Remarks on large-scale effects of smoothing mechanisms in 3D reaction-diffusion equations

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Abstract

In [EX20], we considered a class of reaction-diffusion equations that approximates the dynamical Φ^4_3 model at large scales. These approximations involve very general smoothing mechanisms that are higher order perturbations of the Laplacian. In this note, we discuss assumptions made on the smoothing mechanism in that article. In particular, we remark that the strict positivity condition of the Fourier multiplier of the operator cannot be relaxed without other modifications, not even to allow it to reach zero outside the origin. On the other hand, if we introduce suitable Fourier cutoffs that are compatible with the non-positive part of the operator, then we expect that no smoothing assumption will be needed on its higher order terms. We then explain how this changes the coupling constant of the limiting equation, and how to modify the argument to prove the result.

Contents

1 Introduction

Let $\{\mathcal{L}_\varepsilon\}_{\varepsilon\in[0,1]}$ be a class of operators on the d-dimensional torus $\mathbf{T}^d = (\mathbf{R}/\mathbf{Z})^d$ defined via

$$\mathcal{L}_\varepsilon = -\frac{1}{\varepsilon^2}\mathcal{Q}(i\varepsilon\nabla)$$

in the sense that $\widehat{\mathcal{L}_\varepsilon}(k) = -\varepsilon^{-2}\mathcal{Q}(2\pi\varepsilon k)$ for every $k \in \mathbf{Z}^d$. Here, $\mathcal{Q} : \mathbf{R}^d \to \mathbf{R}$ is a radially symmetric function with $d+2$ continuous derivatives, and its radial version (also denoted by $\mathcal{Q}$) satisfies the following:

1. $\mathcal{Q}(0) = 0$ and $\frac{1}{2}\mathcal{Q}''(0) = 1$.
2. $\mathcal{Q}(z) > 0$ for all $z \neq 0$.
3. There exists $c, \eta > 0$ such that $\mathcal{Q}(z) > c|z|^{3+\eta}$ for all sufficiently large z.
4. For every $\delta \in (0,1)$, there exists $C_\delta > 0$ such that

$$\max_{0\leq\ell\leq5} |z|^\ell |\mathcal{Q}^{(\ell)}(z)| \leq C_\delta |\mathcal{Q}(z)|^{1+\delta}$$

   for all sufficiently large z.

A typical function $\mathcal{Q}$ satisfying the above is an arbitrary even polynomial

$$\mathcal{Q}(z) = \sum_{j=1}^{2q} \nu_j |z|^{2j}$$

with $\nu_1 = 1$, $q \geq 2$ and satisfying the positivity Condition 2 above. In this case, the operator $\mathcal{L}_\varepsilon$ is given by

$$\mathcal{L}_\varepsilon = \sum_{j=1}^{2q} (-1)^{j-1}\nu_j \Delta^j \; .$$

This (general) class of operators are large scale versions of the microscopic smoothing mechanism $\mathcal{L} = -\mathcal{Q}(i\nabla)$. In [EX20], we considered macroscopic continuous phase coexistence models on the three dimensional torus of the type

$$\partial_t \Phi_\varepsilon = (\mathcal{L}_\varepsilon - 1)\Phi_\varepsilon - \varepsilon^{-\frac{3}{2}} V'(\sqrt{\varepsilon}\Phi_\varepsilon) + \xi + C_\varepsilon \Phi_\varepsilon \; , \qquad (t,x) \in \mathbf{R}^+ \times \mathbf{T}^3 \; , \tag{1.1}$$

where $\mathcal{L}_\varepsilon$ satisfies the conditions listed above (with $d = 3$), V is an even polynomial of degree $2n \geq 4$, and ξ is the space-time white noise on $\mathbf{R} \times \mathbf{T}^3$. The process Φ_ε can be obtained from rescaling the microscopic continuous phase coexistence model with smoothing mechanism $\mathcal{L} = -\mathcal{Q}(i\nabla)$ and nonlinearity V'.

The main result of [EX20] is that under the above assumptions on $\mathcal{Q}$, there exists $C_\varepsilon \to +\infty$ such that Φ_ε converges to the solution of the dynamical $\Phi^4_3(\lambda)$ model, formally given by

$$\partial_t \phi = (\Delta - 1)\phi - \lambda\phi^3 + \xi \; , \tag{1.2}$$

and that the coupling constant λ depends on nontrivial interactions of all details of the smoothing operator $\mathcal{Q}$ and the nonlinearity V'. More precisely, one has

$$\lambda = \frac{1}{6}\mathbf{E}V^{(4)}(\mathcal{N}(0,\sigma^2)) \;, \tag{1.3}$$

where

$$\sigma^2 = \frac{1}{2}\int_{\mathbf{R}^3} \frac{1}{\mathcal{Q}(2\pi|\theta|)}\mathrm{d}\theta \;. \tag{1.4}$$

Here, we use $\mathcal{N}(0,\sigma^2)$ to denote a centered Gaussian random variable with variance σ^2. This type of result is referred to as *weak universality*. The study of such continuous models was initiated in [HQ18] with the KPZ equation (see Remark 1.5 below). In our situation, the macroscopic process Φ_ε is described by the dynamical Φ^4_3 equation with the Laplacian as the smoothing operator and a cubic term as nonlinearity, regardless of the microscopic smoothing mechanism and nonlinearities. Yet the interesting feature is that the coupling constant λ depends on all details of $\mathcal{Q}$ and V. We will discuss more on this in Section 3.1 below.

The purpose of this note is to discuss the assumptions made on the microscopic smoothing operator $\mathcal{L} = -\mathcal{Q}(i\nabla)$ for such a result to hold, and also discuss what modifications need to be made if one wishes to remove some of them. These are the assumptions on $\mathcal{Q}$ specified at the beginning of the introduction. These conditions are somewhat overlapping. For example, with the presence of Condition 1, Conditions 2 and 3 can be merged into a single one saying that $\mathcal{Q}(z) > cz^{3+\eta}$ for all z. But we still state these four conditions separately since they represent different prospects of the smoothing effect of $\mathcal{L}_\varepsilon$, and are used relatively independently of each other.

The first condition says that $\mathcal{L}_\varepsilon$ behaves like the Laplacian at low frequencies, and is certainly needed for the limiting operator to be Δ. All other three conditions measure the smoothing properties in different aspects, and can be relaxed/removed if we introduce extra suitable cutoffs into the equation. In general, to remove Condition 3, it suffices to introduce any reasonable Fourier cutoff of the noise, while to remove Condition 4, one also needs to Fourier truncate both the nonlinearity and the initial data. On the other hand, to remove Condition 2 (even to relax it to allow $\mathcal{Q}$ hitting 0), one needs the cutoff function to be carefully chosen so that it is compatible with the non-positive parts of $\mathcal{Q}$.

