Candidate Number: 1234567

Axioms for $\mathbb{R}$ (or any **field**)
- $a+b=b+a$ (+ is comm.)
- $a+(b+c)=(a+b)+c$ (+ assoc.)
- $a+0=a$ (additive identity)
- $a+(-a)=0$ (additive inverses)
- $a\cdot b=b\cdot a$ (· comm.)
- $a.(b.c)=(a.b).c$ (· assoc.)
- $a.1=a$ (multiplicative identity)
- if $a\neq 0$, $a\cdot\frac{1}{a}=1$ (multiplicative inverses)
- $a.(b+c)=a.b+a.c$ (· distributes over +)
- $0\neq 1$ (avoid total collapse)

For $a\in\mathbb{R}\setminus\{0\}$, define $a^0=1$, for $k\in\mathbb{Z}^{\geq 0}$ define $a^{k+1}=a^k.a$, for $\ell\in\mathbb{Z}^{\leq -1}$ define $a^\ell=\frac{1}{a^{-\ell}}$.

Ordering axioms for $\mathbb{R}$: have $\mathbb{P}\subseteq\mathbb{R}$ st
- if $a,b\in\mathbb{P}$ then $a+b\in\mathbb{P}$ (+ and ordering)
- if $a,b\in\mathbb{P}$ then $a.b\in\mathbb{P}$ (· and ordering)
- exactly one of $a\in\mathbb{P}$, $a=0$, $-a\in\mathbb{P}$ holds.

Trichotomy: exactly one of $a<b$, $a=b$, $a>b$ holds.
Reflexivity: $a\leq a$
Antisymmetry: if $a\leq b$ and $b\leq a$ then $a=b$
Transitivity: if $a\leq b$, $b\leq c$ then $a\leq c$ (or < throughout)

Bernoulli's inequality: Take $x\in\mathbb{R}$ with $x>-1$, $n\in\mathbb{Z}^{>0}$. Then $(1+x)^n\geq 1+nx$. Induction on n.
Triangle inequality: $|a+b|\leq|a|+|b|$. **Reverse triangle inequality**: $|a+b|\geq||a|-|b||$.

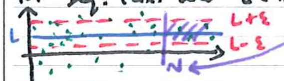
For $S\subseteq\mathbb{R}$, $\alpha\in\mathbb{R}$ is **sup S** if: $s\leq\alpha$ $\forall s\in S$ and if $s\leq b$ $\forall s\in S$ then $\alpha\leq b$. **Inf**: $\geq$ throughout.
Completeness axiom for $\mathbb{R}$: let $S\subseteq\mathbb{R}$ be non-empty and bdd above. Then S has a supremum.
Approximation property: let $S\subseteq\mathbb{R}$ be non-empty & bdd above. Then $\forall\varepsilon>0$ $\exists s_\varepsilon\in S$ st $\sup S-\varepsilon<s_\varepsilon\leq\sup S$.
$\sqrt{2}$ exists: consider $S=\{s\in\mathbb{R}: s>0, s^2<2\}$, show it has a sup α, show $\alpha^2=2$ using trichotomy.
Archimedean property of $\mathbb{N}$: $\mathbb{N}$ is not bdd above. So $\forall\varepsilon>0$ $\exists n\in\mathbb{N}$ st $0<\frac{1}{n}<\varepsilon$.

Set A **finite** if $A=\emptyset$ or $\exists n\in\mathbb{N}$ st $\exists$ bijection $f:A\to\{1,2,\dots,n\}$. A **infinite** if not finite.
Set A **countably infinite** if $\exists$ bijection $f:A\to\mathbb{N}$. **Countable** if $\exists$ injection $f:A\to\mathbb{N}$.
Uncountable if A not countable.
$\mathbb{Q}$ countable, $\mathbb{R}$ uncountable.
↳ Cantor diagonal argument.

injection: different elements map to different elements
surjection: every element hit by map
bijection: injection & surjection.

Real **sequence**: function $a:\mathbb{N}\to\mathbb{R}$. **Tail**: given seq (a_n), (b_n) is a tail if $\exists k\in\mathbb{N}$ st $b_n=a_{n+k}$ $\forall n\geq 1$.
Subsequence: (b_n) where $\exists f:\mathbb{N}\to\mathbb{N}$ strictly increasing st $b_r=a_{f(r)}$ for $r\geq 1$.

For seq. (a_n) and $L\in\mathbb{R}$ (or $L\in\mathbb{C}$), say $a_n\to L$ as $n\to\infty$ if $\forall\varepsilon>0$ $\exists N\in\mathbb{N}$ st $\forall n\geq N$ $|a_n-L|<\varepsilon$.



