

Every Loewner space has the CLP.

Conj. Every compact (self-similar) space with the CLP is QM to a Loewner space.

Thm. The S-carpet and M-sponge have the CLP. So do many hyperbolic Coxeter groups.

QM invariant function spaces

Let G be a hyperbolic group, with Cayley graph Γ.

The boundary $\partial\Gamma$ compactifies Γ:

$$\bar{\Gamma} := \Gamma \cup \partial\Gamma \,.$$

Pick $p > 1$, and consider the collection of continuous functions $u \in C(\partial\Gamma)$ which have continuous extensions

$$\bar{u} : \Gamma \to \mathbb{R}$$

with p-summable gradient, i.e.

$$\sum_{g \in G, \sigma \in \Sigma} |u(g) - u(g\sigma)|^p < \infty \,.$$

This space is QM invariant.

Let $p_{\neq 0}$ be the infimal exponent for which the function space is nontrivial,

and p_{sep} be the infimal p for which the function space separates points.

Thm.

- $p_{sep} = \mathrm{ConfDim}(\partial\Gamma)$.
- If ∂G is QM to a Loewner space, then $p_{\neq 0} = p_{sep}$.

Open problems

- Rigidity for random groups.
- Rigidity for Gromov-Thurson examples.
- CLP implies Loewner.
- S-carpet and M-sponge are Loewner. Conformal dimension?
- Good G-invariant measure class on ∂G if G doesn't virtually split over a virtually cyclic group.