We say that $\rho : \mathbf{R}^d \to \mathbf{R}^+$ is a cutoff function if it takes value in $[0,1]$ and that $\rho = 1$ in a neighbourhood of the origin. For a cutoff function ρ and $\varepsilon \in (0,1)$, define the operator Π^ρ_ε by

$$\widehat{\Pi^\rho_\varepsilon f}(k) := \rho(\varepsilon k)\widehat{f}(k) \;,$$

where $\widehat{f}(k)$ denotes the k-th Fourier coefficient of f. In order to obtain a convergence statement without Conditions 2–4 on $\mathcal{Q}$, we make the following assumption on $\mathcal{Q}$ and the cutoff function ρ.

Assumption 1.1. *Let $\mathcal{Q} : \mathbf{R}^d \to \mathbf{R}$ be radially symmetric having $d+2$ continuous derivatives. Assume its radial version (also denoted by $\mathcal{Q}$) satisfies $\mathcal{Q}(0) = 0$ and $\frac{1}{2}\mathcal{Q}''(0) = 1$.*

Let

$$z_0 = \inf\{\zeta > 0 : \mathcal{Q}(\zeta) = 0\}$$

be the first zero of $\mathcal{Q}$ *outside the origin, and fix* $R_0 < \frac{z_0}{2\pi}$. *Let* $\rho : \mathbf{R}^d \to [0, 1]$ *be a radially symmetric smooth cutoff function that has compact support in the ball* $\{|\zeta| < R_0\}$.

The main convergence statement can then be modified as follows.

Theorem 1.2. *Suppose* $\mathcal{Q}$ *and* ρ *satisfy Assumption 1.1 above (with* $d = 3$*). Consider the process* u_ε *given by*

$$\partial_t u_\varepsilon = (\mathcal{L}_\varepsilon - 1)u_\varepsilon - \Pi_\varepsilon^\rho\Big(\varepsilon^{-\frac{3}{2}}V'(\sqrt{\varepsilon}u_\varepsilon) - \xi - C_\varepsilon u_\varepsilon\Big) , \quad (t, x) \in \mathbf{R}^+ \times \mathbf{T}^3 . \quad (1.5)$$

Under suitable assumptions on the convergence of the initial condition $u_\varepsilon(0, \cdot)$ *to* $\phi(0, \cdot)$*, the process* u_ε *converges to the dynamical* $\Phi^4_3(\lambda)$ *model as given in* (1.2) *with initial data* $\phi(0, \cdot)$*. The coupling constant* λ *is given by*

$$\lambda = \frac{1}{6}\mathbf{E}V^{(4)}(\mathcal{N}(0, \sigma^2)) , \quad (1.6)$$

where the variance σ^2 *of the Gaussian random variable is given by*

$$\sigma^2 = \frac{1}{2}\int_{\mathbf{R}^3} \frac{\rho^2(\theta)}{\mathcal{Q}(2\pi\theta)}\mathrm{d}\theta . \quad (1.7)$$

Remark 1.3. Comparing the right hand sides of (1.7) and (1.4), one sees the effect of the cutoff function on the limiting coupling constant. We will explain why λ and σ^2 take the above form in Section 3 below.

We use the same cutoff function for both the nonlinearity and the noise, but one could use two different ones. If we change the cutoff for the nonlinearity (and the renormalisation term) to $\widetilde{\rho}$ but still use ρ for the noise, the resulting coupling constant λ will not change. It depends on the cutoff function for the noise but not the one in front of the nonlinearity.

On the other hand, if we further add the cutoff $\Pi_\varepsilon^{\widetilde{\rho}}$ inside the nonlinearity V' (that is, replace $V'(\sqrt{\varepsilon}u_\varepsilon)$ by $V'(\sqrt{\varepsilon}\Pi_\varepsilon^{\widetilde{\rho}}u_\varepsilon)$), then the expression for σ^2 would become

$$\sigma^2 = \frac{1}{2}\int_{\mathbf{R}^3} \frac{\rho^2(\theta)\widetilde{\rho}^2(\theta)}{\mathcal{Q}(2\pi\theta)}\mathrm{d}\theta. \quad (1.8)$$

Remark 1.4. The process u_ε in (1.5) will be the main object of focus in this note. On the other hand, if one also wants to study the invariant measure (with a closed form), one could instead look at the process ϕ_ε given by

$$\partial_t \phi_\varepsilon = (\Pi_\varepsilon^{\widetilde{\rho}})^2\Pi_\varepsilon^\rho\Big((\mathcal{L}_\varepsilon - 1)\phi_\varepsilon - (\varepsilon^{-\frac{3}{2}}V'(\sqrt{\varepsilon}\Pi_\varepsilon^\rho\phi_\varepsilon) - C_\varepsilon\Pi_\varepsilon^\rho\phi_\varepsilon)\Big) + \Pi_\varepsilon^{\widetilde{\rho}}\xi ,$$

where ρ and $\widetilde{\rho}$ could be two different cutoffs satisfying Assumption 1.1. Note that in this case the limting dynamics depends on ρ but not $\widetilde{\rho}$! This is because the effect of

$\Pi_\varepsilon^{\widetilde{\rho}}$ on the correlation of the linear part of the solution was exactly cancelled out by the cutoff $(\Pi_\varepsilon^{\widetilde{\rho}})^2$ in front of the operator.

The dynamics ϕ_ε keeps the following measure invariant:

$$\nu_\varepsilon(\mathrm{d}\phi) = \frac{1}{\mathcal{Z}_\varepsilon} \exp\Big[-2 \int_{\mathbf{T}^3} \Big(\varepsilon^{-2} V(\sqrt{\varepsilon}\Pi_\varepsilon^\rho \phi) - \frac{1}{2} C_\varepsilon (\Pi_\varepsilon^\rho \phi)^2 \Big) \mathrm{d}x \Big] \mu_\varepsilon(\mathrm{d}\phi) ,$$

Here μ_ε is the Gaussian measure with covariance operator $(1-\mathcal{L}_\varepsilon)^{-1}$ restricted to the Fourier modes $\{|k| \leq \frac{R_0}{\varepsilon}\}$, and its covariance structure does not depend on the cutoff function. See also [GP16] for the situation of the KPZ equation where one also needs to add a further cutoff in front of the nonlinearity to keep the invariant measure of the dynamics.