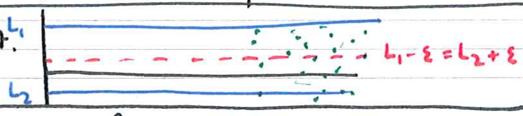
beyond here all terms lie within ε of L.

(a_n) **converges** if $\exists L\in\mathbb{R}$ (or $\mathbb{C}$) st $a_n\to L$ as $n\to\infty$.

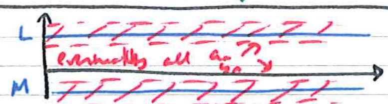
Tails lemma: If seq. (a_n) converges to L then every tail of (a_n) converges, also to L. If a tail of (a_n) converges then (a_n) converges.

If $c\in\mathbb{R}$ and $|c|<1$ then $c^n\to 0$. Write $|c|=\frac{1}{1+y}$ with $y>0$, use Bernoulli. $\frac{n}{2^n}\to 0$. Use $\binom{n}{2}\leq 2^n$.

Uniqueness of limits. A convergent seq. has a unique limit. ※, can't have all terms near L_1 & near L_2 if $L_1\neq L_2$.



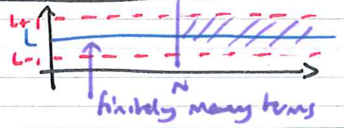
Limits preserve weak inegs (but not strict!). Say $a_n\to L$, $b_n\to M$, $a_n\leq b_n$ $\forall n$. Then $L\leq M$. ※, suppose $L>M$.



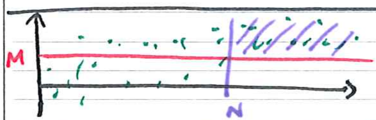
Sandwiching. If (a_n), (b_n), (c_n) real seqs and $a_n\leq b_n\leq c_n$ $\forall n$ and $a_n\to L$ and $c_n\to L$, then $b_n\to L$. (defn of conv.)

$(-1)^n$ bounded but not convergent.

(a_n) **bounded** if $\exists M$ st $|a_n|\leq M$ $\forall n\geq 1$.
A **convergent seq is bounded**.



$M=\max\{|a_1|,|a_2|,\dots,|a_N|,|L|+1\}$.



(a_n) real seq. **tends to ∞** as $n\to\infty$ if $\forall M\in\mathbb{R}$ $\exists N\in\mathbb{N}$ st $\forall n\geq N$ $a_n>M$.

If $\alpha<0$ then $n^\alpha\to 0$ as $n\to\infty$. If $\alpha>0$ then $n^\alpha\to\infty$ as $n\to\infty$. Use $n^\alpha=e^{\alpha\log n}$.
For $c\in\mathbb{R}^{>0}$, if $c<1$ then $c^n\to 0$, if $c=1$ then $c^n\to 1$, if $c>1$ then $c^n\to\infty$.

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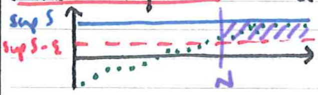
AOL. Take (a_n), (b_n) seq. with $a_n \to L$, $b_n \to M$. Take constant c.
- if $a_n = c$ $\forall n$ then $a_n \to c$. defn
- (ca_n) conv. and $ca_n \to cL$. defn
- $(a_n \pm b_n)$ conv. and $a_n \pm b_n \to L \pm M$. defn, Δ ineq.
- $(|a_n|)$ conv. and $|a_n| \to |L|$. reverse Δ ineq + defn.
- $(a_n b_n)$ conv. and $a_n b_n \to LM$. $|a_n b_n - LM| \leq |a_n||b_n - M| + |M||a_n - L|$.
- if $M \neq 0$ $(\frac{1}{b_n})$ conv. and $\frac{1}{b_n} \to \frac{1}{M}$. eventually b_n close to M so $b_n \neq 0$, $|\frac{1}{b_n} - \frac{1}{M}| = \frac{|b_n - M|}{|M||b_n|}$ sever. num. small, b_n close to M.
- if $M \neq 0$, $(\frac{a_n}{b_n})$ conv. and $\frac{a_n}{b_n} \to \frac{L}{M}$. earlier parts.

Take (z_n) complex seq., with $z_n = x_n + iy_n$ $(x_n, y_n \in \mathbb{R})$. Then (z_n) conv. iff both (x_n), (y_n) conv.

If (a_n) conv. then every subseq. of (a_n) conv., to same limit as (a_n). Use defn, note $n_r \geq r$.

For seq. (a_n), (b_n), write $a_n = O(b_n)$ as $n \to \infty$ if $\exists C \in \mathbb{R}^{>0}$ & $\exists N$ st if $n \geq N$ then $|a_n| \leq C|b_n|$.
If $b_n \neq 0$ $\forall$ suff. large n, write $a_n = o(b_n)$ as $n \to \infty$ if $\frac{a_n}{b_n} \to 0$ as $n \to \infty$.