Remark 1.5. One also expects a similar behaviour for the KPZ equation ([HQ18, Hai12]). Consider the processes h_ε given by

$$\partial_t h_\varepsilon = (\mathcal{L}_\varepsilon - 1) h_\varepsilon + \Pi_\varepsilon^\rho \Big(\varepsilon^{-1} F(\sqrt{\varepsilon}\partial_x h_\varepsilon) + \xi \Big) - C_\varepsilon , \quad (t,x) \in \mathbf{R}^+ \times \mathbf{T} , \tag{1.9}$$

where now ξ is the space-time white noise on the one dimensional torus $\mathbf{T}$, F is an even polynomial, and the operator $\mathcal{L}_\varepsilon$ has the same form as before but also acts on functions on $\mathbf{T}$. The pair $(\mathcal{Q}, \rho)$ satisfies Assumption 1.1 (with $d=1$).

The process h_ε in (1.9) models a macroscopic height fluctuation that can be obtained from a height function (called $\widetilde{h}$) for some microscopic interface from rescaling and recentering, at least when there is no cutoff in the nonlinearity.

One expects that h_ε converges to the solution of the KPZ(λ) equation, formally given by

$$\partial_t h = (\partial_x^2 - 1) h + \lambda (\partial_x h)^2 + \xi ,$$

where the coupling constant λ now has the form

$$\lambda = \frac{1}{2} \mathbf{E} F''(\mathcal{N}(0,\sigma^2)) ,$$

and the variance σ^2 is given by

$$\sigma^2 = 2\pi^2 \int_{\mathbf{R}} \frac{\theta^2 \rho^2(\theta)}{\mathcal{Q}(2\pi\theta)} \mathrm{d}\theta .$$

Here, the extra θ^2 compared to (1.7) comes from the spatial derivative in the equation. The expression can be obtained from similar analysis of the variance of the derivative of the linear solution and the second chaos component of F.

Structure of the article

We will not give a full proof of Theorem 1.2 above, but we will provide the ingredients that are necessary to adapt the proof in [EX20] to the current setting. More precisely, in Section 2, we will discuss why Conditions 1–4 were needed in [EX20], and what needs to be changed to remove them. In Section 3, we give a sketch to prove Theorem 1.2 under the current assumptions. We first explain in Section 3.1 why the constant λ takes the form described in (1.6), and then briefly describe the two (independent) parts of the proof in the following two subsections. Finally in the Appendix we provide the main statements about the semigroup $e^{t\mathcal{L}_\varepsilon}$ that are necessary to adapt the PDE part of the proof.

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2 Some remarks on the smoothing conditions 2–4

2.1 Some general remarks

The mathematical issue in dealing with equations of the type (1.1) is that the noise is too singular so that the solution map fails to be continuous in the topology where the noise lives. As we can see from the statement, different approximations $\mathcal{L}_\varepsilon$ to the Laplacian and different cutoff functions lead to different limiting equations. This is a sign of the singularity of the problem.

The main strategy in dealing with such singular equations are the theories of regularity structures ([Hai14]) and para-controlled distributions ([GIP15]). They have roots in rough path theory ([Lyo98]) which was developed for ODEs driven by rough signals. The procedure in these frameworks is as follows.

1. Construct a collection of enhanced stochastic objects, built from the original noise (in this case ξ), the integration kernel (in this case $(\partial_t - \mathcal{L}_\varepsilon + 1)^{-1}$) and certain nonlinear operations. We refer to this step as the stochastic part.

2. Show that the (deterministic) map

$$(\text{initial condition, enhanced noise, operator}) \longmapsto \text{solution}$$

is jointly continuous in a suitable topology. We refer to this step as the PDE part.

We will now discuss the relevance of Conditions 2–4. As mentioned already in the introduction, Conditions 2–4 describe different aspects of the smoothing effect of $\mathcal{L}_\varepsilon$. Conditions 3 and 4 are growth constraints on the function and its derivatives at infinity. Hence, they govern $\mathcal{L}_\varepsilon$ at frequencies that are much larger than $\frac{1}{\varepsilon}$. On the other hand, Condition 2 sets a constraint for $\mathcal{L}_\varepsilon$ at *all scales*.

We will see below that all these three conditions are essentially necessary, and relaxation / removal requires the introduction of additional cutoff functions as in (1.5).

2.2 On Conditions 3 and 4

In the original equation with space-time white noise, Condition 3 (growth of $\mathcal{Q}$) ensures that the operator has a sufficient smoothing effect in order for Φ_ε to make sense even for fixed ε. It can be removed as soon as one replace the space-time white noise ξ by its regularised version $\Pi_\varepsilon^\rho \xi$ with cutoff function ρ that decays sufficiently fast at infinity. As a consequence, the value of λ in the limiting equation will also need to depend on ρ.

Condition 4 is needed in establishing smoothing properties of the perturbed heat semi-group $e^{t\mathcal{L}_\varepsilon}$. Note that these smoothing properties are based on Besov space regularities. In the case of stationary stochastic processes, additional structures are available. In particular, the bounds for the stochastic objects arising from dynamical Φ_3^4 can all be obtained without Condition 4, while the PDE part of the proof requires it, see the comment at the beginning of Section 4 in [EX20].

The removal for Conditions 3 and 4 do not require the cutoff function to satisfy Assumption 1.1. As long as ρ has compact support, one can establish the convergence in Theorem 1.2 with Conditions 1 and 2.

2.3 The strict positivity condition

Now we turn to Condition 2, namely the strict positivity of $\mathcal{Q}$ except at the origin. The microscopic operator $\mathcal{L} = -\mathcal{Q}(i\nabla)$ is smoothing as long as $\mathcal{Q}(z) \to +\infty$ as $|z| \to +\infty$, and the values of $\mathcal{Q}$ on bounded sets do not change this effect. But this mere assumption on the behaviour of $\mathcal{Q}$ at infinity turned out to be insufficient for the macroscopic operator $\mathcal{L}_\varepsilon$ to have a uniform-in-ε smoothing effect. Indeed, we need that $\mathcal{Q}(z) > 0$ for all $z \neq 0$. This means $\mathcal{L}_\varepsilon$ needs to be smoothing at all scales.

One can already see why this condition is needed from the expression (1.4). If $\mathcal{Q}(z_0) = 0$ for some $z_0 \neq 0$, then the integrand on the right hand side of (1.4) will have a non-integrable singularity on the two dimensional sphere $\{|\theta| = \frac{|z_0|}{2\pi}\}$, and hence σ^2 will not be defined.

But the problem actually occurs at a more fundamental level. Even the linear evolution $e^{t\mathcal{L}_\varepsilon} f$ will not have the desired uniform-in-ε properties if Condition 2 is not satisfied. To see this, we note that the Fourier transform of $e^{t\mathcal{L}_\varepsilon} f$ is given by

$$\widehat{e^{t\mathcal{L}_\varepsilon} f}(k) = e^{-\frac{t}{\varepsilon^2}\mathcal{Q}(2\pi\varepsilon|k|)} \widehat{f}(k) . \tag{2.1}$$

If $\mathcal{Q}(z_0) < 0$ at some $z_0 \neq 0$, then for k such that $|k| \approx \frac{z_0}{2\pi\varepsilon}$, we will have $e^{-\frac{t}{\varepsilon^2}\mathcal{Q}(2\pi\varepsilon|k|)} > e^{\frac{ct}{\varepsilon^2}}$ for some $c > 0$. We see that $\|e^{t\mathcal{L}_\varepsilon} f\|_{L^2}$ diverges even for fixed t and general smooth f, unless $\widehat{f}$ decays faster than Gaussians.

If $\mathcal{Q} \geq 0$ but $\mathcal{Q}(z_0) = 0$, then $e^{t\mathcal{L}_\varepsilon} f$ converges to $e^{t\Delta} f$ in L^2, but will not be in a space of higher regularity than that of f, since the Fourier transform of $e^{t\mathcal{L}_\varepsilon} f$ is of the same size as $\widehat{f}$ at $|k| = \frac{z_0}{2\pi\varepsilon}$. Hence, we should not expect any uniform-in-ε smoothing effect of $e^{t\mathcal{L}_\varepsilon}$. This regularisation effect is the basis for essentially all parts of the analysis, and hence the strict positivity condition cannot be relaxed without further assumptions. This is the reason that one needs to assume that the support of ρ is disjoint with the non-positive parts of $\mathcal{Q}$ (see Assumption 1.1, and the factor 2π is a mere reflection of the choice of the size of the torus).

3 Sketch of the proof of Theorem 1.2

In this section, we briefly explain why the limiting coupling constant λ takes the form in (1.6) and (1.7), and give a sketch on how one could prove Theorem 1.2 with certain modifications from [EX20].

3.1 Value of the coupling constant

Now, let $(\mathcal{Q}, \rho)$ satisfies Assumption 1.1. Then from the above discussions, it is natural to expect that the process u_ε defined in (1.5) should also converge to $\Phi^4_3(\lambda)$. In this section, we give a heuristic derivation of the constant λ, which explains the effects of the smoothing operator, the nonlinearity and also the cutoff function.

Let Ψ_ε be the (space-time) stationary solution to

$$\partial_t \Psi_\varepsilon = (\mathcal{L}_\varepsilon - 1)\Psi_\varepsilon + \Pi^\rho_\varepsilon \xi \ . \tag{3.1}$$

Since the solution u_ε to (1.5) behaves like Ψ_ε at small scales (that is, $u_\varepsilon - \Psi_\varepsilon$ has a higher uniform-in-ε regularity than Ψ_ε), the coupling constant λ in front of the cubic term in the limiting equation should be given by the coefficient of $\Psi_\varepsilon^{\diamond 3}$ in the expansion of the right hand side of (1.5), where $X^{\diamond j}$ denotes the j-th Wick power of the Gaussian random variable X.

To see why higher order nonlinearities in V' contribute to the limiting coefficient, we take the example of a fifth power, which is $\varepsilon u_\varepsilon^5$ (i.e. $V(z) = \frac{z^6}{6}$). The term containing $\Psi_\varepsilon^{\diamond 3}$ comes from the chaos expansion of

$$\varepsilon \Psi_\varepsilon^5 = \varepsilon\big(\Psi_\varepsilon^{\diamond 5} + a_\varepsilon \Psi_\varepsilon^{\diamond 3} + b_\varepsilon \Psi_\varepsilon\big) \ ,$$

where

$$a_\varepsilon = 10 \times \mathbf{E}|\Psi_\varepsilon(t,x)|^2 \ ,$$

which does not depend on (t,x) by stationarity. The contribution to the third chaos component from $\varepsilon \Psi_\varepsilon^5$ is then $\varepsilon a_\varepsilon \Psi_\varepsilon^{\diamond 3}$, which converges to a constant multiple of $\Psi^{\diamond 3}$ as $\varepsilon \to 0$. The diverging term $\varepsilon b_\varepsilon \Psi_\varepsilon$ will be balanced out by the renormalisation $C_\varepsilon u_\varepsilon$.

In the case of a general function V', one can show that the third chaos component of $\varepsilon^{-\frac{3}{2}} V'(\sqrt{\varepsilon}\Psi_\varepsilon)$ is given by

$$\frac{1}{6}\mathbf{E} V^{(4)}(\sqrt{\varepsilon}\Psi_\varepsilon) \cdot \Psi_\varepsilon^{\diamond 3} \ ,$$

and hence the limiting coupling constant λ should be

$$\lambda = \frac{1}{6} \lim_{\varepsilon \to 0} \mathbf{E} V^{(4)}(\sqrt{\varepsilon}\Psi_\varepsilon) \ .$$

It then remains to compute the variance of $\sqrt{\varepsilon}\Psi_\varepsilon$. We have the following proposition.

Proposition 3.1. *Let Ψ_ε be the stationary solution to* (3.1)*, and denote $\widehat{\Psi}_\varepsilon(t,\cdot)$ be its Fourier transform at time t. Then we have*

$$\mathbf{E}\big(\widehat{\Psi}_\varepsilon(s,k)\widehat{\Psi}_\varepsilon(t,\ell)\big) = \delta_{k,-\ell} \cdot \frac{\rho^2(\varepsilon k) \cdot e^{-|t-s|(1+\varepsilon^{-2}\mathcal{Q}(2\pi\varepsilon k))}}{1+\varepsilon^{-2}\mathcal{Q}(2\pi\varepsilon k)} \ .$$

Proof. Since Ψ_ε is stationary in time, we have the expression

$$\widehat{\Psi}_\varepsilon(s,k) = \int_{-\infty}^{s} e^{t(\widehat{\mathcal{L}_\varepsilon}(k)-1)}\rho(\varepsilon k)\widehat{\xi}(r,k)\mathrm{d}r \;.$$

By definition of white noise, the Fourier modes $\widehat{\xi}$ satisfies

$$\mathbf{E}(\widehat{\xi}(r,k)\widehat{\xi}(r',\ell)) = \delta_{k,-\ell}\delta(r-r') \;.$$

The conclusion then follows from that $\widehat{\mathcal{L}_\varepsilon}(k) = -\varepsilon^{-2}\mathcal{Q}(2\pi\varepsilon k)$. □

With the above lemma, we can show that

$$\mathbf{E}|\Psi_\varepsilon(t,x)|^2 = \sum_{k\in\mathbf{Z}^3}\mathbf{E}|\widehat{\Psi}_\varepsilon(t,k)|^2 = \sum_{k\in\mathbf{Z}^3}\frac{\varepsilon^2\rho^2(\varepsilon k)}{2(\varepsilon^2+\mathcal{Q}(2\pi\varepsilon k))} \;.$$

The right hand side above makes sense (and is always positive) since $\rho(\varepsilon k)$ is zero if $\mathcal{Q}(2\pi\varepsilon k)$ is non-positive. From the above expression, we clearly have

$$\mathbf{E}|\sqrt{\varepsilon}\Psi_\varepsilon(t,x)|^2 = \frac{\varepsilon^3}{2}\sum_{k\in\mathbf{Z}^3}\frac{\rho^2(\varepsilon k)}{\varepsilon^2+\mathcal{Q}(2\pi\varepsilon k)} \to \frac{1}{2}\int_{\mathbf{R}^3}\frac{\rho^2(\theta)}{\mathcal{Q}(2\pi\theta)}\mathrm{d}\theta \;,$$

which is the same as (1.7). Hence, we still have $\lambda = \frac{1}{6}V^{(4)}(\mathcal{N}(0,\sigma^2))$, but now the variance σ^2 is given by the right hand side of (1.7).