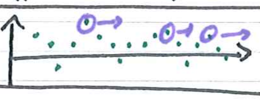
Monotone Seq. Thm. Take (a_n) real seq. If (a_n) inc. & bdd above then (a_n) conv. Similarly dec & bdd below.



Use Approx. property.

For $c \geq 0$, $\sqrt{c}$ exists. Define $a_1 = 1$, $a_{n+1} = \frac{1}{2}\left(a_n + \frac{c}{a_n}\right)$.
Show (a_n) dec & bdd below. $\frac{\log n}{n} \to 0$. dec, bdd below.

Scenic Viewpoints Thm. Let (a_n) be real seq. Then (a_n) has monotone subseq.
Let $V = \{k \in \mathbb{N} : \text{if } m > k \text{ then } a_m < a_k\}$. V infinite or V finite.

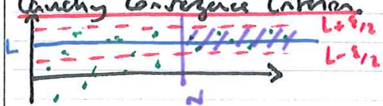


Bolzano-Weierstrass Thm. A bounded real seq. has a convergent subseq. Scenic Viewpoints + MST.

(a_n) Cauchy seq. if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ st $\forall m, n \geq N$ $|a_n - a_m| < \varepsilon$.
If (a_n) Cauchy and (a_{n_r}) conv. then (a_n) converges.

A Cauchy seq. is bdd.
Adapt from conv. seq. bdd.

Cauchy Convergence Criterion. A seq. (a_n) is conv. iff (a_n) Cauchy.



$(\Rightarrow)$ Δ ineq + diagram.
$(\Leftarrow)$ use B-W.

$\sum r^n$ conv. for $|r| < 1$, div. $|r| \geq 1$.
$\sum \frac{1}{k^2}$ converges
$\sum \frac{1}{k(k+1)}$ converges.

$\sum_{k=1}^{\infty} a_k$ converges if seq. (s_n) of partial sums converges. $s_n := \sum_{k=1}^{n} a_k$. Conv. abs. if $\sum |a_k|$ conv.

If $\sum a_k$ converges then $a_k \to 0$ as $k \to \infty$. Converse false eg $\sum \frac{1}{k}$ diverges. (Partial sums not Cauchy).

Comparison Test. Take (a_k), (b_k) real seq. with $0 \leq a_k \leq b_k$ $\forall k$ and $\sum b_k$ conv. Then $\sum a_k$ conv. MST.

Cauchy Convergence Criterion for series. Let (a_n) be a seq., set $s_n = \sum_{k=1}^{n} a_k$. Then $\sum a_k$ converges iff
$\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ st $\forall n > m \geq N$ $|s_n - s_m| = \left|\sum_{k=m+1}^{n} a_k\right| < \varepsilon$.

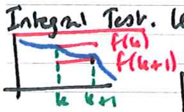
Absolute convergence implies convergence. Use partial sums + Cauchy criterion.

$\sum k^{-p}$ diverges for $p \leq 1$, converges for $p > 1$. Comparison Test, integral test, harmonic series.

Alternating Series Test. for (u_k) real seq, if $u_k \geq 0$ $\forall k$ and (u_k) decreasing and $u_k \to 0$, then $\sum (-1)^{k-1} u_k$ converges. Consider partial sums, get subseqs monotone & bdd. Eg $\sum \frac{(-1)^n}{n}$ conv.

Ratio Test. Take (a_k) real seq. of +ve (nonzero) terms. Assume $\frac{a_{k+1}}{a_k}$ $\left(\left|\frac{a_{k+1}}{a_k}\right|\right) \to L$ as $k \to \infty$.
If $0 \leq L < 1$ then $\sum a_k$ converges. If $L > 1$ then $\sum a_k$ diverges.

Integral Test. Let $f : [1, \infty) \to \mathbb{R}$ be non-neg, dec, with $\int_k^{k+1} f(x)\,dx$ exists $\forall k$. Let $s_n = \sum_{k=1}^{n} f(k)$, $I_n = \int_1^n f(x)\,dx$.



Let $\sigma_n = s_n - I_n$. Then (σ_n) conv. say to σ and $0 \leq \sigma \leq f(1)$. And $\sum f(k)$ conv. iff (I_n) conv.

Let $\gamma_n = \sum_{k=1}^{n} \frac{1}{k} - \log n$. Then (γ_n) converges, to Euler's constant γ.

Radius of convergence of $\sum c_k z^k$: $R := \sup\{|z| \in \mathbb{R} : \sum |c_k z^k| \text{ converges}\}$ if sup exists, ∞ otherwise.
If $R > 0$ and $|z| < R$ then $\sum c_k z^k$ conv. absolutely so conv. If $|z| > R$ then $\sum c_k z^k$ diverges.

Differentiation Thm. Can differentiate a power series term by term inside its circle of convergence.