Remark 3.2. If we put another cutoff $\Pi_\varepsilon^{\widetilde{\rho}}$ into the nonlinearity so that we have $V'(\sqrt{\varepsilon}\Pi_\varepsilon^{\widetilde{\rho}}u_\varepsilon)$ in the equation, then the relevant quantity would be $\sqrt{\varepsilon}\Pi_\varepsilon^{\widetilde{\rho}}\Psi_\varepsilon$. Its variance is given by (1.8) asymptotically.

The coupling constant for the limiting KPZ equation, as given in Remark 1.5, can be derived in the same way.

3.2 The stochastic objects

Recall from (3.1) the definition of the object Ψ_ε. Also recall the value of λ from Theorem 1.2. Define the processes $\square_\varepsilon$, $\circ_\varepsilon$, ${\color{gray}\bullet}_\varepsilon$ and ${\color{blue}\bullet}_\varepsilon$ by

$$\square_\varepsilon := \frac{1}{6\lambda}V^{(4)}(\sqrt{\varepsilon}\Psi_\varepsilon) \;, \qquad \circ_\varepsilon := \frac{1}{6\lambda\sqrt{\varepsilon}}V^{(3)}(\sqrt{\varepsilon}\Psi_\varepsilon)$$

$${\color{gray}\bullet}_\varepsilon := \frac{1}{3\lambda\varepsilon}V''(\sqrt{\varepsilon}\Psi_\varepsilon) - C_\varepsilon^{(1)} \;, \qquad {\color{blue}\bullet}_\varepsilon := \frac{1}{\lambda\varepsilon^{3/2}}V'(\sqrt{\varepsilon}\Psi_\varepsilon) - 3C_\varepsilon^{(1)}\Psi_\varepsilon \;,$$

where the constant $C_\varepsilon^{(1)}$ is given by

$$C_\varepsilon^{(1)} = \frac{1}{3\lambda\varepsilon}\mathbf{E}V''(\sqrt{\varepsilon}\Psi_\varepsilon) \;.$$

Let $\Psi^{\color{gray}\bullet}_\varepsilon$ and $\Psi^{\color{blue}\bullet}_\varepsilon$ be the processes given by

$$\Psi^{\color{gray}\bullet}_\varepsilon(t) = \int_{-\infty}^{t} e^{(t-r)(\mathcal{L}_\varepsilon-1)}(\Pi_\varepsilon^{\rho}\,{\color{gray}\bullet}(r))\mathrm{d}r \;, \qquad \Psi^{\color{blue}\bullet}_\varepsilon(t) = \int_{-\infty}^{t} e^{(t-r)(\mathcal{L}_\varepsilon-1)}(\Pi_\varepsilon^{\rho}\,{\color{blue}\bullet}(r))\mathrm{d}r \;.$$

We then define the three second order processes $\Psi^{(2)}_\varepsilon$, $\Psi^{(3)}_\varepsilon$ and $\Psi^{(4)}_\varepsilon$ by

$$\Psi^{(2)}_\varepsilon := \mathsf{I}(\circledcirc_\varepsilon) \circ \circledcirc_\varepsilon - C^{(2)}_\varepsilon \,, \qquad \Psi^{(3)}_\varepsilon := \mathsf{I}(\bullet_\varepsilon) \circ \circ_\varepsilon - C^{(3)}_\varepsilon \,,$$
$$\Psi^{(4)}_\varepsilon := \mathsf{I}(\bullet_\varepsilon) \circ \circledcirc_\varepsilon - (3C^{(2)}_\varepsilon + 2C^{(3)}_\varepsilon)\mathsf{I}_\varepsilon \,,$$

where $\circ$ denotes the resonance product[1], and the constants $C^{(2)}_\varepsilon$ and $C^{(3)}_\varepsilon$ are respectively given by

$$C^{(2)}_\varepsilon = \mathbf{E}[\mathsf{I}(\circledcirc_\varepsilon) \circ \circledcirc_\varepsilon] \,, \qquad C^{(3)}_\varepsilon = \mathbf{E}[\mathsf{I}(\bullet_\varepsilon) \circ \circ_\varepsilon] \,.$$

All the processes defined above are space-time stationary, and the constants are independent of the space-time point.

Essentially the same argument as in [EX20, Section 4] shows the convergence

$$(\square_\varepsilon \,, \circ_\varepsilon \,, \circledcirc_\varepsilon \,, \mathsf{I}(\bullet_\varepsilon) \,, \Psi^{(3)}_\varepsilon \,, \Psi^{(2)}_\varepsilon \,, \Psi^{(4)}_\varepsilon) \longrightarrow (1 \,, \ldots)$$

as $\varepsilon \to 0$ in a suitable topology, and the right hand side are the collection of stochastic objects that arise from the standard dynamical Φ^4_3 model.

3.3 The PDE part

Recall the definition of the constants $C^{(j)}_\varepsilon$ from the previous section. Let the renormalisation constant C_ε in (1.5) be

$$C_\varepsilon = 3\lambda C^{(1)}_\varepsilon - 9\lambda^2 C^{(2)}_\varepsilon - 6\lambda^2 C^{(3)}_\varepsilon.$$

As in [MW17a] and [EX20], the process $u_\varepsilon - \mathsf{I}_\varepsilon + \lambda \mathsf{I}(\bullet_\varepsilon)$ can be decomposed into

$$u_\varepsilon - \mathsf{I}_\varepsilon + \lambda \mathsf{I}(\bullet_\varepsilon) = v_\varepsilon + w_\varepsilon \,,$$

where the pair $(v_\varepsilon, w_\varepsilon)$ satisfies the system

$$\begin{cases} \partial_t v_\varepsilon = (\mathcal{L}_\varepsilon - 1)v_\varepsilon - 3\lambda\Pi^\rho_\varepsilon\Big((v_\varepsilon + w_\varepsilon - \lambda \mathsf{I}(\bullet_\varepsilon)) \prec \circledcirc_\varepsilon\Big) \,, \\ \partial_t w_\varepsilon = (\mathcal{L}_\varepsilon - 1)w_\varepsilon - \Pi^\rho_\varepsilon\Big(3\lambda(e^{t(\mathcal{L}_\varepsilon - 1)}v_\varepsilon(0) + w_\varepsilon) \circ \circledcirc_\varepsilon - G_\varepsilon(v_\varepsilon + w_\varepsilon)\Big) \,. \end{cases} \tag{3.2}$$

Here the initial data $(v_\varepsilon(0), w_\varepsilon(0))$ are also of the form $(\Pi^\rho_\varepsilon f_\varepsilon, \Pi^\rho_\varepsilon g_\varepsilon)$, and $\prec$ is the para-product given in (A.5). G_ε is an explicit function on $v_\varepsilon + w_\varepsilon$, which depends on the stochastic objects defined above. It is the same as in [EX20, Section 3] except that the commutator involving the (perturbed) heat kernel should be accompanied by the cutoff operator Π^ρ_ε.

One can write down the mild formulation of the system (3.2). With Assumption 1.1 on $\mathcal{Q}$ and the cutoff function ρ, the standard regularisation, continuity and commutator estimates for heat kernels all hold for $e^{t(\mathcal{L}_\varepsilon - 1)}\Pi^\rho_\varepsilon$, uniformly in ε, with the presence of the cutoff Π^ρ_ε (see the appendix for details). The point is that if the cutoff function has support disjoint from non-positive parts of $\mathcal{Q}$, then the analysis will only be concerned with positive parts of $\mathcal{Q}$, which by Assumption 1.1 is quadratic (near the origin).

[1]We have $f \circ g = f \cdot g - f \prec g - g \prec f$, where $f \cdot g$ denotes the standard pointwise product between f and g, and $\prec$ is the para-product given in (A.5).

These heat kernel estimates, which are provided in the appendix, are the proper replacements for the corresponding heat kernel estimates [EX20]. Together with the convergence of the stochastic objects, they are sufficient to guarantee that the solution $(v_\varepsilon, w_\varepsilon)$ to the system (3.2) converges to the corresponding remainder of the standard dynamical $\Phi^4_3(\lambda)$ model. The convergence u_ε to (1.2) then follows.

Appendix A Besov spaces and (perturbed) heat kernel estimates

We very briefly recall the definition of the Besov spaces. A comprehensive account can be found in [BCD11]. A concise and self-contained review can also be found in [GIP15, MW17b]. We present it on the whole space $\mathbf{R}^d$. The statements hold for the torus $\mathbf{T}^d$ as well.

Let $\widetilde{\chi}, \chi$ be two $\mathcal{C}^\infty_c(\mathbf{R}^d)$ functions taking values in $[0,1]$ such that

1. $\mathrm{supp}(\widetilde{\chi}) \subset B(0, \frac{4}{3})$, and $\mathrm{supp}(\chi) \subset B(0, \frac{8}{3}) \setminus B(0, \frac{3}{4})$.
2. $\widetilde{\chi}(\zeta) + \sum_{j=0}^{+\infty} \chi(\zeta/2^j) = 1$ for all $\zeta \in \mathbf{R}^d$.

We also define

$$\chi_{-1} := \widetilde{\chi} \;, \quad \text{and} \quad \chi_j := \chi(\cdot/2^j) \quad \text{for } j \geq 1 \;.$$

For every function/distribution f on $\mathbf{R}^d$, its Fourier transform $\widehat{f} : \mathbf{R}^d \to \mathbf{C}$ is defined by

$$\widehat{f}(\zeta) := \int_{\mathbf{R}^d} f(x) e^{-2\pi i \zeta \cdot x} \mathrm{d}x \;.$$

For every integer $j \geq -1$, we define the operator $\Delta_j : \mathcal{S}(\mathbf{R}^d) \to \mathcal{S}(\mathbf{R}^d)$ by $\widehat{\Delta_j f} = \chi_j \widehat{f}$. For every $\alpha \in \mathbf{R}$ and every Schwartz function $f \in \mathcal{S}(\mathbf{R}^d)$, define the Besov norm $\| \cdot \|_{\mathcal{B}^\alpha(\mathbf{R}^d)}$ of f by

$$\|f\|_{\mathcal{B}^\alpha} = \sup_{j \geq -1} \left(2^{\alpha j} \|\Delta_j f\|_{L^\infty} \right) \;. \tag{A.1}$$

The right hand side above is finite for every $f \in \mathcal{S}(\mathbf{R}^d)$.

Definition A.1. *For every $\alpha \in \mathbf{R}$, the Besov space $\mathcal{B}^\alpha = \mathcal{B}^\alpha(\mathbf{R}^d)$ is the completion of $\mathcal{S}(\mathbf{R}^d)$ functions with respect to the norm $\| \cdot \|_{\mathcal{B}^\alpha}$ given in* (A.1).

In what follows, $\mathcal{Q}$ and ρ are as in Assumption 1.1, and that we assume $\mathcal{Q}$ has $N \geq d+2$ derivatives.

Lemma A.2. *For every multi-index $\ell \in \mathbf{N}^d$ with $|\ell| \leq N = d+2$, there exists c and C such that*

$$\sup_{\zeta : |\zeta| \in [\frac{1}{20}, \frac{11}{3}]} \left| \partial_\zeta^\ell \left(e^{-r\mathcal{Q}(\mu\zeta)} \rho(\mu\zeta/2\pi) \right) \right| \leq C(1+\mu)^N e^{-r\mu^2}$$

for all $r \in [0, +\infty)$ and all $\mu \in \mathbf{R}^+$.

Proof. The proof is essentially the same as in [EX20, Lemma A.5]. We give a sketch here. For every multi-index $\ell \in \mathbf{N}^d$, the quantity $\partial_\zeta^\ell(e^{-r\mathcal{Q}(\mu\zeta)})$ is a linear combination of the quantities

$$\mu^{|\ell_0|}\bigg(\prod_{i=1}^m \Big(r\mu^{|\ell_i|}(\partial^{\ell_i}\mathcal{Q})(\mu\zeta)\Big)^{n_i}\bigg)\cdot e^{-r\mathcal{Q}(\mu\zeta)}\cdot(\partial^{\ell_0}\rho)(\mu\zeta/2\pi)\;,$$

where $n_1,\dots,n_m \in \mathbf{N}$ are integers and $\ell_0,\ell_1,\dots,\ell_m \in \mathbf{N}^d$ are multi-indices, and they satisfy the constraint

$$|\ell_0| + \sum_{i=1}^m n_i|\ell_i| = |\ell|\;.$$

Here, we use $|\ell|$ to denote the sum of all its components. Note that we only need to consider the case when $\frac{\mu\zeta}{2\pi}$ is in the support of $\mathcal{Q}$. By assumption, this says

$$\frac{\mu|\zeta|}{2\pi} < R_0 < \frac{z_0}{2\pi}\;,$$

where R_0 is the radius support of ρ and z_0 is the first zero of $\mathcal{Q}$ outside the origin. Note that the range of ζ is also bounded from below (by $\frac{1}{20}$). Hence, for μ and ζ within the above range, we have

$$\mu^{|\ell_i|}|(\partial^{\ell_i}\mathcal{Q})(\mu\zeta)| \lesssim (\mu|\zeta|)^{|\ell_i|}|(\partial^{\ell_i}\mathcal{Q})(\mu\zeta)| \lesssim |\mathcal{Q}(\mu\zeta)|\;,$$

where in the last inequality we used that $\mathcal{Q}$ has quadratic behaviour before z_0. Since $|\mathcal{Q}(\mu\zeta)| \geq c_0|\mu\zeta|^2$ in this domain, we have

$$\begin{aligned}&\bigg|\mu^{|\ell_0|}\bigg(\prod_{i=1}^m \Big(r\mu^{|\ell_i|}(\partial^{\ell_i}\mathcal{Q})(\mu\zeta)\Big)^{n_i}\bigg)\cdot e^{-r\mathcal{Q}(\mu\zeta)}\cdot(\partial^{\ell_0}\rho)(\mu\zeta/2\pi)\bigg|\\ &\lesssim (1+\mu)^N(r|\mathcal{Q}(\mu\zeta)|)^n e^{-r\mathcal{Q}(\mu\zeta)}\\ &\lesssim (1+\mu)^N e^{-c\mu^2\zeta^2}\\ &\lesssim (1+\mu)^N e^{-c\mu^2}\end{aligned}$$

for possibly different values of c, where in the last bound we have again used $|\zeta| \geq \frac{1}{20}$. □

We now start to give (perturbed) heat kernel estimates that are sufficient to guarantee the well-posedness and stability of the system (3.2). Recall that $\mathcal{L}_\varepsilon = -\varepsilon^{-2}\mathcal{Q}(i\varepsilon\nabla)$ in the sense that

$$\widehat{\mathcal{L}_\varepsilon}(\zeta) = -\frac{1}{\varepsilon^2}\mathcal{Q}(2\pi\varepsilon\zeta)\;.$$

Recall also Π_ε^ρ stands for the Fourier multiplier $\rho(\varepsilon\cdot)$. We also work with with the operator $e^{t\mathcal{L}_\varepsilon}$ instead of $e^{t(\mathcal{L}_\varepsilon-1)}$. This makes no difference to the PDE part since it only changes a constant multiple by e^{-t}. The bounds are used in the PDE part in Section 3.3, and only $t \in [0,T]$ for fixed $T > 0$ are needed. But the statements below are uniform over all $t \in \mathbf{R}^+$.

Theorem A.3. *We have the bound*

$$\|e^{t\mathcal{L}_\varepsilon}\Pi_\varepsilon^\rho f\|_{\mathcal{B}^\gamma} \lesssim \|f\|_{\mathcal{B}^\alpha} \tag{A.2}$$

for all $f \in \mathcal{S}(\mathbf{R}^d)$*. Furthermore, we have*

$$\|(e^{t\mathcal{L}_\varepsilon}\Pi_\varepsilon^\rho - e^{t\Delta})f\|_{\mathcal{B}^\gamma} \lesssim \varepsilon^\delta t^{-\frac{\gamma-\alpha+\delta}{2}}\|f\|_{\mathcal{B}^\alpha} . \tag{A.3}$$

The proportionality constants depend on α*,* γ*,* δ *(in the second claim) and the cutoff function* ρ*, but are independent of* t*,* ε *and* f*.*

Proof. We first prove (A.2). By the definition (A.1), it suffices to show the bound

$$2^{\gamma j}\|\Delta_j e^{t\mathcal{L}_\varepsilon}\Pi_\varepsilon^\rho f\|_{L^\infty} \lesssim t^{-\frac{\gamma-\alpha}{2}} 2^{\alpha j}\|\Delta_j f\|_{L^\infty} \tag{A.4}$$

with the proportionality constant uniform in ε and j.

Let $\varphi \in [0,1]$ be a smooth function with support contained in the annulus $B(0,\frac{10}{3}) \setminus B(0,\frac{3}{5})$ and that $\varphi = 1$ on the smaller annulus $B(0,\frac{8}{3}) \setminus B(0,\frac{3}{4})$. The Fourier transform of $\Delta_j e^{t\mathcal{L}_\varepsilon}\Pi_\varepsilon^\rho f$ will not change if we multiply it by $\varphi(\cdot/2^j)$. Hence, we have

$$\Delta_j e^{t\mathcal{L}_\varepsilon}\Pi_\varepsilon^\rho f = e^{t\mathcal{L}_\varepsilon}\Pi_\varepsilon^\rho \Delta_j f = \phi_{t,j}^{(\varepsilon)} * (\Delta_j f) ,$$

where the Fourier transform of $\phi_{t,j}^{(\varepsilon)}$ is given by

$$\widehat{\phi_{t,j}^{(\varepsilon)}}(\zeta) = e^{t\widehat{\mathcal{L}_\varepsilon}(\zeta)}\rho(\varepsilon\zeta)\varphi(\zeta/2^j) .$$

By Young's inequality, we have

$$\|\Delta_j e^{t\mathcal{L}_\varepsilon}\Pi_\varepsilon^\rho f\|_{L^\infty} \leq \|\phi_{t,j}^{(\varepsilon)}\|_{L^1}\|\Delta_j f\|_{L^\infty} .$$

It then remains to control the L^1-norm of $\phi_{t,j}^{(\varepsilon)}$. Proceeding in the same way as [GIP15, Lemma A.5], we have

$$\|\phi_{t,j}^{(\varepsilon)}\|_{L^1} \lesssim \max_{\ell \in \mathbf{N}^d : |\ell| \leq d+2} \sup_{\zeta : |\zeta| \in [\frac{3}{5}, \frac{10}{3}]} \left|\partial_\zeta^\ell \big(e^{-\frac{t}{\varepsilon^2}\mathcal{Q}(2\pi \cdot 2^j \varepsilon\zeta)}\rho(2^j\varepsilon\zeta)\big)\right| .$$

Note that since ζ is bounded from below by $|\zeta| \geq \frac{3}{5}$, and that ρ has compact support within a ball of radius R_0, so the above quantity is zero if $\frac{3}{5} \times 2^j\varepsilon > R_0$. Hence, we only need to consider the range of j and ε such that

$$2^j\varepsilon \leq \frac{5R_0}{3}.$$

Applying Lemma A.2 with $r = \frac{t}{\varepsilon^2}$ and $\mu = 2\pi \cdot 2^j\varepsilon \leq \frac{10R_0}{3}$, we get

$$\|\phi_{t,j}^{(\varepsilon)}\|_{L^1} \lesssim e^{-ct2^{2j}} = t^{-\frac{\gamma-\alpha}{2}} 2^{-(\gamma-\alpha)j}\Big((t2^{2j})^{\frac{\gamma-\alpha}{2}} e^{-ct2^{2j}}\Big) .$$

The quantity in the parenthisis on the right hand side above is uniformly bounded in t and j, and hence we obtain (A.2).

The bound (A.3) can be obtained in essentially the same way, except that one needs to control the L^1-norm of $\phi_{t,j}^{(\varepsilon)} - \phi_{t,j}$ where $\phi_{t,j}$ is the $\varepsilon = 0$ limit. This would be reduced to controlling the quantity

$$\max_{\ell:|\ell|\leq d+2} \sup_{\zeta:|\zeta|\in[\frac{3}{5},\frac{10}{3}]} \left|\partial_\zeta^\ell \big(e^{-\frac{t}{\varepsilon^2}\mathcal{Q}(2\pi\cdot 2^j\varepsilon\zeta)}\rho(2^j\varepsilon\zeta) - e^{-4\pi^2 t|\zeta|^2}\big)\right| ,$$

since the Fourier transform of the Laplacian is $-4\pi^2|\zeta|^2$. A modification of Lemma A.2 will give the desired bound for the above difference. We omit the details. □

With such cutoffs, similar statements can also be shown for continuity estimates of the heat kernel (as $t \to 0$) and commutator between the heat kernel and para-products.

Theorem A.4. *For every α, γ with $\alpha - 2 \leq \gamma \leq \alpha$ and every $\delta \in (0,1)$, we have*

$$\|(e^{t\mathcal{L}_\varepsilon}\Pi_\varepsilon^\rho - \mathrm{id})f\|_{\mathcal{B}^\gamma} \lesssim t^{\frac{\alpha-\gamma}{2}}\|f\|_{\mathcal{B}^\alpha} ,$$

and

$$\|(e^{t\mathcal{L}_\varepsilon}\Pi_\varepsilon^\rho - e^{t\Delta})f\|_{\mathcal{B}^\gamma} \lesssim \varepsilon^\delta t^{\frac{\alpha-\gamma-\delta}{2}}\|f\|_{\mathcal{B}^\alpha} .$$

Both proportionality constants are independent of ε, f and g.

We omit the proof of the continuity estimate. Define the para-product "$\prec$" by

$$f \prec g := \sum_{i\leq j-2} \Delta_i f \Delta_j g =: \sum_j \mathcal{S}_{j-1}f \cdot \Delta_j g . \tag{A.5}$$

We have the following commutator estimate.

Theorem A.5. *Let $\alpha \in (0,1)$, $\beta \in \mathbf{R}$, and $\gamma > \alpha + \beta$. Then we have*

$$\|e^{t\mathcal{L}_\varepsilon}\Pi_\varepsilon^\rho(f \prec g) - f \prec (e^{t\mathcal{L}_\varepsilon}\Pi_\varepsilon^\rho g)\|_\gamma \lesssim t^{-\frac{\gamma-\alpha-\beta}{2}}\|f\|_{\mathcal{B}^\alpha}\|g\|_{\mathcal{B}^\beta} . \tag{A.6}$$

Furthermore, for $\delta \in (0,1)$ small enough such that $\gamma - \alpha - \beta > \delta$, we have

$$\begin{aligned}&\|e^{t\mathcal{L}_\varepsilon}\Pi_\varepsilon^\rho(f \prec g) - f \prec (e^{t\mathcal{L}_\varepsilon}\Pi_\varepsilon^\rho g) - (e^{t\Delta}(f \prec g) - f \prec (e^{t\Delta}g))\|_{\mathcal{B}^\gamma}\\ &\lesssim \varepsilon^{\frac{\delta^2}{\alpha+\delta}} t^{-\frac{\gamma-\alpha-\beta+\delta}{2}}\|f\|_{\mathcal{B}^\alpha}\|g\|_{\mathcal{B}^\beta} .\end{aligned}$$

Both proportionality constants are uniform in ε, f and g.

Proof. We first decompose the left hand side of (A.6) by

$$e^{t\mathcal{L}_\varepsilon}\Pi_\varepsilon^\rho(f \prec g) - f \prec (e^{t\mathcal{L}_\varepsilon}\Pi_\varepsilon^\rho g) = \sum_{k\geq 0} h_{\varepsilon,k} ,$$

where

$$h_{\varepsilon,k} = e^{t\mathcal{L}_\varepsilon}(\mathcal{S}_{k-1}f_\varepsilon \cdot \Delta_k g_\varepsilon) - \mathcal{S}_{k-1}f_\varepsilon \cdot \Delta_k(e^{t\mathcal{L}_\varepsilon}g) ,$$

with $\mathcal{S}_{k-1} = \sum_{i\leq k-2} \Delta_i$ being the operator given in (A.5). Note that the Fourier transform of $h_{\varepsilon,k}$ is supported in

$$\Big\{\zeta : \frac{2^k}{12} < |\zeta| < \frac{10\times 2^k}{3}\Big\} ,$$

which is an annulus of size 2^k. Hence, by [BCD11, Lemma 2.84], it suffices to show that

$$\sup_k \Big(2^{\gamma k}\|h_{\varepsilon,k}\|_{L^\infty}\Big) \lesssim t^{-\frac{\gamma-\alpha-\beta}{2}}\|f_\varepsilon\|_{\mathcal{B}^\alpha}\|g_\varepsilon\|_{\mathcal{B}^\beta} .$$

Similar as before, this can be reduced to controlling the supremum norm of derivatives (in ζ) of the function

$$e^{t\widehat{\mathcal{L}_\varepsilon}(2^k\zeta)}\rho(2^k\varepsilon\zeta)\varphi(\zeta) ,$$

but now with φ supported in the slightly larger annulus $B(0,\frac{11}{3})\setminus B(0,\frac{1}{20})$, and is constantly 1 on the smaller annulus $B(0,\frac{10}{3})\setminus B(0,\frac{1}{12})$.

Also, since ρ is compactly supported, we only need to consider those k's such that $2^k\varepsilon \lesssim 1$. Hence, we can again apply Lemma A.2 to get the desired bound.

The second claim follows directly by combining (A.3) and (A.6). □

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